Research Article

Fractional Differential Equations with Fractional Impulsive and Nonseparated Boundary Conditions

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This paper studies the existence results for nonseparated boundary value problems of fractional differential equations with fractional impulsive conditions. By means of Schaefer fixed point theorem, Banach fixed point theorem, and nonlinear alternative of Leray-Schauder type, some existence results are obtained. Examples are given to illustrate the results.

1. Introduction

The subject of fractional differential equations has recently evolved as an interesting and popular field of research. In fact, fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. More and more researchers have found that fractional differential equations play important roles in many research areas, such as physics, chemical technology, population dynamics, biotechnology, and economics (see [1–4]). Hence, there are many literatures devoted to solving fractional differential equations through theoretical analysis or numerical methods (e.g., see [5–9]).

The theory of impulsive differential equations of integer order has found its extensive applications in realistic mathematical modeling of a wide variety of practical situations and has emerged as an important area of investigation in recent years. For the general theory and applications of impulsive differential equations, see [10–15] and so forth. However, impulsive differential equations of fractional order have not been much studied and many aspects of these equations are yet to be explored. For some recent work on impulsive fractional differential equations, we can refer to [16–24] and the references therein.

In this paper, we investigate the existence and uniqueness of solutions for fractional differential equations with fractional impulsive conditions

$$^{c}D^{\alpha}x(t) = f(t, x(t)),$$

$$t \in J := [0, T], \quad t \neq t_k, \quad k = 1, 2, \ldots, m,$$

$$\Delta x(t_k) = I_k(x(t_k^-)), \qquad \Delta(^{c}D^{\gamma}x(t_k)) = I_k^*(x(t_k^-)),$$

$$k = 1, 2, \ldots, m, \tag{1}$$

and nonseparated boundary conditions

$$a_1 x(0) + b_1 x(T) = c_1, \qquad a_2 x'(0) + b_2 x'(T) = c_2, \tag{2}$$

$$a_1 x(0) + b_1 x(T) = c_1,$$

$$a_2(^{c}D^{\gamma}x(0)) + b_2(^{c}D^{\gamma}x(T)) = c_2, \tag{3}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (1, 2)$, $0 < \gamma < 1$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ with $x(t_k^+) = \lim_{\epsilon \to 0^+} x(t_k + \epsilon)$, $x(t_k^-) = \lim_{\epsilon \to 0^-} x(t_k + \epsilon)$ representing the right and left limits of $x(t)$ at $t = t_k$, $\Delta(^{c}D^{\gamma}x(t_k))$ has a similar meaning for $^{c}D^{\gamma}x(t_k)$, and $a_i, b_i, c_i, \ i = 1, 2$, are real constants

such that $(a_1 + b_1)(a_2 + b_2) \neq 0$ for (2) and $(a_1 + b_1)b_2 \neq 0$ for (3).

We note that, as pointed out in the papers [22–24], the concept of piecewise continuous solutions used in some already published works to handle the impulsive fractional differential equations is not appropriate (see counterexamples given in Lemma 3.1 of [22], Section 1 in [23], and Section 3 in [24]). The papers on this topic cited above except [22] all deal with the Caputo derivative and the impulsive conditions only involve integer order derivatives. Here we study the fractional differential equations with fractional impulsive conditions and nonseparated boundary conditions [25–27].

If we let $a_1 = b_1 = 1$, $a_2 = b_2 = 1$, and $c_1 = c_2 = 0$ in boundary conditions (2) and (3), then they reduce to antiperiodic boundary conditions $x(0) = -x(T)$, $x'(0) = -x'(T)$ and $x(0) = -x(T)$, ${}^c D^\gamma x(0) = -{}^c D^\gamma x(T)$.

The rest of the paper is organized as follows. In Section 2 we introduce some notations and definitions needed in the following sections and give the appropriate formula of solutions for our problems. In Section 3 we present the existence results for the problems (1), (2) and (1), (3). Two examples are presented in Section 4 to illustrate the results. Concluding remarks are provided in Section 5.

