

Research Article

The Hybrid Steepest Descent Method for Split Variational Inclusion and Constrained Convex Minimization Problems

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We introduced an implicit and an explicit iteration method based on the hybrid steepest descent method for finding a common element of the set of solutions of a constrained convex minimization problem and the set of solutions of a split variational inclusion problem.

1. Introduction

Fixed-point optimization methods are very popular methods for solving the nonlinear problems such as variational inequality problems, optimization problems, inverse problems, and equilibrium problems. The convex feasibility problem (CFP) is used for modeling inverse problems which arise from phase retrieval problems and the intensity-modulated radiation therapy. Moreover, the well-known special case of CEP is a split feasibility problem (SFP).

Let H_1 and H_2 be two real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2 , respectively. Now, we recall that the split feasibility problem (SFP) is to find a point x^* with the following property:

$$x^* \in C, \quad Ax^* \in Q, \quad (1)$$

where $A \in B(H_1, H_2)$ and $B(H_1, H_2)$ denotes the family of all bounded linear operators H_1 to H_2 . In 1994, the SFP was first introduced by Censor and Elfving [1], in finite-dimensional Hilbert spaces, for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. A number of image reconstruction problems can be formulated as the SFP; see, for example, [2] and the references therein. Recently, it is found that the SFP can also be applied to study intensity modulated radiation therapy; see, for example, [3] and the references therein.

A special case of the SFP is following a *convex constrained linear inverse problem* [4] of finding an element x^* such that

$$x^* \in C, \quad Ax^* = b. \quad (2)$$

Recall that a mapping S of C is said to be a *nonexpansive mapping* such that

$$\|Sx - Sy\| \leq \|x - y\| \quad (3)$$

for all $x, y \in C$. Further, we consider the following fixed point problem (FPP) for a nonexpansive mapping $S : H_1 \rightarrow H_1$. Find $x \in H_1$ such that

$$Sx = x. \quad (4)$$

The solution set of FPP (4) is denoted by $F(S)$. It is well known that if $F(S) \neq \emptyset$, $F(S)$ is closed and convex. A mapping $T : H_1 \rightarrow H_1$ is said to be an *averaged mapping* if it can be written as the average of an identity I and a nonexpansive mapping S ; that is,

$$T = (1 - \alpha)I + \alpha S, \quad (5)$$

where α is a number in $(0, 1)$. More precisely, we say that T is α -averaged. It is known that the projection is $(1/2)$ -averaged. Consider the following constrained convex minimization problem:

$$\text{minimize } \{f(x) : x \in C\}, \quad (6)$$

where C is a closed and convex subset of a Hilbert space H_1 and $f : C \rightarrow \mathbb{R}$ is a real valued convex function. If f is Fréchet differentiable, then the gradient-projection algorithm (GPA) generates a sequence $\{x_n\}_{n=0}^{\infty}$ according to the following recursive formula:

$$x_{n+1} = P_C(x_n - \gamma \nabla f(x_n)), \quad \forall n \geq 0, \quad (7)$$

or more generally,

$$x_{n+1} = P_C(x_n - \gamma_n \nabla f(x_n)), \quad \forall n \geq 0, \quad (8)$$

where, in both (7) and (8), the initial guess x_0 is taken from C arbitrarily and the parameters, γ or γ_n , are positive real numbers satisfying certain conditions. The convergence of the algorithms (7) and (8) depends on the behavior of the gradient ∇f . It is known that if ∇f is α -strongly monotone and L -Lipschitzian with constants $\alpha, L \geq 0$, such that

$$\begin{aligned} \langle \nabla f(x) - \nabla f(y), x - y \rangle &\geq \alpha \|x - y\|^2, \quad x, y \in C, \\ \|\nabla f(x) - \nabla f(y)\| &\leq L \|x - y\|, \quad x, y \in C, \end{aligned} \quad (9)$$

then, for $0 < \gamma < 2\alpha/L^2$, the operator

$$T := P_C(I - \gamma \nabla f) \quad (10)$$

is a contraction; hence, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the GPA (7) converges in norm to the unique solution of (6). More generally, if the sequence $\{\gamma_n\}_{n=0}^{\infty}$ is chosen to satisfy the property

$$0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < \frac{2\alpha}{L^2}, \quad (11)$$

then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by the GPA (8) converges in norm to the unique minimizer of (6).

However, if the gradient ∇f fails to be strongly monotone, the operator T defined in (10) may fail to be contractive; consequently, the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the algorithm (7) may fail to converge strongly [5]. If ∇f is Lipschitzian, then the algorithms (7) and (8) can still converge in the weak topology under certain conditions [6, 7].

In 2011, Xu [5] gave an alternative operator-oriented approach to algorithm (8), namely, an averaged mapping approach. He gave his averaged mapping approach to the GPA (8) and the relaxed GPA. Moreover, he constructed an example which shows that the algorithm (7) does not converge in norm in an infinite-dimensional space and also presented two modifications of GPA which are shown to have strong convergence.

Given a mapping $A : C \rightarrow H_1$, the classical variational inequality problem (VIP) is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \quad (12)$$

The solution set of VIP (12) is denoted by $VI(C, A)$. It is well known that $x^* \in VI(C, A)$ if and only if $x^* = P_C(x^* - \lambda Ax^*)$ for some $\lambda > 0$. The variational inequality was first discussed by Lions [8] and now is well known. The variational inequality theory has been studied quite extensively and has emerged

as an important tool in the study of a wide class of obstacle, unilateral, free, moving, and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework.

Yamada [9] introduced the *hybrid steepest descent method* as follows:

$$x_{n+1} = Sx_n - \alpha_n \mu BSx_n, \quad (13)$$

where $x_1 = x \in H_1$, $\{\alpha_n\} \subset (0, 1)$, $B : H_1 \rightarrow H_1$ is a strongly monotone and Lipschitz continuous mapping, and μ is a positive real number. He considered the variational inequality problem over the set of common fixed points of a finite family of nonexpansive mappings and proved strong convergence of the sequence generated by the method. Later, Tian [10] considered the following iterative method for a nonexpansive mapping $T : H_1 \rightarrow H_1$ with $F(T) \neq \emptyset$:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n F)Tx_n, \quad n \geq 1, \quad (14)$$

where F is a k -Lipschitzian and η -strongly monotone operator. He proved that the sequence $\{x_n\}$ generated by (14) converges to a fixed point $q \in F(T)$, which is the unique solution of the variational inequality

$$\langle (\gamma f - \mu F)q, p - q \rangle \leq 0, \quad p \in F(T). \quad (15)$$

Recently, Moudafi [11] introduced the following split monotone variational inclusion problem (SMVIP). Find $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*), \quad (16)$$

and such that

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in f_2(y^*) + B_2(y^*), \quad (17)$$

where $B_1 : H_1 \rightarrow 2^{H_1}$ and $B_2 : H_2 \rightarrow 2^{H_2}$ are multivalued maximal monotone mappings.

Moudafi [11] introduced an iterative method for solving SMVIP (16)-(17), which can be seen an important generalization of an iterative method given by Censor et al. [12] for split variational inequality problem. As Moudafi noted in [11], SMVIP (16)-(17) includes a special case, the split common fixed point problem, split variational inequality problem, split zero problem, and split feasibility problem [1, 3, 11, 12] which have already been studied and used in practice as a model in intensity-modulated radiation therapy treatment planning; see [1, 3]. This formalism is also at the core of modeling of many inverse problems arising for phase retrieval and other real-world problems, for instance, in sensor networks, in computerized tomography, and in data compression; see, for example, [2, 13].

If $f_1 \equiv 0$ and $f_2 \equiv 0$ then SMVIP (16)-(17) reduces to the following split variational inclusion problem (SVIP). Find $x^* \in H_1$ such that

$$0 \in B_1(x^*), \quad (18)$$

and such that

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in B_2(y^*). \quad (19)$$

When looked separately, (18) is the variational inclusion problem and we denoted its solution set by SOLVIP (B_1) . The SVIP (18)-(19) constitutes a pair of variational inclusion problems which have to be solved so that the image $y^* = Ax^*$ under a given bounded linear operator A of the solution x^* of SVIP (18) in H_1 is the solution of another SVIP (19) in another space H_2 ; we denote the solution set of SVIP (19) by SOLVIP (B_2) . The solution set of SVIP (18)-(19) is denoted by $\Gamma = \{x^* \in H_1 : x^* \in \text{SOLVIP}(B_1) \text{ and } Ax^* \in \text{SOLVIP}(B_2)\}$.

