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Research Article

On Symplectic Analysis for the Plane Elasticity Problem of Quasicrystals with Point Group 12 mm

Hua Wang, Jianrui Chen, and Xiaoyu Zhang

College of Sciences, Inner Mongolia University of Technology, Hohhot 010051, China

Correspondence should be addressed to Hua Wang; huawang96@yahoo.com

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The symplectic approach, the separation of variables based on Hamiltonian systems, for the plane elasticity problem of quasicrystals with point group 12 mm is developed. By introducing appropriate transformations, the basic equations of the problem are converted to two independent Hamiltonian dual equations, and the associated Hamiltonian operator matrices are obtained. The study of the operator matrices shows the feasibility of the method. Without any assumptions, the general solution is presented for the problem with mixed boundary conditions.

1. Introduction

Quasicrystals (QCs), a new material and structure, were first discovered by the authors in [1] in 1984. QCs that exhibit excellent physical and mechanical properties, such as low friction, high hardness, and high wear resistance, have promising potential applications (cf. [2]). It is well known that the general solution of quasicrystal elasticity is very important, but it is difficult to be obtained because of the complexity of the basic governing equations. So far, many methods and techniques have been developed to seek for the general solution (see, e.g., [3–8]). However, some problems of quasicrystal elasticity have not been solved well due to the complicated assumptions of the solution, and the symplectic approach, developed by Zhong [9], may be helpful in those problems.

The symplectic approach has advantages of avoiding the difficulty of solving high order differential equations and having no any further assumptions and has been applied into various research fields such as elasticity [10–12], viscoelasticity [13], fluid mechanics [14], piezoelectric material [15], and functionally graded effects [16]. In this method, one needs to transform the considered problem into Hamiltonian dual equations and then obtains the desired Hamiltonian operator matrix. Based on the eigenvalue analysis and eigenfunction expansion, the analytical solution of the problem can be explicitly presented. It should be noted that the feasibility of this method depends on the completeness of eigenfunction systems of the corresponding Hamiltonian operator matrices.

To the best of the author's knowledge, there are no reports of the method on the analysis of QCs. The objective of this paper is to propose the symplectic approach for the plane elasticity problem of quasicrystals with point group 12 mm. After derivation of two independent Hamiltonian dual equations of the problem, we prove the completeness of eigenfunction systems for the corresponding Hamiltonian operator matrices. Finally, we obtain the analytical solution with the use of the eigenfunction expansion.

2. Basic Equations and Their Hamiltonian Dual Equations

According to the quasicrystal elasticity theory, we have the deformation geometry equations of the plane elasticity problem of quasicrystals with point group 12 mm

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad w_{ij} = \frac{\partial w_i}{\partial x_j}, \quad i, j = 1, 2, \quad (1)$$

the equilibrium equations

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = 0, \qquad \frac{\partial H_{ij}}{\partial x_j} + g_i = 0, \tag{2}$$

and the generalized Hooke's law

$$\begin{aligned}
\sigma_{xx} &= C_{12}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + 2C_{66}\varepsilon_{xx},\\
\sigma_{yy} &= C_{12}\left(\varepsilon_{xx} + \varepsilon_{yy}\right) + 2C_{66}\varepsilon_{yy},\\
\sigma_{xy} &= \sigma_{yx} = 2C_{66}\varepsilon_{xy},\\
H_{xx} &= K_1 w_{xx} + K_2 w_{yy},\\
H_{yy} &= K_1 w_{yy} + K_2 w_{xx},\\
H_{xy} &= \left(K_1 + K_2 + K_3\right) w_{xy} + K_3 w_{yx},\\
H_{yx} &= \left(K_1 + K_2 + K_3\right) w_{yx} + K_3 w_{xy}.
\end{aligned} \tag{3}$$

Here u_i and w_i are the phonon and phason displacements, σ_{ij} and ε_{ij} are the phonon stresses and strains, H_{ij} and w_{ij} are the phason stresses and strains, C_{12}, C_{66}, K_1, K_2, and K_3 are the elastic constants, and f_i and g_i are the body and generalized body forces, respectively.