2. Preliminaries

Let us set $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \ldots, J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, t_{m+1}]$, and $J' := J \setminus \{t_1, t_2, \ldots, t_m\}$ and introduce the space $PC(J, \mathbb{R}) := \{u : J \to \mathbb{R} \mid u \in C(J_k, \mathbb{R}), k = 0, 1, 2, \ldots, m,$ and there exist $u(t_k^+)$ and $u(t_k^-), k = 1, 2, \ldots, m,$ with $u(t_k^-) = u(t_k)\}$. It is clear that $PC(J, \mathbb{R})$ is a Banach space with the norm $\|u\| = \sup\{|u(t)| : t \in J\}$.

Definition 1 (see [4]). The Riemann-Liouville fractional integral of order q for a function f is defined as

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} f(s) \, ds, \quad q > 0, \quad (4)$$

provided the integral exists.

Definition 2 (see [4]). For a continuous function f, the Caputo derivative of order q is defined as

$$
{}^c D^q f(t) = \frac{1}{\Gamma(n - q)} \int_0^t (t - s)^{n-q-1} f^{(n)}(s) \, ds,
$$
$$n - 1 < q < n, \ n = [q] + 1, \quad (5)$$

where $[q]$ denotes the integer part of the real number q.

Lemma 3 (see [4]). *Let $\alpha > 0$; then the differential equation*

$$
{}^c D^\alpha h(t) = 0 \quad (6)
$$

has solutions $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$ and

$$I^\alpha {}^c D^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}, \quad (7)$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [\alpha] + 1$.

Definition 4. A function $x \in PC(J, \mathbb{R})$ with its α-derivative existing on J' is said to be a solution of the problems (1), (2) (or the problems (1), (3)), if x satisfies the equation ${}^c D^\alpha x(t) = f(t, x(t))$ on J', the impulsive conditions

$$
\Delta x(t_k) = I_k\left(x\left(t_k^-\right)\right), \qquad \Delta\left({}^c D^\gamma x(t_k)\right) = I_k^*\left(x\left(t_k^-\right)\right),
$$
$$k = 1, 2, \ldots, m, \quad (8)$$

and the nonseparated boundary conditions (2) (or (3)).

Lemma 5. *Let $y \in PC(J, \mathbb{R})$. A function x is a solution of the fractional integral equation*

$$
x(t)
$$
$$
= \begin{cases}
\displaystyle\int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds + d_0 + e_0 t, & t \in J_0; \\[2ex]
\displaystyle\int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds + d_0 + I_1\left(x\left(t_1^-\right)\right) \\
\quad - \Gamma(2 - \gamma) t_1^\gamma I_1^*\left(x\left(t_1^-\right)\right) \\
\quad + e_0 t + \Gamma(2 - \gamma) t_1^{\gamma-1} I_1^*\left(x\left(t_1^-\right)\right) t, & t \in J_1; \\
\qquad\qquad\vdots \\
\displaystyle\int_0^t \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} y(s) \, ds + d_0 + \sum_{i=1}^k I_i\left(x\left(t_i^-\right)\right) \\
\quad - \Gamma(2 - \gamma) \sum_{i=1}^k t_i^\gamma I_i^*\left(x\left(t_i^-\right)\right) + e_0 t \\
\quad + \Gamma(2 - \gamma) t \sum_{i=1}^k t_i^{\gamma-1} I_i^*\left(x\left(t_i^-\right)\right), & t \in J_k, \\
\qquad\qquad k = 1, 2, \ldots, m,
\end{cases} \quad (9)
$$

where

$$
d_0 = \frac{c_1 - b_1 \int_0^T \left((T - s)^{\alpha-1}/\Gamma(\alpha)\right) y(s) \, ds - b_1 A}{a_1 + b_1}
$$
$$
- \left(b_1 c_2 T + a_2 b_1 T B \right.
$$
$$
\left. - b_1 b_2 T \int_0^T \frac{(T - s)^{\alpha-2}}{\Gamma(\alpha - 1)} y(s) \, ds \right) \quad (10)
$$
$$
\times \left((a_1 + b_1)(a_2 + b_2)\right)^{-1},
$$

$$
e_0 = \frac{c_2 - b_2 \int_0^T \left((T - s)^{\alpha-2}/\Gamma(\alpha - 1)\right) y(s) \, ds - b_2 B}{a_2 + b_2},
$$

$$
A = \sum_{i=1}^m I_i\left(x\left(t_i^-\right)\right) - \Gamma(2 - \gamma) \sum_{i=1}^m t_i^\gamma I_i^*\left(x\left(t_i^-\right)\right),
$$