Very recently, Byrne et al. [14] studied the weak and strong convergence of the following iterative method for SVIP (18)-(19): for given $x_0 \in H_1$, compute the iterative sequence $\{x_n\}$ generated by the following iterative scheme:

$$x_{n+1} = J_\lambda^{B_1} (x_n + \gamma A^* (J_\lambda^{B_2} - I) Ax_n), \tag{20}$$

for $\lambda > 0$. In 2013, Kazmi and Rizvi [15] studied the strong convergence of the following iterative method:

$$\begin{aligned} u_n &= J_\lambda^{B_1} (x_n + \gamma A^* (J_\lambda^{B_2} - I) Ax_n); \\ x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n) Su_n, \end{aligned} \tag{21}$$

where $\lambda > 0$ and $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A , and A^* is the adjoint of A . He proved the sequence $\{x_n\}$ generated by (21) strongly convergence to fixed point of nonexpansive mapping S and the solution set Γ of SVIP (18)-(19).

In this paper, motivated by the work of Xu [5], Yamada [9], Tian [10], Byrne et al. [14], and Kazmi and Rizvi [15], we proved the strong convergence theorems for finding a common element of the set of solutions of a constrained convex minimization problem and the set of solutions of a split variational inclusion problem (18)-(19).

2. Preliminaries

Throughout this paper, we always write \rightharpoonup for weak convergence and \rightarrow for strong convergence. We need some facts and tools in a real Hilbert space H_1 , which are listed below. For any $x \in H_1$, there exists a unique nearest point in C denoted by $P_C(x)$ such that

$$\|x - P_C(x)\| \leq \|x - y\|, \quad \forall y \in C. \tag{22}$$

The mapping P_C is called the *metric projection* of H_1 onto C . We know that P_C is a nonexpansive mapping from H_1 onto C . It is also known that $P_C(x) \in C$ and satisfied

$$\langle x - y, P_Cx - P_Cy \rangle \geq \|P_Cx - P_Cy\|^2, \quad \forall x, y \in H_1. \tag{23}$$

Moreover, P_Cx is characterized by the fact that $P_Cx \in C$ and

$$\begin{aligned} \langle x - P_C(x), y - P_C(x) \rangle &\leq 0, \\ \|x - y\|^2 &\geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \\ &\forall x \in H_1, y \in C. \end{aligned} \tag{24}$$

It is known that every nonexpansive operator $T : H_1 \rightarrow H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$, the inequality

$$\begin{aligned} \langle (x - T(x)) - (y - T(y)), T(y) - T(x) \rangle \\ \leq \frac{1}{2} \|(T(x) - x) - (T(y) - y)\|^2 \end{aligned} \tag{25}$$

and therefore, we get, for all $(x, y) \in H_1 \times F(T)$,

$$\langle x - T(x), y - T(x) \rangle \leq \frac{1}{2} \|T(x) - x\|^2 \tag{26}$$

(see, e.g., Theorem 3 in [16] and Theorem 1 in [17]).

Lemma 1. *Let H_1 be a real Hilbert space. There hold the following identities:*

- (i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \forall x, y \in H_1;$
- (ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \forall t \in [0, 1], \forall x, y \in H_1.$

Lemma 2 (see [7]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n) a_n + \delta_n, \tag{27}$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

- (i) $\sum_{n=1}^\infty \gamma_n = \infty;$
- (ii) $\limsup_{n \rightarrow \infty} (\delta_n / \gamma_n) \leq 0$ or $\sum_{n=1}^\infty |\delta_n| < \infty.$

Then, $\lim_{n \rightarrow \infty} \alpha_n = 0.$

Lemma 3 (see [18]). *Let $B : H_1 \rightarrow H_1$ be an L -Lipschitzian and η -strongly monotone operator on a Hilbert space H_1 with $L > 0, \eta > 0, 0 < \mu < 2\eta/L^2$, and $0 < t < 1$. Then $S = (I - t\mu B) : H_1 \rightarrow H_1$ is a contraction with contractive coefficients $1 - t\tau$ and $\tau = (1/2)\mu(2\eta - \mu L^2).$*

Lemma 4. *A nonlinear mapping B whose domain is $D(B) \subseteq H_1$ and range is $R(B) \subseteq H_1$ is said to be*

- (i) *monotone, if*

$$\langle Bx - By, x - y \rangle \geq 0, \quad \forall x, y \in D(B); \tag{28}$$

- (ii) *β -strongly monotone if there exists a constant $\beta > 0$ such that*

$$\langle Bx - By, x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in D(B); \tag{29}$$

- (iii) *v -inverse strongly monotone (or, v -ism), if there exists a constant $v > 0$ such that*

$$\langle Bx - By, x - y \rangle \geq v \|Bx - By\|^2, \quad \forall x, y \in D(B); \tag{30}$$

- (iv) *firmly nonexpansive, if*

$$\langle Bx - By, x - y \rangle \geq \|Bx - By\|^2, \quad \forall x, y \in D(B). \tag{31}$$

A multivalued mapping $M : H_1 \rightarrow 2^{H_1}$ is called *monotone* if, for all $x, y \in H_1$, $u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v \rangle \geq 0. \quad (32)$$

A monotone mapping $M : H_1 \rightarrow 2^{H_1}$ is *maximal* if $\text{Graph}(M)$ is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping M is maximal if and only if, for $(x, u) \in H_1 \times H_1$, $\langle x - y, u - v \rangle \geq 0$, for every $(y, v) \in \text{Graph}(M)$ implies that $u \in Mx$.

Let $M : H_1 \rightarrow 2^{H_1}$ be a multivalued maximal monotone mapping. Then, the resolvent mapping $J_\lambda^M : H_1 \rightarrow H_1$ associated with M is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \quad \forall x \in H_1, \quad (33)$$

for some $\lambda > 0$, where I stands for identity operator on H_1 .

We note that for all $\lambda > 0$ the resolvent operator J_λ^M is single-valued, nonexpansive, and firmly nonexpansive.

Lemma 5 (see [15]). *SVIP (18)-(19) is equivalent to find $x^* \in H_1$ such that $y^* = Ax^* \in H_2$,*

$$x^* = J_\lambda^{B_1}(x^*), \quad y^* = J_\lambda^{B_2}(y^*), \quad \text{for some } \lambda > 0. \quad (34)$$

Lemma 6 (see [19]). *Let $V : C \rightarrow H_1$ be an l -Lipschitz mapping with coefficient $l \geq 0$ and $B : C \rightarrow H_1$ a strong positive bounded linear operator with $\bar{\gamma} > 0$. Then for $0 < \gamma < (\mu\bar{\gamma}/l)$,*

$$\begin{aligned} & \langle x - y, (\mu B - \gamma V)x - (\mu B - \gamma V)y \rangle \\ & \geq (\mu\bar{\gamma} - \gamma l) \|x - y\|^2, \quad x, y \in H_1. \end{aligned} \quad (35)$$

This is, $\mu B - \gamma V$ is strongly monotone with coefficient $\mu\bar{\gamma} - \gamma l$.

Proposition 7 (see [20]). *We have the following assertions.*

- (i) *T is nonexpansive if and only if the complement $I - T$ is $(1/2)$ -ism.*
- (ii) *If T is ν -ism and $\gamma > 0$, then γT is (ν/γ) -ism.*
- (iii) *T is averaged if and only if the complement $I - T$ is ν -ism, for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α -averaged if and only if $I - T$ is $(1/2\alpha)$ -ism.*

Proposition 8 (see [20, 21]). *We have the following assertions.*

- (i) *If $T = (1 - \alpha)S + \alpha V$, for some $\alpha \in (0, 1)$, S is averaged and V is nonexpansive, and then T is averaged.*
- (ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*
- (iii) *If $T = (1 - \alpha)S + \alpha V$, for some $\alpha \in (0, 1)$, S is firmly nonexpansive and V is nonexpansive, and then T is averaged.*
- (iv) *The composite of finite many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \circ T_2 \circ \dots \circ T_N$. In particular, if T_1 is α_1 -averaged and T_2 is α_2 -averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite $T_1 T_2$ is α -averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2$.*

Lemma 9 (see [18]). *Let H_1 be a Hilbert space, C a nonempty closed convex subset of H_1 , and $T : C \rightarrow C$ a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y , then $(I - T)x = y$.*

3. Main Results

Throughout the rest of this paper, we always assume that V is an l -Lipschitzian mapping with coefficient $l \geq 0$, and D is a strongly positive bounded linear operator with coefficient $\bar{\zeta}$. Then we obtain that D is $\|D\|$ -Lipschitzian and $\bar{\zeta}$ -strongly monotone. Let $f : C \rightarrow \mathbb{R}$ be a real-valued convex function and assume that ∇f is an L -Lipschitzian mapping with $L \geq 0$.