Substituting (1) and (3) into (2), we get the displacement equilibrium equations

$$\begin{aligned}
&C_{66}\nabla^2 u_x + \left(C_{12} + C_{66}\right)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + f_1 = 0,\\
&C_{66}\nabla^2 u_y + \left(C_{12} + C_{66}\right)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + f_2 = 0,\\
&K_1\nabla^2 w_x + \left(K_2 + K_3\right)\frac{\partial}{\partial y}\left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right) + g_1 = 0,\\
&K_1\nabla^2 w_y + \left(K_2 + K_3\right)\frac{\partial}{\partial x}\left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right) + g_2 = 0,
\end{aligned} \tag{4}$$

where $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$. Let

$$\begin{aligned}
q_1 &= \frac{C_{66}}{C_{12} + 2C_{66}}\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right),\\
q_2 &= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y},\\
q_3 &= \frac{K_1 + K_2 + K_3}{K_1}\left(\frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x}\right),\\
q_4 &= -\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y}.
\end{aligned} \tag{5}$$

Then (4) can be expressed in the following matrix forms:

$$\frac{\partial}{\partial y} Z_1 = H_1 Z_1 + F_1, \tag{6}$$

$$\frac{\partial}{\partial y} Z_2 = H_2 Z_2 + F_2, \tag{7}$$

where $Z_1 = (u_x, u_y, q_1, q_2)^T$, $Z_2 = (w_x, w_y, q_3, q_4)^T$, $F_1 = -(1/(C_{12} + 2C_{66}))(0, 0, f_1, f_2)^T$, $F_2 = (-1/K_1)(0, 0, g_1, g_2)^T$,

$$\begin{aligned}
H_1 &= \begin{pmatrix} 0 & \frac{\partial}{\partial x} & \frac{C_{12} + 2C_{66}}{C_{66}} & 0 \\ -\frac{\partial}{\partial x} & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial x} & 0 \end{pmatrix},\\
H_2 &= \begin{pmatrix} 0 & -\frac{\partial}{\partial x} & \frac{K_1}{K_1 + K_2 + K_3} & 0 \\ \frac{\partial}{\partial x} & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & -\frac{\partial}{\partial x} & 0 \end{pmatrix}.
\end{aligned} \tag{8}$$

Obviously, $H_1^T = JH_1J$ and $H_2^T = JH_2J$, in which $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ is the symplectic matrix with I_2 being the 2×2 identity matrix. Thus, H_1 and H_2 are both Hamiltonian operator matrices and (6) and (7) are exactly the Hamiltonian dual equations for the plane elasticity problem of quasicrystals with point group 12 mm.

We consider the problem satisfying the mixed boundary conditions

$$\begin{aligned}
u_x &= 0, \quad \sigma_{xy} = 0, \quad \text{for } x = 0,\ x = h,\\
w_x &= 0, \quad H_{xy} = 0, \quad \text{for } x = 0,\ x = h.
\end{aligned} \tag{9}$$

From (9) and (3), we have

$$q_1 = q_3 = 0, \quad \text{for } x = 0,\ x = h. \tag{10}$$

3. Theoretical Analysis

In the following, we only discuss (6), and the analysis for (7) is similar.

First, considering the homogeneous equation of (6),

$$\frac{\partial}{\partial y} Z_1 = H_1 Z_1. \tag{11}$$

Applying the method of separation of variables to the above equation, we write the solution as

$$Z_1 = X(x)\, Y(y), \tag{12}$$

in which $Y(y) = e^{\mu y}$, and μ and $X(x)$ are the eigenvalue of the Hamiltonian operator matrix H_1 and its associated eigenvector, respectively. They are determined by the equation

$$H_1 X(x) = \mu X(x). \tag{13}$$

Solving (13) with the boundary conditions (9) and (10) at $x = 0, h$, we obtain the eigenvalues of H_1:

$$\mu_0 = 0, \qquad \mu_n = \frac{n\pi}{h}, \qquad \mu_{-n} = -\frac{n\pi}{h}, \qquad n = 1, 2, \ldots, \tag{14}$$

and the associated eigenvectors of μ_0, μ_n, and μ_{-n} are

$$X_0^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad X_n^0 = \begin{pmatrix} \sin(\mu_n x) \\ -\cos(\mu_n x) \\ 0 \\ 0 \end{pmatrix}, \qquad X_{-n}^0 = \begin{pmatrix} -\sin(\mu_n x) \\ -\cos(\mu_n x) \\ 0 \\ 0 \end{pmatrix}, \tag{15}$$