$$
B = \Gamma(2 - \gamma) \sum_{i=1}^m t_i^{\gamma-1} I_i^*\left(x\left(t_i^-\right)\right) \quad (11)
$$

if and only if x is a solution of the impulsive fractional boundary value problem

$$^cD^\alpha x(t) = y(t), \quad t \in J', \ 1 < \alpha < 2,$$

$$\Delta x(t_k) = I_k(x(t_k^-)), \qquad \Delta(^cD^\gamma x(t_k)) = I_k^*(x(t_k^-)),$$

$$k = 1, 2, \dots, m,$$

$$a_1 x(0) + b_1 x(T) = c_1, \qquad a_2 x'(0) + b_2 x'(T) = c_2. \tag{12}$$

Proof. For $1 < \alpha < 2$, by Lemma 3, we know that a general solution of the equation $^cD^\alpha x(t) = y(t)$ on each interval J_k ($k = 0, 1, 2, \dots, m$) is given by

$$x(t) = I^\alpha y(t) + d_k + e_k t$$

$$= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + d_k + e_k t, \quad t \in J_k, \tag{13}$$

where d_k and $e_k \in \mathbb{R}$ are arbitrary constants. Then we have

$$x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds + e_k, \quad t \in J_k. \tag{14}$$

Since $^cD^\gamma C = 0$ (C is a constant), $^cD^\gamma t = t^{1-\gamma}/\Gamma(2-\gamma)$, and $^cD^\gamma I^\alpha y(t) = I^{\alpha-\gamma} y(t)$ (see [4]), then from (13), we have

$$^cD^\gamma x(t) = \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)\, ds + \frac{e_k t^{1-\gamma}}{\Gamma(2-\gamma)}, \quad t \in J_k. \tag{15}$$

Applying the boundary conditions of (12), we get (since $0 \in J_0$ and $T \in J_m$)

$$a_1 \times d_0 + b_1 \times \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + d_m + e_m T \right) = c_1,$$

$$a_2 \times e_0 + b_2 \times \left(\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds + e_m \right) = c_2. \tag{16}$$

Next using the impulsive conditions in (12), we obtain that, for $k = 1, 2, \dots, m$,

$$d_k - d_{k-1} + (e_k - e_{k-1}) t_k = I_k(x(t_k^-)),$$

$$(e_k - e_{k-1}) \frac{t_k^{1-\gamma}}{\Gamma(2-\gamma)} = I_k^*(x(t_k^-)). \tag{17}$$

That is to say, we have, for $k = 1, 2, \dots, m$,

$$d_k = d_0 + \sum_{i=1}^k I_i(x(t_i^-)) - \Gamma(2-\gamma) \sum_{i=1}^k t_i^\gamma I_i^*(x(t_i^-)),$$

$$e_k = e_0 + \Gamma(2-\gamma) \sum_{i=1}^k t_i^{\gamma-1} I_i^*(x(t_i^-)). \tag{18}$$

Now the above two equations together with (16) imply that

$$d_0 = \frac{c_1 - b_1 \int_0^T \left((T-s)^{\alpha-1}/\Gamma(\alpha)\right) y(s)\, ds \ - b_1 A}{a_1 + b_1}$$

$$- \Bigg(b_1 c_2 T + a_2 b_1 T B$$

$$- b_1 b_2 T \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds \Bigg) \tag{19}$$

$$\times \left((a_1 + b_1)(a_2 + b_2)\right)^{-1},$$

$$e_0 = \frac{c_2 - b_2 \int_0^T \left((T-s)^{\alpha-2}/\Gamma(\alpha-1)\right) y(s)\, ds - b_2 B}{a_2 + b_2},$$

where

$$A = \sum_{i=1}^m I_i(x(t_i^-)) - \Gamma(2-\gamma) \sum_{i=1}^m t_i^\gamma I_i^*(x(t_i^-)),$$

$$B = \Gamma(2-\gamma) \sum_{i=1}^m t_i^{\gamma-1} I_i^*(x(t_i^-)). \tag{20}$$

Hence, for $k = 0, 1, 2, \dots, m$, we have

$$d_k + e_k t = d_0 + \sum_{i=1}^k I_i(x(t_i^-)) - \Gamma(2-\gamma) \sum_{i=1}^k t_i^\gamma I_i^*(x(t_i^-))$$

$$+ e_0 t + \Gamma(2-\gamma) t \sum_{i=1}^k t_i^{\gamma-1} I_i^*(x(t_i^-)). \tag{21}$$

Substituting the value of d_0 and e_0 in (21), and by (13), we get the fractional integral equation (9).