Note that ∇f is L -Lipschitzian; it implies that ∇f is $(1/L)$ -ism, which then implies that $\gamma \nabla f$ is $(1/\gamma L)$ -ism. So by Proposition 7, its complement $I - \gamma \nabla f$ is $(\gamma L/2)$ -averaged. Since P_C is $(1/2)$ -averaged, we obtain from Proposition 8 that the composition $P_C(I - \gamma \nabla f)$ is $((2 + \gamma L)/4)$ -averaged for $0 < \gamma < 2/L$. Hence we have that, for each n , $P_C(I - \gamma_n \nabla f)$ is $((2 + \gamma_n L)/4)$ -averaged. Therefore, we can write

$$P_C(I - \gamma_n \nabla f) = \frac{2 - \gamma_n L}{4} + \frac{2 + \gamma_n L}{4} S_n, \quad (36)$$

where S_n is nonexpansive.

Suppose that minimization problem (6) is consistent and let U denote its solution set. Assume that $0 < \mu < 2\bar{\zeta}/\|D\|^2$ and $0 < \zeta < (\mu(\bar{\zeta} - \mu\|D\|^2/2)/l) = \tau/l$.

Define a mapping $W_n = S_n J_\lambda^{B_1}(I + \xi A^*(J_\lambda^{B_2} - I)A)$. Since both $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are firmly nonexpansive, they are averaged mappings. For $\xi \in (0, 1/L)$, the mapping $(I + \xi A^*(J_\lambda^{B_2} - I)A)$ is averaged. It follows from Proposition 8(iv) that the mapping $J_\lambda^{B_1}(I + \xi A^*(J_\lambda^{B_2} - I)A)$ is averaged and hence that is nonexpansive mapping. It is easy to see that W_n is also nonexpansive mapping.

Consider the following mapping G_n on H_1 defined by

$$\begin{aligned} G_n x &= \alpha_n \zeta V \left(J_\lambda^{B_1} \left(x + \xi A^* \left(J_\lambda^{B_2} - I \right) Ax \right) \right) \\ &+ (I - \alpha_n \mu D) W_n x, \quad \forall x \in H_1, n \in \mathbb{N}, \end{aligned} \quad (37)$$

where $\alpha_n \in (0, 1)$. From Lemma 3, we have

$$\begin{aligned} & \|G_n x - G_n y\| \\ &= \left\| \alpha_n V \left(J_\lambda^{B_1} \left(x + \xi A^* \left(J_\lambda^{B_2} - I \right) Ax \right) \right) + (I - \alpha_n \mu D) W_n x \right. \\ &\quad \left. - \left[\alpha_n V \left(J_\lambda^{B_1} \left(y + \xi A^* \left(J_\lambda^{B_2} - I \right) Ay \right) \right) \right. \right. \\ &\quad \left. \left. + (I - \alpha_n \mu D) W_n y \right] \right\| \end{aligned}$$

$$\begin{aligned}
 &\leq \alpha_n \varsigma \|V (J_\lambda^{B_1} (I + \xi A^* (J_\lambda^{B_2} - I) A) x) \\
 &\quad - V (J_\lambda^{B_1} (I + \xi A^* (J_\lambda^{B_2} - I) A) y)\| \\
 &\quad + (1 - \alpha_n \tau) \|W_n x - W_n y\| \\
 &\leq \alpha_n \varsigma \|Vx - Vy\| + (1 - \alpha_n \tau) \|W_n x - W_n y\| \\
 &\leq \alpha_n \varsigma l \|x - y\| + (1 - \alpha_n \tau) \|x - y\| \\
 &= (1 - \alpha_n (\tau - \varsigma l)) \|x - y\|.
 \end{aligned} \tag{38}$$

Since $0 < 1 - \alpha_n (\tau - \varsigma l) < 1$, it follows that G_n is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed point $x_n^Z \in H_1$ such that

$$\begin{aligned}
 x_n^Z &= \alpha_n \varsigma V (J_\lambda^{B_1} (x_n^Z + \xi (J_\lambda^{B_2} - I) A x_n^Z)) \\
 &\quad + (I - \alpha_n \mu D) W_n x_n^Z.
 \end{aligned} \tag{39}$$

For simplicity, we will write x_n for x_n^Z , provided no confusion occurs. Next, we prove that the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega := U \cap \Gamma$ which solves the variational inequality

$$\langle (\varsigma V - \mu D) x^*, p - x^* \rangle \leq 0, \quad \forall p \in \Omega. \tag{40}$$

Equivalently, $x^* = P_\Omega (I - \mu D + \varsigma V) x^*$.

3.1. An Implicit Iteration Method

Theorem 10. *Let H_1 and H_2 be two real Hilbert spaces and let $A : H_1 \rightarrow H_1$ be a bounded linear operator, $f : C \rightarrow \mathbb{R}$ a real-valued convex function, and ∇f an L -Lipschitzian mapping with $L \geq 0$. Assume that $\Omega := U \cap \Gamma \neq \emptyset$. Let $V : C \rightarrow C$ be an l -Lipschitzian mapping with $l \geq 0$ and let D be a strongly positive bounded linear operator with coefficients $\bar{\varsigma} \geq 0$, $0 < \mu < (2\bar{\varsigma}/\|D\|^2)$, and $0 < \varsigma < (\mu(\bar{\varsigma} - (\mu\|D\|^2/2)))/l = \tau/l$. Given $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ and $\{u_n\}$ be a sequence generated by the following algorithm:*

$$\begin{aligned}
 u_n &= J_\lambda^{B_1} (x_n + \xi A^* (J_\lambda^{B_2} - I) A x_n); \\
 x_n &= \alpha_n \varsigma V u_n + (I - \mu \alpha_n D) S_n u_n,
 \end{aligned} \tag{41}$$

where $\xi \in (0, 1/L)$, $P_C(I - \gamma_n \nabla f) = \beta_n I + (1 - \beta_n) S_n$, S_n is nonexpansive, $\beta_n = (2 - \gamma_n L)/4$, $\gamma_n \in (0, 2/L)$, and A^* is the adjoint of A and $\{\alpha_n\}, \{\beta_n\}$ satisfying the following conditions:

- (i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
- (ii) $\{\beta_n\} \subset (0, 1/2)$, $\lim_{n \rightarrow \infty} \beta_n = 0$ ($\gamma_n \rightarrow 2/L$).

Then, the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, which solves the variational inequality (40).

Proof. Consider the following.

Step 1. Show first that $\{x_n\}$ is bounded.

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, we can assume that $\alpha_n \in (0, \|D\|^{-1})$. By Lemma 3, we have $\|I - \alpha_n \mu D\| \leq 1 - \alpha_n \tau$.

Let $p \in \Omega$; we have $p = J_\lambda^{B_1} p$, $Ap = J_\lambda^{B_2} (Ap)$. We estimate

$$\begin{aligned}
 &\|u_n - p\|^2 \\
 &= \|J_\lambda^{B_1} (x_n + \xi A^* (J_\lambda^{B_2} - I) A x_n) - p\|^2 \\
 &= \|J_\lambda^{B_1} (x_n + \xi A^* (J_\lambda^{B_2} - I) A x_n) - J_\lambda^{B_1} p\|^2 \\
 &\leq \|x_n + \xi A^* (J_\lambda^{B_2} - I) A x_n - p\|^2 \\
 &\leq \|x_n - p\|^2 + \xi^2 \|A^* (J_\lambda^{B_2} - I) A x_n\|^2 \\
 &\quad + 2\xi \langle x_n - p, A^* (J_\lambda^{B_2} - I) A x_n \rangle.
 \end{aligned} \tag{42}$$

Thus, we have

$$\begin{aligned}
 &\|u_n - p\|^2 \\
 &\leq \|x_n - p\|^2 + \xi^2 \langle (J_\lambda^{B_2} - I) A x_n, A A^* (J_\lambda^{B_2} - I) A x_n \rangle \\
 &\quad + 2\xi \langle x_n - p, A^* (J_\lambda^{B_2} - I) A x_n \rangle.
 \end{aligned} \tag{43}$$

Now, we have

$$\begin{aligned}
 &\xi^2 \langle (J_\lambda^{B_2} - I) A x_n, A A^* (J_\lambda^{B_2} - I) A x_n \rangle \\
 &\leq L \xi^2 \langle (J_\lambda^{B_2} - I) A x_n, (J_\lambda^{B_2} - I) A x_n \rangle \\
 &= L \xi^2 \|(J_\lambda^{B_2} - I) A x_n\|^2.
 \end{aligned} \tag{44}$$