respectively. From

$$H_1 X_n^1(x) = \mu_n X_n^1(x) + X_n^0(x), \tag{16}$$

and the imposed boundary conditions, the first-order Jordan form eigenvectors of μ_0, μ_n, and μ_{-n} can be solved as

$$X_0^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$X_n^1 = \begin{pmatrix} \sin(\mu_n x) \\ \left(\dfrac{C_{12}+3C_{66}}{(C_{12}+C_{66})\mu_n} - 1\right)\cos(\mu_n x) \\ \dfrac{2C_{66}}{C_{12}+C_{66}}\sin(\mu_n x) \\ \dfrac{2C_{66}}{C_{12}+C_{66}}\cos(\mu_n x) \end{pmatrix},$$

$$X_{-n}^1 = \begin{pmatrix} -\sin(\mu_n x) \\ -\left(\dfrac{C_{12}+3C_{66}}{(C_{12}+C_{66})\mu_n} + 1\right)\cos(\mu_n x) \\ -\dfrac{2C_{66}}{C_{12}+C_{66}}\sin(\mu_n x) \\ \dfrac{2C_{66}}{C_{12}+C_{66}}\cos(\mu_n x) \end{pmatrix}, \tag{17}$$

respectively. Besides, we can verify that there are no other high-order Jordan form eigenvectors in every chain.

It is easy to prove that the above eigenvectors and Jordan form eigenvectors satisfy the symplectic conjugacy and orthogonality; that is,

$$\int_0^h X_0^{0T} J X_0^0 dx = \int_0^h X_0^{0T} J X_n^1 dx = \int_0^h X_0^{1T} J X_n^0 dx = \int_0^h X_0^{1T} J X_n^1 dx = 0,$$

$$\int_0^h X_n^{0T} J X_n^0 dx = \int_0^h X_n^{0T} J X_n^1 dx = \int_0^h X_n^{0T} J X_{-n}^0 dx = \int_0^h X_n^{1T} J X_{-n}^1 dx = 0,$$

$$\int_0^h X_0^{0T} J X_0^1 dx = h, \qquad \int_0^h X_n^{0T} J X_{-n}^1 dx = -\frac{2C_{66}}{C_{12}+C_{66}} h, \qquad n = \pm 1, \pm 2, \ldots. \tag{18}$$

Next, we will prove the symplectic orthogonal expansion theorem, that is, the completeness theorem of the generalized eigenvector system (i.e., the collection of all the eigenvectors and Jordan form eigenvectors), which shows that the symplectic method can be adopted to solve the title problem.

Theorem 1. *The generalized eigenvector system*

$$\left\{X_0^0, X_0^1\right\} \cup \left\{X_n^0, X_n^1 \mid n = \pm 1, \pm 2, \ldots\right\} \tag{19}$$

of the Hamiltonian operator matrix H_1 is complete in the Hilbert space X; that is, there exist constant sequences $\{c_0^0, c_0^1\}$, $\{c_n^i\}_{n=1}^{\infty}$, and $\{c_{-n}^i\}_{n=1}^{\infty}$ $(i = 0, 1)$ such that

$$\Phi = c_0^0 X_0^0 + c_0^1 X_0^1 + \sum_{n=1}^{+\infty}\left(c_n^0 X_n^0 + c_n^1 X_n^1 + c_{-n}^0 X_{-n}^0 + c_{-n}^1 X_{-n}^1\right) \tag{20}$$

for each $\Phi = (\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x))^T \in X$, where $X = L^2[0,h] \times L^2[0,h] \times L^2[0,h] \times L^2[0,h]$.