Conversely, assume that x satisfies the fractional integral equation (9); that is, for $t \in J_k$, $k = 0, 1, 2, \dots, m$, we have

$$x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + d_0 + \sum_{i=1}^k I_i(x(t_i^-))$$

$$- \Gamma(2-\gamma) \sum_{i=1}^k t_i^\gamma I_i^*(x(t_i^-)) \tag{22}$$

$$+ e_0 t + \Gamma(2-\gamma) t \sum_{i=1}^k t_i^{\gamma-1} I_i^*(x(t_i^-)),$$

$$x'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds$$

$$+ e_0 + \Gamma(2-\gamma) \sum_{i=1}^k t_i^{\gamma-1} I_i^*(x(t_i^-)). \tag{23}$$

Since $1 < \alpha < 2$, we have $^cD^\alpha C = 0$ (C is a constant) and $^cD^\alpha t = 0$. Using the fact that $^cD^\alpha$ is the left inverse of I^α, we get

$$^cD^\alpha x(t) = y(t), \quad t \in J', \tag{24}$$

which means that x satisfies the first equation of the impulsive fractional boundary value problem (12). Next we will verify that x satisfies the impulsive conditions. Taking fractional derivative $^cD^\gamma$ of (22), we have, for $t \in J_k$,

$$
\begin{aligned}
^cD^\gamma x(t) = {} & \int_0^t \frac{(t-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)\, ds \\
& + \frac{e_0 t^{1-\gamma}}{\Gamma(2-\gamma)} + t^{1-\gamma} \sum_{i=1}^k t_i^{\gamma-1} I_i^* \left(x(t_i^-) \right).
\end{aligned}
\tag{25}
$$

From (22), we obtain

$$
\begin{aligned}
x(t_k^+) = {} & \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + d_0 + \sum_{i=1}^k I_i \left(x(t_i^-) \right) \\
& - \Gamma(2-\gamma) \sum_{i=1}^k t_i^\gamma I_i^* \left(x(t_i^-) \right) + e_0 t_k \\
& + \Gamma(2-\gamma) t_k \sum_{i=1}^k t_i^{\gamma-1} I_i^* \left(x(t_i^-) \right), \\
x(t_k^-) = {} & \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + d_0 + \sum_{i=1}^{k-1} I_i \left(x(t_i^-) \right) \\
& - \Gamma(2-\gamma) \sum_{i=1}^{k-1} t_i^\gamma I_i^* \left(x(t_i^-) \right) + e_0 t_k \\
& + \Gamma(2-\gamma) t_k \sum_{i=1}^{k-1} t_i^{\gamma-1} I_i^* \left(x(t_i^-) \right).
\end{aligned}
\tag{26}
$$

The two equations above imply that, for $k = 1, 2, \ldots, m$,

$$
\begin{aligned}
\Delta x(t_k) = {} & I_k \left(x(t_k^-) \right) - \Gamma(2-\gamma) t_k^\gamma I_k^* \left(x(t_k^-) \right) \\
& + \Gamma(2-\gamma) t_k t_k^{\gamma-1} I_k^* \left(x(t_k^-) \right) = I_k \left(x(t_k^-) \right).
\end{aligned}
\tag{27}
$$

Similarly, from (25), we have

$$
\begin{aligned}
^cD^\gamma x(t_k^+) = {} & \int_0^{t_k} \frac{(t_k-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)\, ds + \frac{e_0 t_k^{1-\gamma}}{\Gamma(2-\gamma)} \\
& + t_k^{1-\gamma} \sum_{i=1}^k t_i^{\gamma-1} I_i^* \left(x(t_i^-) \right), \\
^cD^\gamma x(t_k^-) = {} & \int_0^{t_k} \frac{(t_k-s)^{\alpha-\gamma-1}}{\Gamma(\alpha-\gamma)} y(s)\, ds \\
& + \frac{e_0 t_k^{1-\gamma}}{\Gamma(2-\gamma)} + t_k^{1-\gamma} \sum_{i=1}^{k-1} t_i^{\gamma-1} I_i^* \left(x(t_i^-) \right).
\end{aligned}
\tag{28}
$$