Setting $\Lambda := 2\xi \langle x_n - p, A^* (J_\lambda^{B_2} - I) A x_n \rangle$ and using (26), we have

$$\begin{aligned}
 \Lambda &= 2\xi \langle x_n - p, A^* (J_\lambda^{B_2} - I) A x_n \rangle \\
 &= 2\xi \langle A(x_n - p), (J_\lambda^{B_2} - I) A x_n \rangle \\
 &= 2\xi \langle A(x_n - p) + (J_\lambda^{B_2} - I) A x_n \\
 &\quad - (J_\lambda^{B_2} - I) A x_n, (J_\lambda^{B_2} - I) A x_n \rangle \\
 &= 2\xi \left\{ \langle J_\lambda^{B_2} A x_n - Ap, (J_\lambda^{B_2} - I) A x_n \rangle - \|(J_\lambda^{B_2} - I) A x_n\|^2 \right\} \\
 &\leq 2\xi \left\{ \frac{1}{2} \|(J_\lambda^{B_2} - I) A x_n\|^2 - \|(J_\lambda^{B_2} - I) A x_n\|^2 \right\} \\
 &\leq -\xi \|(J_\lambda^{B_2} - I) A x_n\|^2.
 \end{aligned} \tag{45}$$

Using (43), (44), and (45), we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + \xi (L\xi - 1) \|(J_\lambda^{B_2} - I) A x_n\|^2. \tag{46}$$

Since $\xi \in (0, 1/L)$, we obtain

$$\|u_n - p\|^2 \leq \|x_n - p\|^2. \tag{47}$$

Thus, by (41) and Lemma 3, we derive that

$$\begin{aligned}
& \|x_n - p\| \\
&= \|\alpha_n \zeta V u_n + (I - \mu \alpha_n D) S_n u_n - p\| \\
&= \|(I - \mu \alpha_n D) S_n u_n - (I - \mu \alpha_n D) S_n p \\
&\quad + \alpha_n \zeta V u_n - \alpha_n \zeta V p + \alpha_n \zeta V p - \alpha_n \mu D p\| \\
&\leq (1 - \alpha_n \tau) \|u_n - p\| + \alpha_n \zeta l \|u_n - p\| \\
&\quad + \alpha_n \|(\zeta V - \mu D) p\| \\
&\leq (1 - \alpha_n (\tau - \zeta l)) \|x_n - p\| + \alpha_n \|(\zeta V - \mu D) p\| \\
&\leq (1 - \alpha_n (\tau - \zeta l)) \|x_n - p\| + \alpha_n (\tau - \zeta l) \frac{\|(\zeta V - \mu D) p\|}{\tau - \zeta l}.
\end{aligned} \tag{48}$$

It follows that $\|x_n - p\| \leq \|(\zeta V - \mu D) p\| / (\tau - \zeta l)$.

Hence $\{x_n\}$ is bounded and so is $\{u_n\}$. It follows from the Lipschitz of $D, \nabla f, V$ that $\{Dx_n\}, \{Du_n\}, \{\nabla f u_n\}$, and $\{Vu_n\}$ are also bounded. From the nonexpansivity of S_n , it follows that $\{S_n u_n\}$ is also bounded.

Step 2. Show that

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \tag{49}$$

Next, from (46) and (47), we will show that

$$\begin{aligned}
& \|x_n - p\|^2 \\
&= \|\alpha_n \zeta V u_n + (I - \mu \alpha_n D) S_n u_n - p\|^2 \\
&= \|(I - \mu \alpha_n D) S_n u_n - (I - \mu \alpha_n D) S_n p \\
&\quad + \alpha_n \zeta V u_n - \alpha_n \mu D p\|^2 \\
&\leq (1 - \alpha_n \tau)^2 \|u_n - p\|^2 + 2\alpha_n \langle \zeta V u_n - \mu D p, x_n - p \rangle \\
&= (1 - \alpha_n \tau)^2 \|u_n - p\|^2 \\
&\quad + 2\alpha_n \langle \zeta V u_n - \zeta V p + \zeta V p - \mu D p, x_n - p \rangle \\
&\leq (1 - \alpha_n \tau)^2 \|u_n - p\|^2 + 2\alpha_n \langle \zeta V u_n - \zeta V p, x_n - p \rangle \\
&\quad + 2\alpha_n \langle \zeta V p - \mu D p, x_n - p \rangle \\
&\leq (1 - \alpha_n \tau)^2 (\|x_n - p\|^2 + \xi (L\xi - 1) \|(J_\lambda^{B_2} - I) A x_n\|^2) \\
&\quad + 2\alpha_n \zeta l \|x_n - p\|^2 + 2\alpha_n \|\zeta V p - \mu D p\| \|x_n - p\| \\
&= (1 - \alpha_n \tau)^2 \|x_n - p\|^2 - (1 - \alpha_n \tau)^2 \xi (1 - L\xi) \\
&\quad \times \|(J_\lambda^{B_2} - I) A x_n\|^2 + 2\alpha_n \zeta l \|x_n - p\|^2 \\
&\quad + 2\alpha_n \|\zeta V p - \mu D p\| \|x_n - p\|.
\end{aligned} \tag{50}$$

(51)

Therefore

$$\begin{aligned}
& (1 - \alpha_n \tau)^2 \xi (1 - L\xi) \|(J_\lambda^{B_2} - I) A x_n\|^2 \\
&\leq (\alpha_n \tau)^2 \|x_n - p\|^2 + 2\alpha_n \zeta l \|x_n - p\|^2 \\
&\quad + 2\alpha_n \|\zeta V p - \mu D p\| \|x_n - p\|.
\end{aligned} \tag{52}$$

Since $(1 - \alpha_n \tau)^2 \xi (1 - L\xi) > 0$ and $\alpha_n \rightarrow 0$, we get

$$\lim_{n \rightarrow \infty} \|(J_\lambda^{B_2} - I) A x_n\| = 0. \tag{53}$$

Next, we estimate

$$\begin{aligned}
& \|u_n - p\|^2 \\
&= \|J_\lambda^{B_1} (x_n + \xi A^* (J_\lambda^{B_2} - I) A x_n) - p\|^2 \\
&= \|J_\lambda^{B_1} (x_n + \xi A^* (J_\lambda^{B_2} - I) A x_n) - J_\lambda^{B_1} p\|^2 \\
&\leq \langle u_n - p, x_n + \xi A^* (J_\lambda^{B_2} - I) A x_n - p \rangle \\
&= \frac{1}{2} \left\{ \|u_n - p\|^2 + \|x_n + \xi A^* (J_\lambda^{B_2} - I) A x_n - p\|^2 \right. \\
&\quad \left. - \|(u_n - p) - [x_n + \xi A^* (J_\lambda^{B_2} - I) A x_n - p]\|^2 \right\} \\
&= \frac{1}{2} \left\{ \|u_n - p\|^2 + \|x_n - p\|^2 \right. \\
&\quad + \xi (L\xi - 1) \|(J_\lambda^{B_2} - I) A x_n - p\|^2 \\
&\quad \left. - \|u_n - x_n - \xi A^* (J_\lambda^{B_2} - I) A x_n\|^2 \right\} \\
&\leq \frac{1}{2} \left\{ \|u_n - p\|^2 + \|x_n - p\|^2 \right. \\
&\quad - [\|u_n - x_n\|^2 + \xi^2 \|A^* (J_\lambda^{B_2} - I) A x_n\|^2 \\
&\quad \left. - 2\xi \langle u_n - x_n, A^* (J_\lambda^{B_2} - I) A x_n \rangle] \right\} \\
&\leq \frac{1}{2} \left\{ \|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 \right. \\
&\quad \left. + 2\xi \|A (u_n - x_n)\| \|(J_\lambda^{B_2} - I) A x_n\| \right\}.
\end{aligned} \tag{54}$$

So, we obtain

$$\begin{aligned}
& \|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2 \\
&\quad + 2\xi \|A (u_n - x_n)\| \|(J_\lambda^{B_2} - I) A x_n\|.
\end{aligned} \tag{55}$$

Observe that, from (50) and (55), we get

$$\begin{aligned} & \|x_n - p\|^2 \\ & \leq (1 - \alpha_n \tau)^2 \\ & \quad \times (\|x_n - p\|^2 - \|u_n - x_n\|^2 \\ & \quad + 2\xi \|A(u_n - x_n)\| \|(J_\lambda^{B_2} - I)Ax_n\|) \\ & \quad + 2\alpha_n \varsigma l \|x_n - p\|^2 + 2\alpha_n \|\varsigma Vp - \mu Dp\| \|x_n - p\| \\ & = (1 - \alpha_n \tau)^2 \|x_n - p\|^2 - (1 - \alpha_n \tau)^2 \|u_n - x_n\|^2 \\ & \quad + 2(1 - \alpha_n \tau)^2 \xi \|A(u_n - x_n)\| \|(J_\lambda^{B_2} - I)Ax_n\| \\ & \quad + 2\alpha_n \varsigma l \|x_n - p\|^2 + 2\alpha_n \|\varsigma Vp - \mu Dp\| \|x_n - p\|. \end{aligned} \tag{56}$$

Therefore

$$\begin{aligned} & (1 - \alpha_n \tau)^2 \|u_n - x_n\|^2 \\ & \leq (\alpha_n \tau)^2 \|x_n - p\|^2 \\ & \quad + 2(1 - \alpha_n \tau)^2 \xi \|A(u_n - x_n)\| \|(J_\lambda^{B_2} - I)Ax_n\| \\ & \quad + 2\alpha_n \varsigma l \|x_n - p\|^2 + 2\alpha_n \|\varsigma Vp - \mu Dp\| \|x_n - p\|. \end{aligned} \tag{57}$$

Since $(1 - \alpha_n \tau)^2 > 0$, $\alpha_n \rightarrow 0$, and as seen in (53), we get that (49) holds.