Proof. For any $\Phi \in V$, in order to prove equality (20), we set

$$c_0^0 = \frac{\int_0^h \Phi^T J X_0^1 dx}{\int_0^h X_0^{0T} J X_0^1 dx} = \frac{\int_0^h \phi_2 dx}{h},$$

$$c_0^1 = \frac{\int_0^h \Phi^T J X_0^0 dx}{\int_0^h X_0^{1T} J X_0^0 dx} = \frac{\int_0^h \phi_4 dx}{h},$$

$$c_n^0 = \frac{\int_0^h \Phi^T J X_{-n}^1 dx}{\int_0^h X_n^{0T} J X_{-n}^1 dx} = \frac{\int_0^h \phi_1 \sin(\mu_n x)\, dx - \int_0^h \phi_2 \cos(\mu_n x)\, dx}{h} - \left(\int_0^h \phi_3 \sin(\mu_n x)\, dy + \left(\frac{(C_{12}+3C_{66})}{(C_{12}+C_{66})\mu_n} + 1\right) \times \int_0^h \phi_4 \cos(\mu_n x)\, dx\right) \times \left(\frac{2C_{66}}{C_{12}+C_{66}} h\right)^{-1},$$

$$c_n^1 = \frac{\int_0^h \Phi^T J X_{-n}^0 dx}{\int_0^h X_n^{1T} J X_{-n}^0 dx} = \frac{\int_0^h \phi_3 \sin(\mu_n x)\, dx + \int_0^h \phi_4 \cos(\mu_n x)\, dx}{\left(2C_{66}/\left(C_{12}+C_{66}\right)\right) h} \tag{21}$$

by the symplectic orthogonal relationship (18). Then,

$$c_0^0 X_0^0 + c_0^1 X_0^1 + \sum_{n=1}^{+\infty} \left(c_n^0 X_n^0 + c_n^1 X_n^1 + c_{-n}^0 X_{-n}^0 + c_{-n}^1 X_{-n}^1 \right)$$
$$= \begin{pmatrix} \sum_{n=1}^{+\infty} \frac{2}{h} \int_0^h \left(\phi_1 \sin \frac{n\pi x}{h} dx \right) \sin \frac{n\pi x}{h} \\ \frac{1}{h} \int_0^h \phi_2 dx + \sum_{n=1}^{+\infty} \frac{2}{h} \left(\int_0^h \phi_2 \cos \frac{n\pi x}{h} dx \right) \cos \frac{n\pi x}{h} \\ \sum_{n=1}^{+\infty} \frac{2}{h} \int_0^1 \left(\phi_3 \sin \frac{n\pi x}{h} dx \right) \sin \frac{n\pi x}{h} \\ \frac{1}{h} \int_0^h \phi_4 dx + \sum_{n=1}^{+\infty} \frac{2}{h} \left(\int_0^h \phi_4 \cos \frac{n\pi x}{h} dx \right) \cos \frac{n\pi x}{h} \end{pmatrix}, \tag{22}$$

in which the four components of the above expression are the corresponding Fourier series of $\phi_1, \phi_2, \phi_3, \phi_4$ associated with the orthogonal function system $\{\sin(n\pi x/h)\}_{n=1}^{+\infty}$ or $\{\cos(n\pi x/h)\}_{n=0}^{+\infty}$ in $L^2[0,h]$. Therefore, equality (20) is valid, which means that the generalized eigenvector system of H_1 is complete in the Hilbert space X. □

Similarly, for the Hamiltonian operator matrix H_2, we also have the following completeness theory.

Theorem 2. *The generalized eigenvector system*

$$\left\{ \overline{X}_0^0, \overline{X}_0^1 \right\} \cup \left\{ \overline{X}_n^0, \overline{X}_n^1 \mid n = \pm 1, \pm 2, \ldots \right\} \tag{23}$$

of the Hamiltonian operator matrix H_2 is complete in the Hilbert space X, where

$$\overline{X}_0^0 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \overline{X}_n^0 = \begin{pmatrix} \sin(\lambda_n x) \\ \cos(\lambda_n x) \\ 0 \\ 0 \end{pmatrix},$$
$$\overline{X}_0^1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{24}$$
$$\overline{X}_n^1 = \begin{pmatrix} \sin(\lambda_n x) \\ \left(1 + \frac{2K_1 + K_2 + K_3}{\lambda_n (K_2 + K_3)} \right) \cos(\lambda_n x) \\ -\left(\frac{2K_1}{K_2 + K_3} + 2 \right) \sin(\lambda_n x) \\ \left(\frac{2K_1}{K_2 + K_3} + 2 \right) \cos(\lambda_n x) \end{pmatrix}$$

are the associated eigenvectors and the first-order Jordan form eigenvectors of the eigenvalue $\lambda_0 = 0$ and $\lambda_n = n\pi/h$ of H_2.