Therefore, for $k = 1, 2, \ldots, m$, we have

$$
\Delta \left(^cD^\gamma x(t_k) \right) = t_k^{1-\gamma} t_k^{\gamma-1} I_k^* \left(x(t_k^-) \right) = I_k^* \left(x(t_k^-) \right).
\tag{29}
$$

Finally, it follows from (22) and (23) that (since $0 \in J_0$, $T \in J_m$)

$$
\begin{aligned}
x(0) &= d_0, \\
x(T) &= \int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + d_0 + A + e_0 T + TB, \\
x'(0) &= e_0, \\
x'(T) &= \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds + e_0 + B.
\end{aligned}
\tag{30}
$$

Hence, we get

$$
\begin{aligned}
& a_1 x(0) + b_1 x(T) \\
&= a_1 d_0 + b_1 \left(\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + d_0 + A + e_0 T + TB \right) \\
&= c_1 + b_1 TB + b_1 e_0 T \\
& \quad - \Bigg(b_1 c_2 T + a_2 b_1 TB \\
& \qquad - b_1 b_2 T \int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds \Bigg) \\
& \quad \times (a_2 + b_2)^{-1} = c_1, \\
& a_2 x'(0) + b_2 x'(T) \\
&= a_2 e_0 + b_2 \left(\int_0^T \frac{(T-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)\, ds + e_0 + B \right) = c_2.
\end{aligned}
\tag{31}
$$

Therefore, x given by (9) satisfies the impulsive fractional boundary value problem (12). The proof is complete. $\square$

Lemma 6. *Let $y \in PC(J, \mathbb{R})$. A function x is a solution of the fractional integral equation*

$$
x(t)
$$

$$
= \begin{cases}
\displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + d_0 + e_0 t, & t \in J_0; \\[2ex]
\displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + d_0 + I_1 \left(x(t_1^-) \right) \\
\quad - \Gamma(2-\gamma) t_1^\gamma I_1^* \left(x(t_1^-) \right) \\
\quad + e_0 t + \Gamma(2-\gamma) t_1^{\gamma-1} I_1^* \left(x(t_1^-) \right) t, & t \in J_1; \\[2ex]
\qquad\qquad \vdots \\[1ex]
\displaystyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + d_0 + \sum_{i=1}^k I_i \left(x(t_i^-) \right) \\
\quad - \Gamma(2-\gamma) \sum_{i=1}^k t_i^\gamma I_i^* \left(x(t_i^-) \right) + e_0 t \\
\quad + \Gamma(2-\gamma) t \sum_{i=1}^k t_i^{\gamma-1} I_i^* \left(x(t_i^-) \right), & t \in J_k, \\
\qquad\qquad\qquad\qquad\qquad k = 1, 2, \ldots, m,
\end{cases}
\tag{32}
$$

where

$$d_0 = -\frac{b_1}{a_1 + b_1}(C + A) + \frac{c_1 b_2 - b_1 c_2 T^\gamma \Gamma(2 - \gamma)}{(a_1 + b_1) b_2},$$

$$e_0 = -\frac{\Gamma(2 - \gamma)}{T^{1-\gamma}}\left(\int_0^T \frac{(T - s)^{\alpha - \gamma - 1}}{\Gamma(\alpha - \gamma)} y(s) ds - \frac{c_2}{b_2}\right) - B \tag{33}$$

with A and B defined by (11) *and*

$$C = \int_0^T \frac{(T - s)^{\alpha - 1}}{\Gamma(\alpha)} y(s) ds$$

$$- T^\gamma \Gamma(2 - \gamma) \int_0^T \frac{(T - s)^{\alpha - \gamma - 1}}{\Gamma(\alpha - \gamma)} y(s) ds, \tag{34}$$

if and only if x is a solution of the impulsive fractional boundary value problem

$${}^c D^\alpha x(t) = y(t), \quad t \in J', \ 1 < \alpha < 2,$$

$$\Delta x(t_k) = I_k(x(t_k^-)), \qquad \Delta({}^c D^\gamma x(t_k)) = I_k^*(x(t_k^-)),$$