Step 3. Show that

$$\lim_{n \rightarrow \infty} \|x_n - S_n x_n\| = 0. \tag{58}$$

Observe that

$$\begin{aligned} & \|x_n - S_n x_n\| \\ & = \|x_n - S_n u_n + S_n u_n - S_n x_n\| \\ & \leq \|x_n - S_n u_n\| + \|S_n u_n - S_n x_n\| \\ & \leq \alpha_n \|\varsigma V u_n - \mu D S_n u_n\| + \|u_n - x_n\|. \end{aligned} \tag{59}$$

Since $\alpha_n \rightarrow 0$ and as seen in (49), we get that (58) holds.

Thus,

$$\begin{aligned} & \|u_n - S_n u_n\| \\ & = \|u_n - x_n + x_n - S_n x_n + S_n x_n - S_n u_n\| \\ & \leq \|u_n - x_n\| + \|x_n - S_n x_n\| + \|S_n x_n - S_n u_n\| \\ & \leq 2 \|u_n - x_n\| + \|x_n - S_n x_n\|. \end{aligned} \tag{60}$$

From (49) and (58), we get $\lim_{n \rightarrow \infty} \|u_n - S_n u_n\| = 0$.

Note that

$$\begin{aligned} & \|P_C(I - \gamma_n \nabla f)u_n - u_n\| \\ & = \|\beta_n u_n + (1 - \beta_n)S_n u_n - u_n\| \\ & = \|(1 - \beta_n)S_n u_n - (1 - \beta_n)u_n\| \\ & = (1 - \beta_n) \|u_n - S_n u_n\| \leq \|u_n - S_n u_n\|, \end{aligned} \tag{61}$$

where $\beta_n \in (0, 1/2)$. Since $\gamma_n \in (0, 2/L)$ and as seen in (61), we get

$$\begin{aligned} & \|P_C(I - \gamma_n \nabla f)u_n - u_n\| \\ & = \|P_C\left(I - \frac{2}{L}\nabla f\right)u_n - u_n\| \\ & \leq \|P_C\left(I - \frac{2}{L}\nabla f\right)u_n - P_C(I - \gamma_n \nabla f)u_n\| \\ & \quad + \|P_C(I - \gamma_n \nabla f)u_n - u_n\| \\ & \leq \left\| \left(I - \frac{2}{L}\nabla f\right)u_n - (I - \gamma_n \nabla f)u_n \right\| \\ & \quad + \|P_C(I - \gamma_n \nabla f)u_n - u_n\| \\ & \leq \left(\frac{2}{L} - \gamma_n\right) \|\nabla f(u_n)\| + \|S_n u_n - u_n\|. \end{aligned} \tag{62}$$

Since the boundedness of $\{\nabla f(u_n)\}$, $\beta_n \rightarrow 0$, and $\|S_n u_n - u_n\| \rightarrow 0$, we conclude that

$$\lim_{n \rightarrow \infty} \|P_C\left(I - \frac{2}{L}\nabla f\right)u_n - u_n\| = 0. \tag{63}$$

So we conclude that

$$\lim_{n \rightarrow \infty} \|P_C(I - \gamma_n \nabla f)u_n - u_n\| = 0. \tag{64}$$

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_j}\}$ which converges weakly to x^* .

Step 4. Show that $x^* \in \Omega$.

Since C is closed and convex, C is weakly closed so we have $x^* \in \Omega$. By Lemma 9 and (63), we have $x^* \in U$.

Next, show that $x^* \in \Gamma$.

Consider that $u_{n_k} = J_\lambda^{B_1}(x_{n_k} + \xi A^*(J_\lambda^{B_2} - I)Ax_{n_k})$ can be rewritten as

$$\frac{(x_{n_k} - u_{n_k}) + A^*(J_\lambda^{B_2} - I)Ax_{n_k}}{\lambda} \in B_1 u_{n_k}. \tag{65}$$

Taking limit $k \rightarrow \infty$ in (65) and by taking into account (49) and (53) and the fact that the graph of a maximal monotone operator is weakly strongly closed, we obtain $0 \in B_1(x^*)$; that is, $x^* \in \text{SOLVIP}(B_1)$. Furthermore, since $\{x_n\}$ and $\{u_n\}$ have the same asymptotical behavior, $\{Ax_{n_k}\}$ weakly converges to Ax^* . Again, by (53) and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive and Lemma 9, we obtain that $Ax^* \in B_2(Ax^*)$; that is, $Ax^* \in \text{SOLVIP}(B_2)$. Thus, $x^* \in \Omega$.

Step 5. Show that $x_n \rightarrow x^*$, where $x^* = P_\Omega(I - \mu D + \varsigma V)x^*$,

$$\begin{aligned} x_n - x^* & = \alpha_n (\varsigma V u_n - \mu D x^*) + (I - \mu \alpha_n D) S_n u_n \\ & \quad - (I - \mu \alpha_n D) x^*. \end{aligned} \tag{66}$$

Hence, we obtain

$$\begin{aligned} & \|x_n - x^*\|^2 \\ &= \alpha_n \langle \zeta V u_n - \mu D x^*, x_n - x^* \rangle \\ &\quad + \langle (I - \mu \alpha_n D) S_n u_n - (I - \mu \alpha_n D) x^*, x_n - x^* \rangle \\ &\leq \alpha_n \langle \zeta V u_n - \mu D x^*, x_n - x^* \rangle + (1 - \alpha_n \tau) \|x_n - x^*\|^2. \end{aligned} \quad (67)$$

It follows that

$$\begin{aligned} & \|x_n - x^*\|^2 \\ &\leq \frac{1}{\tau} \langle \zeta V u_n - \mu D x^*, x_n - x^* \rangle \\ &= \frac{1}{\tau} \langle \zeta V u_n - \zeta V x^* + \zeta V x^* - \mu D x^*, x_n - x^* \rangle \\ &= \frac{1}{\tau} (\zeta \langle V u_n - V x^*, x_n - x^* \rangle \\ &\quad + \langle \zeta V x^* - \mu D x^*, x_n - x^* \rangle) \\ &\leq \frac{1}{\tau} (\zeta l \|x_n - x^*\|^2 + \langle \zeta V x^* - \mu D x^*, x_n - x^* \rangle). \end{aligned} \quad (68)$$

This implies that

$$\|x_n - x^*\|^2 \leq \frac{\langle \zeta V x^* - \mu D x^*, x_n - x^* \rangle}{\tau - \zeta l}. \quad (69)$$

In particular,

$$\|x_{n_j} - x^*\| \leq \frac{\langle \zeta V x^* - \mu D x^*, x_{n_j} - x^* \rangle}{\tau - \zeta l}. \quad (70)$$

Since $x_{n_j} \rightarrow x^*$, it follows from (70) that $x_{n_j} \rightarrow x^*$ as $j \rightarrow \infty$.

Next, we show that x^* solves the variational inequality (40). By the algorithm (41), we have

$$x_n = \alpha_n \zeta V u_n + (I - \mu \alpha_n D) S_n u_n. \quad (71)$$

Therefore, we have

$$\begin{aligned} & \alpha_n (\mu D - \zeta V) x_n \\ &= \alpha_n (\mu D - \zeta V) (I - W_n) x_n + \alpha_n (\mu D - \zeta V) S_n u_n \\ &\quad + \alpha_n \zeta V u_n + (I - \mu \alpha_n D) S_n u_n - x_n \\ &= \alpha_n (\mu D - \zeta V) (I - W_n) x_n + \alpha_n \zeta (V u_n - V S_n u_n) \\ &\quad + S_n u_n - x_n \end{aligned}$$

$$\begin{aligned} &= \alpha_n (\mu D - \zeta V) (I - W_n) x_n + \alpha_n \zeta (V u_n - V S_n u_n) \\ &\quad + S_n J_\lambda^{B_1} (x_n + \xi A^* (J_\lambda^{B_2} - I) A x_n) - x_n \\ &= \alpha_n (\mu D - \zeta V) (I - W_n) x_n + \alpha_n \zeta (V u_n - V S_n u_n) \\ &\quad - [x_n - S_n J_\lambda^{B_1} (I + \xi A^* (J_\lambda^{B_2} - I) A) x_n] \\ &= \alpha_n (\mu D - \zeta V) (I - W_n) x_n + \alpha_n \zeta (V u_n - V S_n u_n) \\ &\quad - (x_n - W_n x_n) \\ &= \alpha_n (\mu D - \zeta V) (I - W_n) x_n + \alpha_n \zeta (V u_n - V S_n u_n) \\ &\quad - (I - W_n) x_n; \end{aligned} \quad (72)$$

that is,

$$\begin{aligned} (\mu D - \zeta V) x_n &= (\mu D - \zeta V) (I - W_n) x_n \\ &\quad + \zeta (V u_n - V S_n u_n) - \frac{1}{\alpha_n} (I - W_n) x_n. \end{aligned} \quad (73)$$