4. General Solution

By completeness Theorem 1, the general solution of the inhomogeneous equation (6) is represented in the form

$$Z_1(x,y) = X_0^0 Y_0^0(y) + X_0^1 Y_0^1(y) + \sum_{n=1}^{+\infty} \left(X_n^0 Y_n^0(y) + X_n^1 Y_n^1(y) + X_{-n}^0 Y_{-n}^0(y) + X_{-n}^1 Y_{-n}^1(y) \right). \tag{25}$$

The vector F_1 can also be expanded as

$$F_1 = X_0^0 F_0^0(y) + X_0^1 F_0^1(y) + \sum_{n=1}^{+\infty} \left(X_n^0 F_n^0(y) + X_n^1 F_n^1(y) + X_{-n}^0 F_{-n}^0(y) + X_{-n}^1 F_{-n}^1(y) \right). \tag{26}$$

Multiplying both sides of (26) by $X_0^{1T} J$, $X_0^{0T} J$, $X_{-n}^{1\ T} J$, $X_{-n}^{0\ T} J$, $X_n^{1T} J$, and $X_n^{0T} J$ and then integrating by x from 0 to h, respectively, we have

$$F_0^0(y) = \frac{\int_0^h X_0^{1T} J F_1 dx}{\int_0^h X_0^{1T} J X_0^0 dx} = 0,$$
$$F_0^1(y) = \frac{\int_0^h X_0^{0T} J F_1 dx}{\int_0^h X_0^{0T} J X_0^1 dx} = -\frac{\int_0^h f_2 dx}{\left(C_{12} + 2C_{66}\right) h},$$
$$F_n^0(y) = \frac{\int_0^h X_{-n}^{1\ T} J F_1 dx}{\int_0^h X_{-n}^{1\ T} J X_n^0 dx} = \left(\int_0^h f_1 \sin(\mu_n x)\, dx + \left(\frac{C_{12} + 3C_{66}}{\left(C_{12} + C_{66}\right) \mu_n} + 1 \right) \times \int_0^h f_2 \cos(\mu_n x)\, dx \right) \times \left(\frac{2C_{66}(C_{12} + 2C_{66})}{C_{12} + C_{66}} h \right)^{-1},$$
$$F_n^1(y) = \frac{\int_0^h X_{-n}^{0\ T} J F_1 dx}{\int_0^h X_{-n}^{0\ T} J X_n^1 dx} = \frac{-\int_0^h f_1 \sin(\mu_n x)\, dx - \int_0^h f_2 \cos(\mu_n x)\, dx}{\left(2C_{66}\left(C_{12} + 2C_{66}\right)/\left(C_{12} + C_{66}\right)\right) h}$$
$$n = \pm 1, \pm 2, \ldots. \tag{27}$$

TABLE 1: The computed results.

(x, y)	(0.1464, 0.8536)	(1.4, 0.55)	(2.1, 0.34)	(3.42, 0.89)	(4.7, 0.27)
u_x	0.204	−0.08131	−0.237	0.4822	−0.2427
u_y	−0.06272	−0.01618	−0.02746	−0.0399	−0.04801
w_x	0.2965	−0.243	−0.583	0.6355	−0.4975
w_y	−0.02621	−0.006761	−0.01148	−0.01667	−0.02006

Then, substituting (25) and (26) into (6) yields

$$\frac{dY_0^1(y)}{dy} = F_0^1(y), \qquad \frac{dY_0^0(y)}{dy} = Y_0^1(y),$$
$$\frac{dY_n^1(y)}{dy} = \mu_n Y_n^1(y) + F_n^1(y),$$
$$\frac{dY_n^0(y)}{dy} = \mu_n Y_n^0(y) + Y_n^1(y) + F_n^0(y). \tag{28}$$