$$k = 1, 2, \ldots, m,$$

$$a_1 x(0) + b_1 x(T) = c_1,$$

$$a_2({}^c D^\gamma x(0)) + b_2({}^c D^\gamma x(T)) = c_2. \tag{35}$$

Proof. The proof is similar to that of **Lemma 5**. Let the notations be given as in the proof of **Lemma 5**. Applying the boundary conditions in (35), from the relations (13), (15), and (17), we have

$$a_1 \times d_0 + b_1 \times \left(\int_0^T \frac{(T - s)^{\alpha - 1}}{\Gamma(\alpha)} y(s) ds + d_m + e_m T\right) = c_1,$$

$$a_2 \times 0 + b_2 \times \left(\int_0^T \frac{(T - s)^{\alpha - \gamma - 1}}{\Gamma(\alpha - \gamma)} y(s) ds + \frac{e_m T^{1-\gamma}}{\Gamma(2 - \gamma)}\right) = c_2. \tag{36}$$

After a direct computation, we obtain that

$$d_0 = -\frac{b_1}{a_1 + b_1}(C + A) + \frac{c_1 b_2 - b_1 c_2 T^\gamma \Gamma(2 - \gamma)}{(a_1 + b_1) b_2},$$

$$e_0 = -\frac{\Gamma(2 - \gamma)}{T^{1-\gamma}}\left(\int_0^T \frac{(T - s)^{\alpha - \gamma - 1}}{\Gamma(\alpha - \gamma)} y(s) ds - \frac{c_2}{b_2}\right) - B \tag{37}$$

with A and B defined by (11) and

$$C = \int_0^T \frac{(T - s)^{\alpha - 1}}{\Gamma(\alpha)} y(s) ds$$

$$- T^\gamma \Gamma(2 - \gamma) \int_0^T \frac{(T - s)^{\alpha - \gamma - 1}}{\Gamma(\alpha - \gamma)} y(s) ds. \tag{38}$$

The remaining part of proof is the same as that of **Lemma 5**. $\square$

Remark 7. We note that the solution expression (32) of the problem (35) does not depend on the parameter a_2 appearing in the boundary conditions. Thus, by **Lemma 6**, we conclude that the parameter a_2 is of arbitrary nature of the problem (35).

Remark 8. Our approach for the construction of solutions for impulsive fractional differential equations is general, which provides an effective way to deal with such problems. Taking the problem (1) for example, on each interval J_k, we use

$$x(t) = \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, x(s)) ds + d_k + e_k t \tag{I}$$

instead of

$$x(t) = \int_{t_k}^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, x(s)) ds + d_k + e_k(t - t_k). \tag{II}$$

From the counterexamples given in Lemma 3.1 of [22], Section 1 in [23], and Section 3 in [24], we know that if the case (II) was chosen to construct solutions, the solution formula obtained in the fractional integral equation form is not equivalent to the original impulsive fractional differential equation. This is the main difference between impulsive fractional differential equations and impulsive ordinary differential equations. Observe that fractional calculus has memory property.

Theorem 9 (nonlinear alternative of Leray-Schauder type [28]). *Let X be a Banach space, C a nonempty convex subset of X, and U a nonempty open subset of C with $0 \in U$. Suppose that $P : \overline{U} \to C$ is a continuous and compact map. Then either* (a) *P has a fixed point in $\overline{U}$ or* (b) *there exist a $x \in \partial U$ (the boundary of U) and $\lambda \in (0, 1)$ with $x = \lambda P(x)$.*

Theorem 10 (Schaefer fixed point theorem [29]). *Let X be a normed space and P a continuous mapping of X into X which is compact on each bounded subset B of X. Then either* (I) *the equation $x = \lambda Px$ has a solution for $\lambda = 1$ or* (II) *the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.*

3. Main Results

This section deals with the existence and uniqueness of solutions for the problem (1), (2) and the problem (1), (3).

First we study the existence results for the problem (1), (2). In view of **Lemma 5**, we define an operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$(Fx)(t) = \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} f(s, x(s)) ds + d_0^x + \sum_{i=1}^k I_i(x(t_i^-))$$

$$- \Gamma(2 - \gamma) \sum_{i=1}^k t_i^\gamma I_i^*(x(t_i^-)) + e_0^x t$$

$$+ \Gamma (2 - \gamma) t \sum_{i=1}^{k} t_i^{\gamma-1} I_i^* (x (t_i^-)),$$

$$t \in J_k, \ k = 0, 1, 2, \ldots, m,$$

$$(39)$$

where

$$d_0^x = \frac{c_1 - b_1 \int_0^T \left((T - s)^{\alpha-1}/\Gamma (\alpha) \right) f (s, x (s)) \, ds - b_1 A^x}{a_1 + b_1}$$