Due to the nonexpansivity of W_n , we have that $I - W_n$ is monotone; that is, $\langle x - y, (I - W_n)x - (I - W_n)y \rangle \geq 0$, $\forall x, y \in H_1$. Consider

$$\begin{aligned} & \langle (\mu D - \zeta V) x_n, x_n - p \rangle \\ &= -\frac{1}{\alpha_n} \langle (I - W_n) x_n - (I - W_n) p, x_n - p \rangle \\ &\quad + \langle (\mu D - \zeta V) (I - W_n) x_n, x_n - p \rangle \\ &\quad + \zeta \langle V u_n - V S_n u_n, x_n - p \rangle \\ &\leq \langle (\mu D - \zeta V) (I - W_n) x_n, x_n - p \rangle \\ &\quad + \zeta \|V u_n - V S_n u_n\| \|x_n - p\| \\ &\leq \langle (\mu D - \zeta V) (I - W_n) x_n, x_n - p \rangle \\ &\quad + \zeta l \|u_n - S_n u_n\| \|x_n - p\|. \end{aligned} \quad (74)$$

Now, by replacing n in (74) with n_j and taking $j \rightarrow \infty$, we get

$$\begin{aligned} & \langle (\mu D - \zeta V) x^*, x^* - p \rangle \\ &= \lim_{j \rightarrow \infty} \langle (\mu D - \zeta V) x_{n_j}, x_{n_j} - p \rangle \\ &\leq \lim_{j \rightarrow \infty} \langle (\mu D - \zeta V) (x_{n_j} - S_{n_j} u_{n_j}), x_{n_j} - p \rangle = 0. \end{aligned} \quad (75)$$

That is, $x^* \in \Omega$ is a solution of the variational inequality (40).

Further, by the uniqueness of the solution of the variational inequality (40), we conclude that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. We rewrite (40) as

$$\langle (I - \mu D + \zeta V) x^* - x^*, x^* - p \rangle \geq 0, \quad \forall p \in \Omega. \quad (76)$$

This is equivalent to the fixed point equation

$$P_\Omega (I - \mu D + \zeta V) x^* = x^*. \quad (77)$$

□

3.2. An Explicit Iteration Method

Theorem 11. Let H_1 and H_2 be two real Hilbert spaces and let $A : H_1 \rightarrow H_1$ be a bounded linear operator, $f : C \rightarrow \mathbb{R}$ a real-value convex function, and ∇f an L -Lipschitzian mapping with $L \geq 0$. Assume that $\Omega := U \cap \Gamma \neq \emptyset$. Let $V : C \rightarrow C$ be an l -Lipschitzian mapping with $l \geq 0$ and let D be a strongly positive bounded linear operator with coefficients $\bar{\zeta} \geq 0, 0 < \mu < 2\bar{\zeta}/\|D\|^2$, and $0 < \zeta < \mu(\bar{\zeta} - (\mu\|D\|^2/2))/l = \tau/l$. Given $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ and $\{u_n\}$ be a sequence generated by the following algorithm:

$$\begin{aligned} u_n &= J_\lambda^{B_1} (x_n + \xi A^* (J_\lambda^{B_2} - I) Ax_n); \\ x_{n+1} &= \alpha_n \zeta V u_n + (I - \mu \alpha_n D) S_n u_n, \end{aligned} \tag{78}$$

where $\xi \in (0, 1/L)$, $P_C(I - \gamma_n \nabla f) = \beta_n I + (1 - \beta_n) S_n$, S_n is nonexpansive, $\beta_n = (2 - \gamma_n L)/4$, $\gamma_n \in (0, 2/L)$, and A^* is the adjoint of A and $\{\alpha_n\}, \{\beta_n\}$ satisfying the following conditions:

- (i) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$;
- (ii) $\{\beta_n\} \subset (0, 1/2)$, $\lim_{n \rightarrow \infty} \beta_n = 0$ and $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, which solves the variational inequality (40).

Proof. The proof is divided into several steps.

Step 1. Show first that $\{x_n\}$ is bounded.

Let $p \in \Omega$; we have $p = J_\lambda^{B_1} p$ and $Ap = J_\lambda^{B_2} (Ap)$. We have

$$\|u_n - p\|^2 \leq \|x_n - p\|^2. \tag{79}$$

Next, we derive that

$$\begin{aligned} &\|x_{n+1} - p\| \\ &= \|\alpha_n \zeta V u_n + (I - \mu \alpha_n D) S_n u_n - p\| \\ &= \|(I - \mu \alpha_n D) S_n u_n - (I - \mu \alpha_n D) S_n p \\ &\quad + \alpha_n \zeta V u_n - \alpha_n \zeta V p + \alpha_n \zeta V p - \alpha_n \mu D p\| \\ &\leq (1 - \alpha_n \tau) \|u_n - p\| + \alpha_n \zeta l \|u_n - p\| \\ &\quad + \alpha_n \|\zeta V p - \mu D p\| \\ &\leq (1 - \alpha_n (\tau - \zeta l)) \|x_n - p\| + \alpha_n \|\zeta V p - \mu D p\| \\ &= (1 - \alpha_n (\tau - \zeta l)) \|x_n - p\| + \alpha_n (\tau - \zeta l) \frac{\|\zeta V p - \mu D p\|}{\tau - \zeta l} \\ &\leq \max \left\{ \|x_n - p\|, \frac{\|\zeta V p - \mu D p\|}{\tau - \zeta l} \right\}. \end{aligned} \tag{80}$$

By induction, we obtain $\|x_n - p\| \leq \max\{\|x_1 - p\|, \|\zeta V p - \mu D p\|/(\tau - \zeta l)\}, n \geq 1$. Hence, $\{x_n\}$ is bounded and so is $\{u_n\}$. It follows from the Lipschitz continuity of $D, \nabla f$, and

V that $\{Dx_n\}, \{Du_n\}, \{\nabla f(u_n)\}$, and $\{Vu_n\}$ are also bounded. From the nonexpansivity of S_n , it follows that $\{S_n u_n\}$ is also bounded.

Step 2. Show that

$$\|x_{n+1} - x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{81}$$

By (78), we have

$$\begin{aligned} &\|x_{n+1} - x_n\| \\ &= \|\alpha_n \zeta V u_n + (I - \mu \alpha_n D) S_n u_n - \alpha_{n-1} \zeta V u_{n-1} \\ &\quad + (I - \mu \alpha_{n-1} D) S_{n-1} u_{n-1}\| \\ &\leq \alpha_n \zeta \|V u_n - V u_{n-1}\| + |\alpha_n - \alpha_{n-1}| \zeta \|V u_{n-1}\| \\ &\quad + \|(I - \mu \alpha_n D) S_n u_n - (I - \mu \alpha_n D) S_n u_{n-1}\| \\ &\quad + \|(I - \mu \alpha_n D) S_n u_{n-1} - (I - \mu \alpha_n D) S_{n-1} u_{n-1}\| \\ &\quad + \|(I - \mu \alpha_n D) S_{n-1} u_{n-1} - (I - \mu \alpha_{n-1} D) S_{n-1} u_{n-1}\| \\ &\leq \alpha_n \zeta l \|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}| \\ &\quad \times (\|\zeta V u_{n-1}\| + \|\mu D S_{n-1} u_{n-1}\|) \\ &\quad + (1 - \alpha_n \tau) (\|u_n - u_{n-1}\| + \|S_n u_{n-1} - S_{n-1} u_{n-1}\|) \\ &\leq (1 - \alpha_n (\tau - \zeta l)) \|u_n - u_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| (\|\zeta V u_{n-1}\| + \|\mu D S_{n-1} u_{n-1}\|) \\ &\quad + (1 - \alpha_n \tau) \|S_n u_{n-1} - S_{n-1} u_{n-1}\|. \end{aligned} \tag{82}$$