Thus, we obtain

$$Y_0^1(y) = c_0^1 + \int_0^y F_0^1(\xi)\, d\xi,$$
$$Y_0^0(y) = c_0^0 + c_0^1 y + \int_0^y \int_0^\tau F_0^1(\xi)\, d\xi\, d\tau,$$
$$Y_n^1(y) = c_n^1 e^{\mu_n y} + \int_0^y F_n^1(\xi)\, e^{\mu_n(y-\xi)} d\xi, \tag{29}$$
$$Y_n^0(y) = \left(c_n^0 + c_n^1 y\right) e^{\mu_n y} + \int_0^y F_n^0(\xi)\, e^{\mu_n(y-\xi)} d\xi + \int_0^y \int_0^\tau F_n^1(\xi)\, e^{\mu_n(y-\xi)} d\xi\, d\tau,$$

where c_0^1, c_0^0, c_n^1, and c_n^0 are unknown constants to be determined by imposing the remaining boundary conditions at y. Substituting (29) into (25), we have the analytical solutions u_x and u_y of (4) given by

$$\begin{aligned} u_x = \sum_{n=1}^{+\infty} \Bigg[& \left(c_n^0 + c_n^1 + c_n^1 y\right) e^{\mu_n y} - \left(c_{-n}^0 + c_{-n}^1 + c_{-n}^1 y\right) e^{-\mu_n y} \\ & + \int_0^y \left(\left(F_n^0(\xi) + F_n^1(\xi)\right) e^{\mu_n(y-\xi)} - \left(F_{-n}^0(\xi) + F_{-n}^1(\xi)\right) e^{-\mu_n(y-\xi)} \right) d\xi \\ & + \int_0^y \int_0^\tau \left(F_n^1(\xi)\, e^{\mu_n(y-\xi)} - F_{-n}^1(\xi)\, e^{-\mu_n(y-\xi)} \right) d\xi\, d\tau \Bigg] \sin \mu_n x, \end{aligned}$$

$$\begin{aligned} u_y = {} & c_0^0 + c_0^1 y + \int_0^y \int_0^\tau F_0^1(\xi)\, d\xi\, d\tau \\ & - \sum_{n=1}^{+\infty} \Bigg[\left(c_n^0 - \left(\frac{C_{12} + 3C_{66}}{(C_{12} + C_{66})\mu_n} - 1 \right) c_n^1 + c_n^1 y \right) e^{\mu_n y} \\ & + \left(c_{-n}^0 + \left(\frac{C_{12} + 3C_{66}}{(C_{12} + C_{66})\mu_n} + 1 \right) c_{-n}^1 + c_{-n}^1 y \right) e^{-\mu_n y} \\ & + \int_0^y \Bigg(\left(F_n^0(\xi) - \left(\frac{C_{12} + 3C_{66}}{(C_{12} + C_{66})\mu_n} - 1 \right) \times F_n^1(\xi) \right) e^{\mu_n(y-\xi)} \\ & + \left(F_{-n}^0(\xi) + \left(\frac{C_{12} + 3C_{66}}{(C_{12} + C_{66})\mu_n} + 1 \right) \times F_{-n}^1(\xi) \right) e^{-\mu_n(y-\xi)} \Bigg) d\xi \\ & + \int_0^y \int_0^\tau \left(F_n^1(\xi)\, e^{\mu_n(y-\xi)} + F_{-n}^1(\xi)\, e^{-\mu_n(y-\xi)} \right) d\xi\, d\tau \Bigg] \times \cos \mu_n x. \end{aligned} \tag{30}$$

According to the above procedure for (7), the analytical solutions w_x and w_y of (4) can be obtained:

$$\begin{aligned} w_x = \sum_{n=1}^{+\infty} \Bigg[& \left(d_n^0 + d_n^1 + d_n^1 y\right) e^{\mu_n y} - \left(d_{-n}^0 + d_{-n}^1 + d_{-n}^1 y\right) e^{-\lambda_n y} \\ & + \int_0^y \left(\left(\overline{F}_n^0(\xi) + \overline{F}_n^1(\xi)\right) e^{\lambda_n(y-\xi)} - \left(\overline{F}_{-n}^0(\xi) + \overline{F}_{-n}^1(\xi)\right) e^{-\lambda_n(y-\xi)} \right) d\xi \\ & + \int_0^y \int_0^\tau \left(\overline{F}_n^1(\xi)\, e^{\lambda_n(y-\xi)} - \overline{F}_{-n}^1(\xi)\, e^{-\lambda_n(y-\xi)} \right) d\xi\, d\tau \Bigg] \sin \lambda_n x, \end{aligned}$$