$$- \left(b_1 c_2 T + a_2 b_1 T B^x - b_1 b_2 T \right.$$

$$\left. \times \int_0^T \frac{(T - s)^{\alpha-2}}{\Gamma (\alpha - 1)} f (s, x (s)) \, ds \right)$$

$$\times \left((a_1 + b_1) (a_2 + b_2) \right)^{-1},$$

$$e_0^x = \frac{c_2 - b_2 \int_0^T \left((T - s)^{\alpha-2}/\Gamma (\alpha - 1) \right) f (s, x (s)) \, ds - b_2 B^x}{a_2 + b_2},$$

$$A^x = \sum_{i=1}^{m} I_i (x (t_i^-)) - \Gamma (2 - \gamma) \sum_{i=1}^{m} t_i^{\gamma} I_i^* (x (t_i^-)),$$

$$B^x = \Gamma (2 - \gamma) \sum_{i=1}^{m} t_i^{\gamma-1} I_i^* (x (t_i^-)).$$

$$(40)$$

Here d_0^x, e_0^x, A^x, and B^x mean that d_0, e_0, A, and B defined in Lemma 5 are related to $x \in PC(J, \mathbb{R})$. It is obvious that F is well defined because of the continuity of f, I_k, and I_k^* and that the problem (1), (2) has solutions if and only if the operator equation $Fx = x$ has fixed points.

For the sake of convenience, we set

$$\Delta = (a_1 + b_1) (a_2 + b_2).$$

$$(41)$$

Lemma 11. *The operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by* (39) *is completely continuous.*

Proof. Since f, I_k, and I_k^* are continuous, it is easy to check that F is continuous on $PC(J, \mathbb{R})$.

Let $B \subseteq PC(J, \mathbb{R})$ be bounded; then there exist positive constants N_i, $i = 1, 2, 3$, such that $|f(t, x(t))| \leq N_1$, $|I_k(x(t_k^-))| \leq N_2$, and $|I_k^*(x(t_k^-))| \leq N_3$, for all $t \in J$, $x \in B$, $k = 1, 2, \ldots, m$. Thus, for $x \in B$ and $t \in J$, we have

$$|(Fx) (t)| \leq \frac{N_1 T^{\alpha}}{\Gamma (\alpha + 1)} + |d_0^x| + m N_2 + \Gamma (2 - \gamma) N_3 \sum_{i=1}^{m} t_i^{\gamma}$$

$$+ |e_0^x| T + \Gamma (2 - \gamma) T N_3 \sum_{i=1}^{m} t_i^{\gamma-1},$$

$$|d_0^x| \leq \frac{|c_1|}{|a_1 + b_1|} + \frac{|b_1|}{|a_1 + b_1|}$$

$$\times \left(\frac{N_1 T^{\alpha}}{\Gamma (\alpha + 1)} + m N_2 + \Gamma (2 - \gamma) N_3 \sum_{i=1}^{m} t_i^{\gamma} \right)$$

$$+ \frac{|b_1 c_2| T}{|\Delta|} + \frac{|a_2 b_1| T}{|\Delta|} \Gamma (2 - \gamma) N_3 \sum_{i=1}^{m} t_i^{\gamma-1}$$

$$+ \frac{|b_1 b_2| N_1 T^{\alpha}}{|\Delta| \Gamma (\alpha)},$$

$$(42)$$

$$|e_0^x| \leq \frac{|c_2|}{|a_2 + b_2|}$$

$$+ \frac{|b_2|}{|a_2 + b_2|} \left(\frac{N_1 T^{\alpha-1}}{\Gamma (\alpha)} + \Gamma (2 - \gamma) N_3 \sum_{i=1}^{m} t_i^{\gamma-1} \right).$$

$$(43)$$

Therefore, we can deduce that, for all $x \in B$ and $t \in J$,

$$|(Fx) (t)| \leq \frac{N_1 T^{\alpha}}{\Gamma (\alpha + 1)} \left(1 + \frac{|b_1|}{|a_1 + b_1|} \right)$$

$$+ \frac{N_1 T^{\alpha}}{\Gamma (\alpha)} \left(\frac{|b_1 b_2|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|} \right)$$