Next, we estimate $\|S_n u_{n-1} - S_{n-1} u_{n-1}\|$. Observe that

$$\begin{aligned} &\|S_n u_{n-1} - S_{n-1} u_{n-1}\| \\ &= \left\| \frac{P_C(I - \gamma_n \nabla f) - \beta_n I}{1 - \beta_n} u_{n-1} \right. \\ &\quad \left. - \frac{P_C(I - \gamma_{n-1} \nabla f) - \beta_{n-1} I}{1 - \beta_{n-1}} u_{n-1} \right\| \\ &= \left\| \frac{4P_C(I - \gamma_n \nabla f) - (2 - \lambda_n L) I}{2 + \gamma_n L} u_{n-1} \right. \\ &\quad \left. - \frac{4P_C(I - \gamma_{n-1} \nabla f) - (2 - \gamma_{n-1} L) I}{2 + \gamma_{n-1} L} u_{n-1} \right\| \\ &\leq \left\| \frac{4P_C(I - \gamma_n \nabla f)}{2 + \lambda_n L} u_{n-1} - \frac{4P_C(I - \gamma_{n-1} \nabla f)}{2 + \gamma_{n-1} L} u_{n-1} \right\| \\ &\quad + \left\| \frac{2 - \gamma_{n-1} L}{2 + \gamma_{n-1} L} u_{n-1} - \frac{2 - \gamma_n L}{2 + \gamma_n L} u_{n-1} \right\| \end{aligned}$$

$$\begin{aligned}
&= \left\| (4(2 + \gamma_{n-1}L)P_C(I - \gamma_n \nabla f)u_{n-1} \right. \\
&\quad \left. - 4(2 + \gamma_n L)P_C(I - \gamma_{n-1} \nabla f)u_{n-1}) \right. \\
&\quad \left. \times ((2 + \gamma_n L)(2 + \lambda_{n-1}L))^{-1} \right\| \\
&\quad + \frac{4L|\gamma_n - \gamma_{n-1}|}{(2 + \gamma_n L)(2 + \gamma_{n-1}L)} \|u_{n-1}\| \\
&\leq \left\| \frac{4L(\gamma_{n-1} - \gamma_n)P_C(I - \gamma_n \nabla f)u_{n-1}}{(2 + \gamma_n L)(2 + \gamma_{n-1}L)} \right. \\
&\quad \left. + (4(2 + \gamma_n L)(P_C(I - \gamma_n \nabla f)u_{n-1} \right. \\
&\quad \quad \left. - P_C(I - \gamma_{n-1} \nabla f)u_{n-1})) \right. \\
&\quad \left. \times ((2 + \lambda_n L)(2 + \gamma_{n-1}L))^{-1} \right\| \\
&\quad + \frac{4L|\gamma_n - \gamma_{n-1}|}{(2 + \gamma_n L)(2 + \gamma_{n-1}L)} \|u_{n-1}\| \\
&\leq \frac{4L|\gamma_n - \gamma_{n-1}| \|P_C(I - \gamma_n \nabla f)u_{n-1}\|}{(2 + \gamma_n L)(2 + \gamma_{n-1}L)} \\
&\quad + \frac{4 \|P_C(I - \gamma_n \nabla f)u_{n-1} - P_C(I - \gamma_{n-1} \nabla f)u_{n-1}\|}{2 + \gamma_{n-1}L} \\
&\quad + \frac{4L|\gamma_n - \gamma_{n-1}|}{(2 + \gamma_n L)(2 + \gamma_{n-1}L)} \|u_{n-1}\| \\
&\leq |\gamma_n - \gamma_{n-1}| \\
&\quad \times (L \|P_C(I - \gamma_n \nabla f)u_{n-1}\| + 2 \|\nabla f(u_{n-1})\| + L \|u_{n-1}\|) \\
&= \frac{4}{L} |\beta_n - \beta_{n-1}| \\
&\quad \times (L \|P_C(I - \gamma_n \nabla f)u_{n-1}\| + 2 \|\nabla f(u_{n-1})\| + L \|u_{n-1}\|) \\
&\leq M_1 |\beta_n - \beta_{n-1}|, \tag{83}
\end{aligned}$$

where $M_1 = \sup_n \{4 \|P_C(I - \gamma_n \nabla f)u_{n-1}\| + (8/L) \|\nabla f(u_{n-1})\| + 4 \|u_{n-1}\|\}$.

Substitute (83) into (82); we get

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\leq (1 - \alpha_n(\tau - \zeta l)) \|u_n - u_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| (\|\zeta V u_{n-1}\| + \|\mu D S_{n-1} u_{n-1}\|) \\
&\quad + M_1 |\beta_n - \beta_{n-1}| \\
&\leq (1 - \alpha_n(\tau - \zeta l)) \|u_n - u_{n-1}\| \\
&\quad + (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\
&\quad \times (\|\zeta V u_{n-1}\| + \|\mu D S_{n-1} u_{n-1}\| + M_1)
\end{aligned}$$

$$\begin{aligned}
&\leq (1 - \alpha_n(\tau - \zeta l)) \|u_n - u_{n-1}\| \\
&\quad + M_2 (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|), \tag{84}
\end{aligned}$$

for some approximate positive constant M_2 such that

$$M_2 = \sup_n \{\|\zeta V u_{n-1}\| + \|\mu D S_{n-1} u_{n-1}\| + M_1\}. \tag{85}$$

Since, for $\xi \in (0, 1/L)$, the mapping $J_\lambda^{B_1}(I + \xi A^*(J_\lambda^{B_2} - I)A)$ is averaged and hence nonexpansive, we obtain

$$\begin{aligned}
&\|u_n - u_{n-1}\| \\
&\leq \|J_\lambda^{B_1}(x_n + \xi A^*(J_\lambda^{B_2} - I)Ax_n) \\
&\quad - J_\lambda^{B_1}(x_{n-1} + \xi A^*(J_\lambda^{B_2} - I)Ax_{n-1})\| \\
&\leq \|J_\lambda^{B_1}(I + \xi A^*(J_\lambda^{B_2} - I)A)x_n \\
&\quad - J_\lambda^{B_1}(I + \xi A^*(J_\lambda^{B_2} - I)A)x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\|. \tag{86}
\end{aligned}$$

Substitute (86) into (84); we get

$$\begin{aligned}
&\|x_{n+1} - x_n\| \leq (1 - \alpha_n(\tau - \zeta l)) \|x_n - x_{n-1}\| \\
&\quad + M_2 (|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|). \tag{87}
\end{aligned}$$

By Lemma 2, it follows from conditions (i) to (ii) that (81) holds. Further, from (86), this implies that

$$\|u_{n+1} - u_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{88}$$

Step 3. Show that

$$\|x_n - S_n x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{89}$$

From (55) and (78), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|\alpha_n \zeta V u_n + (I - \mu \alpha_n D) S_n u_n - p\|^2 \\
&= \|(I - \mu \alpha_n D) S_n u_n - (I - \mu \alpha_n D) S_n p \\
&\quad + \alpha_n \zeta V u_n - \alpha_n \mu D p\|^2 \\
&\leq (1 - \alpha_n \tau)^2 \|u_n - p\|^2 \\
&\quad + 2(1 - \alpha_n \tau) \alpha_n \|u_n - p\| \|\zeta V u_n - \mu D p\| \\
&\quad + \alpha_n^2 \|\zeta V u_n - \mu D p\|^2
\end{aligned}$$

$$\begin{aligned}
 &\leq (1 - \alpha_n \tau)^2 (\|x_n - p\|^2 - \|u_n - x_n\|^2 \\
 &\quad + 2\xi \|A(u_n - x_n)\| \|(J_\lambda^{B_2} - I)Ax_n\|) \\
 &\quad + 2(1 - \alpha_n \tau) \alpha_n \|u_n - p\| \|\zeta V u_n - \mu D p\| \\
 &\quad + \alpha_n^2 \|\zeta V u_n - \mu D p\|^2 \\
 &= (1 - \alpha_n \tau)^2 \|x_n - p\|^2 - (1 - \alpha_n \tau)^2 \|u_n - x_n\|^2 \\
 &\quad + 2(1 - \alpha_n \tau)^2 \xi \|A(u_n - x_n)\| \|(J_\lambda^{B_2} - I)Ax_n\| \\
 &\quad + 2(1 - \alpha_n \tau) \alpha_n \|u_n - p\| \|\zeta V u_n - \mu D p\| \\
 &\quad + \alpha_n^2 \|\zeta V u_n - \mu D p\|^2.
 \end{aligned} \tag{90}$$

This implies that

$$\begin{aligned}
 &(1 - \alpha_n \tau)^2 \|u_n - x_n\|^2 \\
 &\leq \|x_n - p\|^2 + (\alpha_n \tau)^2 \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
 &\quad + 2(1 - \alpha_n \tau)^2 \xi \|A(u_n - x_n)\| \|(J_\lambda^{B_2} - I)Ax_n\| \\
 &\quad + 2(1 - \alpha_n \tau) \alpha_n \|u_n - p\| \|\zeta V u_n - \mu D p\| \\
 &\quad + \alpha_n^2 \|\zeta V u_n - \mu D p\|^2 \\
 &\leq (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| \\
 &\quad + (\alpha_n \tau)^2 \|x_n - p\|^2 \\
 &\quad + 2(1 - \alpha_n \tau)^2 \xi \|A(u_n - x_n)\| \|(J_\lambda^{B_2} - I)Ax_n\| \\
 &\quad + 2(1 - \alpha_n \tau) \alpha_n \|u_n - p\| \|\zeta V u_n - \mu D p\| \\
 &\quad + \alpha_n^2 \|\zeta V u_n - \mu D p\|^2.
 \end{aligned} \tag{91}$$