$$
\begin{aligned}
w_y &= d_0^0 + d_0^1 y + \int_0^y \int_0^\tau \overline{F}_0^1(\xi)\,d\xi\,d\tau \\
&+ \sum_{n=1}^{+\infty} \Bigg[\left(d_n^0 + \left(1 + \frac{2K_1+K_2+K_3}{\lambda_n(K_2+K_3)}\right) d_n^1 + d_n^1 y \right) e^{\lambda_n y} \\
&\quad + \left(d_{-n}^0 + \left(1 - \frac{2K_1+K_2+K_3}{\lambda_n(K_2+K_3)}\right) d_{-n}^1 + d_{-n}^1 y \right) e^{-\lambda_n y} \\
&\quad + \int_0^y \Bigg(\left(\overline{F}_n^0(\xi) + \left(1 + \frac{2K_1+K_2+K_3}{\lambda_n(K_2+K_3)}\right) \overline{F}_n^1(\xi) \right) \times e^{\lambda_n(y-\xi)} \\
&\qquad + \left(\overline{F}_{-n}^0(\xi) + \left(1 - \frac{2K_1+K_2+K_3}{\lambda_n(K_2+K_3)}\right) \overline{F}_{-n}^1(\xi) \right) \times e^{-\lambda_n(y-\xi)} \Bigg) d\xi \\
&\quad + \int_0^y \int_0^\tau \left(\overline{F}_n^1(\xi) e^{\lambda_n(y-\xi)} + \overline{F}_{-n}^1(\xi) e^{-\lambda_n(y-\xi)} \right) d\xi\,d\tau \Bigg] \cos\lambda_n x,
\end{aligned}
\tag{31}
$$

where d_0^1, d_0^0, d_n^1, and d_n^0 are unknown constants to be determined by imposing the remaining boundary conditions at y and

$$
\begin{aligned}
\overline{F}_0^1 &= \frac{\int_0^h g_2\,dx}{K_1 h}, \\
\overline{F}_n^1 &= \frac{-\int_0^h g_1 \sin\lambda_n x\,dx + \int_0^h g_2 \cos\lambda_n x\,dx}{\left((2K_1+K_2+K_3)/(K_2+K_3)\right)h}, \\
\overline{F}_n^0 &= \left(\int_0^h g_1 \sin\lambda_n x\,dy - \left(1 - \frac{2K_1+K_2+K_3}{\lambda_n(K_2+K_3)}\right) \int_0^h g_2 \cos\lambda_n x\,dx \right) \times \left(\frac{2K_1+K_2+K_3}{K_2+K_3} h \right)^{-1}.
\end{aligned}
\tag{32}
$$

5. Numerical Calculations

Compared with [17], the present paper is devoted to the symplectic analysis of the plane elasticity problem of quasicrystals. To guarantee the feasibility of our method, we also prove the completeness for the eigenfunction system of the associated Hamiltonian operator matrices. Note that the completeness does not always hold for the Hamiltonian operator matrices.

In order to determine the unknown constants c_n^k and d_n^k of the analytical solution in (30) and (31), we consider the boundary conditions at $y = 0$, $y = l$ given by

$$
\begin{aligned}
u_x &= \sin\frac{4\pi}{h}x, \qquad u_y = 0, \quad \text{for } y = 0,\ y = l, \\
w_x &= \sin\frac{4\pi}{h}x, \qquad w_y = 0, \quad \text{for } y = 0,\ y = l.
\end{aligned}
\tag{33}
$$

In the following, let $h = 5$, $l = 1$, and we take the constants $C_{12} = 0.5714$, $C_{66} = 0.88445$, $K_1 = 1.22$, $K_2 = 0.24$, and $K_3 = 0.6$. The computed results are listed in Table 1 for illustrating previous main results, and the data is the same as that of using the treatment in [17].

6. Conclusions

The symplectic approach is established for the plane elasticity problem of quasicrystals with point group 12 mm satisfying the mixed boundary conditions. The corresponding Hamiltonian operator matrix plays an important role in this method, whose eigenvalues and eigenfunctions need to be obtained. Through calculations, the eigenfunction system is symplectic orthogonal. Based on this, we further verify the feasibility of this approach. Then the exact analytical solution is given with the use of the symplectic eigenfunction method. We can know that the method is totally rational and gives us a systematic way to solve physical problems. In addition, this approach is expected to apply to other quasicrystal problems.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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