Since $(1 - \alpha_n \tau)^2 > 0$, $\alpha_n \rightarrow 0$, and as seen in (53) and in (81), we get

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \tag{92}$$

Next,

$$\begin{aligned}
 &\|x_n - S_n x_n\| \\
 &= \|x_n - x_{n+1} + x_{n+1} - S_n u_n + S_n u_n - S_n x_n\| \\
 &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - S_n u_n\| \\
 &\quad + \|S_n u_n - S_n x_n\| \\
 &\leq \|x_n - x_{n+1}\| + \alpha_n \|\zeta V u_n - \mu D S_n u_n\| \\
 &\quad + \|u_n - x_n\|.
 \end{aligned} \tag{93}$$

It follows from condition $\alpha_n \rightarrow 0$, (81), and (92) that (89) holds. Furthermore we have

$$\begin{aligned}
 &\|u_n - S_n u_n\| \\
 &= \|u_n - x_n + x_n - S_n x_n + S_n x_n - S_n u_n\| \\
 &\leq \|u_n - x_n\| + \|x_n - S_n x_n\| + \|x_n - u_n\| \\
 &\leq 2 \|u_n - x_n\| + \|x_n - S_n x_n\|.
 \end{aligned} \tag{94}$$

It follows from (89) and (92) that $\|u_n - S_n u_n\| \rightarrow 0$.

Step 4. Show that

$$\limsup_{n \rightarrow \infty} \langle (\zeta V - \mu D)x^*, x_n - x^* \rangle \leq 0, \tag{95}$$

where $x^* = P_\Omega(I - \mu D + \zeta V)x^*$ is a unique solution of the variational inequality (40). Indeed, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\begin{aligned}
 &\limsup_{n \rightarrow \infty} \langle (\zeta V - \mu D)x^*, x_n - x^* \rangle \\
 &= \lim_{j \rightarrow \infty} \langle (\zeta V - \mu D)x^*, x_{n_j} - x^* \rangle.
 \end{aligned} \tag{96}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ which converges weakly to q . Without loss of generality, we can assume that $x_{n_j} \rightarrow q$. Since $x^* = P_\Omega(I - \mu D + \zeta V)x^*$, it follows that

$$\begin{aligned}
 &\limsup_{n \rightarrow \infty} \langle (\zeta V - \mu D)x^*, x_n - x^* \rangle \\
 &= \lim_{j \rightarrow \infty} \langle (\zeta V - \mu D)x^*, x_{n_j} - x^* \rangle \\
 &= \langle (\zeta V - \mu D)x^*, q - x^* \rangle \leq 0.
 \end{aligned} \tag{97}$$

This implies that (95) holds.

Step 5. Show that

$$x_n \rightarrow x^*, \tag{98}$$

since

$$\begin{aligned}
 &\langle (\zeta V - \mu D)x^*, x_{n+1} - x^* \rangle \\
 &= \langle (\zeta V - \mu D)x^*, x_{n+1} - x_n + x_n - x^* \rangle \\
 &= \langle (\zeta V - \mu D)x^*, x_{n+1} - x_n \rangle \\
 &\quad + \langle (\zeta V - \mu D)x^*, x_n - x^* \rangle \\
 &\leq \|(\zeta V - \mu D)x^*\| \|x_{n+1} - x_n\| \\
 &\quad + \langle (\zeta V - \mu D)x^*, x_n - x^* \rangle.
 \end{aligned} \tag{99}$$

It follows from (81) and (95) that

$$\limsup_{n \rightarrow \infty} \langle (\zeta V - \mu D)x^*, x_{n+1} - x^* \rangle \leq 0. \tag{100}$$

Observe that

$$\begin{aligned}
 & \|x_{n+1} - x^*\|^2 \\
 &= \|\alpha_n \zeta V u_n + (I - \mu \alpha_n D) S_n u_n - x^*\|^2 \\
 &= \|(I - \mu \alpha_n D) S_n u_n - (I - \mu \alpha_n D) x^*\|^2 \\
 &\quad + \alpha_n (\zeta V u_n - \mu D x^*)^2 \\
 &\leq \|(I - \mu \alpha_n D) S_n u_n - (I - \mu \alpha_n D) x^*\|^2 \\
 &\quad + 2\alpha_n \langle \zeta V u_n - \mu D x^*, x_{n+1} - x^* \rangle \\
 &\leq (1 - \alpha_n \tau)^2 \|u_n - x^*\|^2 \\
 &\quad + 2\alpha_n \langle \zeta V u_n - \gamma V x^*, x_{n+1} - x^* \rangle \\
 &\quad + 2\alpha_n \langle (\zeta V - \mu D) x^*, x_{n+1} - x^* \rangle \\
 &\leq (1 - \alpha_n \tau)^2 \|x_n - x^*\|^2 \\
 &\quad + \alpha_n \zeta l (\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
 &\quad + 2\alpha_n \langle (\zeta V - \mu D) x^*, x_{n+1} - x^* \rangle.
 \end{aligned} \tag{101}$$

This implies that

$$\begin{aligned}
 & \|x_{n+1} - x^*\|^2 \\
 &\leq \frac{(1 - \alpha_n \tau)^2 + \alpha_n \zeta l}{1 - \alpha_n \zeta l} \|x_n - x^*\|^2 \\
 &\quad + \frac{2\alpha_n}{1 - \alpha_n \zeta l} \langle (\zeta V - \mu D) x^*, x_{n+1} - x^* \rangle \\
 &\leq \left(1 - \frac{2\alpha_n (\tau - \zeta l)}{1 - \alpha_n \zeta l}\right) \|x_n - x^*\|^2 \\
 &\quad + \frac{2\alpha_n}{1 - \alpha_n \zeta l} \langle (\zeta V - \mu D) x^*, x_{n+1} - x^* \rangle + \frac{(\alpha_n \tau)^2}{1 - \alpha_n \zeta l} M_3 \\
 &= (1 - \varphi_n) \|x_n - x^*\|^2 \\
 &\quad + \frac{2\alpha_n}{1 - \alpha_n \zeta l} \langle (\zeta V - \mu D) x^*, x_{n+1} - x^* \rangle + \frac{(\alpha_n \tau)^2}{1 - \alpha_n \zeta l} M_3,
 \end{aligned} \tag{102}$$

where $M_3 = \sup_n \|x_n - x^*\|^2, n \geq 1$. It is easy to see that $\varphi_n = 2\alpha_n (\tau - \zeta l) / (1 - \alpha_n \zeta l)$. Hence by Lemma 2, the sequence $\{x_n\}$ converges strongly to x^* . \square

Setting $Vu_n = u$ in (78) in Theorem 11, we have the following result.

Corollary 12. *Let H_1 and H_2 be two real Hilbert spaces and let $A : H_1 \rightarrow H_1$ be a bounded linear operator, $f : C \rightarrow \mathbb{R}$ a real-value convex function, and ∇f an L -Lipschitzian mapping with $L \geq 0$. Let $u \in C$ be a fixed point in C . Assume that $\Omega := U \cap \Gamma \neq \emptyset$. Let D be a strongly positive bounded linear operator*

with coefficients $\bar{\zeta} \geq 0, 0 < \mu < 2\bar{\zeta} / \|D\|^2$, and $0 < \zeta < (\mu(\bar{\zeta} - (\mu\|D\|^2/2)) / l) = \tau/l$. Given $u, x_0 \in H_1$ arbitrarily, let $\{x_n\}$ and $\{u_n\}$ be a sequence generated by the following algorithm:

$$\begin{aligned}
 u_n &= J_\lambda^{B_1} (x_n + \xi A^* (J_\lambda^{B_2} - I) A x_n); \\
 x_{n+1} &= \alpha_n \zeta u + (I - \mu \alpha_n D) S_n u_n,
 \end{aligned} \tag{103}$$

where $\xi \in (0, 1/L), P_C(I - \gamma_n \nabla f) = \beta_n I + (1 - \beta_n) S_n, S_n$ is nonexpansive, $\beta_n = (2 - \gamma_n L) / 4, \gamma_n \in (0, 2/L)$, and A^ is the adjoint of A and $\{\alpha_n\}, \{\beta_n\}$ satisfying conditions (i)-(ii) in Theorem 11 Then, the sequence $\{x_n\}$ converges strongly to a point $x^* \in \Omega$, which solves the variational inequality (40).*

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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