$$+ m N_2 \left(1 + \frac{|b_1|}{|a_1 + b_1|} \right)$$

$$+ \Gamma (2 - \gamma) N_3 \sum_{i=1}^{m} t_i^{\gamma} \left(1 + \frac{|b_1|}{|a_1 + b_1|} \right)$$

$$+ \Gamma (2 - \gamma) T N_3 \sum_{i=1}^{m} t_i^{\gamma-1} \left(1 + \frac{|a_2 b_1|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|} \right)$$

$$+ \frac{|c_1|}{|a_1 + b_1|} + \frac{|b_1 c_2| T}{|\Delta|} + \frac{|c_2| T}{|a_2 + b_2|},$$

$$(44)$$

which implies that the operator F is uniformly bounded on B.

On the other hand, let $x \in B$ and, for any $\tau_1, \tau_2 \in J_k$, $k = 0, 1, 2, \ldots, m$, with $\tau_1 < \tau_2$, we have

$$|(Fx) (\tau_2) - (Fx) (\tau_1)|$$

$$\leq \left| \int_0^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma (\alpha)} f (s, x (s)) \, ds \right.$$

$$\left. - \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1}}{\Gamma (\alpha)} f (s, x (s)) \, ds \right|$$

$$+ |e_0^x| (\tau_2 - \tau_1) + \Gamma (2 - \gamma) \sum_{i=1}^{k} t_i^{\gamma-1} |I_i^* (x (t_i^-))| (\tau_2 - \tau_1)$$

$$\leq \frac{N_1 \left(\tau_2^\alpha - \tau_1^\alpha\right)}{\Gamma(\alpha + 1)} + \left|e_0^x\right|(\tau_2 - \tau_1)$$

$$+ \Gamma(2 - \gamma) N_3 \sum_{i=1}^{k} t_i^{\gamma - 1} (\tau_2 - \tau_1).$$

$$(45)$$

From (43) and the above inequality, we deduce that $|(Fx)(\tau_2) - (Fx)(\tau_1)|$ tends to zero as $\tau_2 \to \tau_1$. This implies that F is equicontinuous on the interval J_k. Hence, by PC-type Arzela-Ascoli Theorem (see Theorem 2.1 [15]), the operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is completely continuous.

(H1): (1) there exist $h \in L^\infty(J, \mathbb{R}^+)$ and $\varphi : [0, \infty) \to (0, \infty)$ continuous nondecreasing such that $|f(t, x)| \leq h(t)\varphi(|x|)$ for $(t, x) \in J \times \mathbb{R}$; (2) there exist ψ and $\psi^* : [0, \infty) \to (0, \infty)$ continuous nondecreasing such that $|I_k(x)| \leq \psi(|x|)$ and $|I_k^*(x)| \leq \psi^*(|x|)$ for all $x \in \mathbb{R}$ and $k = 1, 2, \ldots, m$.

$\square$

Theorem 12. *Let (H1) hold and there exists a constant $M > 0$ such that*

$$\frac{M}{\|h\|_{L^\infty} \varphi(M) P + \Gamma(2 - \gamma) \psi^*(M) R + \psi(M) Q + H} > 1,$$

$$(46)$$

where

$$P = \frac{T^\alpha}{\Gamma(\alpha + 1)} \left(1 + \frac{|b_1|}{|a_1 + b_1|}\right)$$

$$+ \frac{T^\alpha}{\Gamma(\alpha)} \left(\frac{|b_1 b_2|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|}\right),$$

$$Q = m \left(1 + \frac{|b_1|}{|a_1 + b_1|}\right),$$

$$H = \frac{|c_1|}{|a_1 + b_1|} + \frac{|b_1 c_2| T}{|\Delta|} + \frac{|c_2| T}{|a_2 + b_2|}, \qquad (47)$$

$$R = \left(1 + \frac{|b_1|}{|a_1 + b_1|}\right) \sum_{i=1}^{m} t_i^\gamma$$

$$+ T \left(1 + \frac{|a_2 b_1|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|}\right) \sum_{i=1}^{m} t_i^{\gamma - 1}.$$

Then the impulsive fractional boundary value problem (1), (2) has at least one solution on J.

Proof. We will show that the operator F defined by (39) satisfies the assumptions of the nonlinear alternative of Leray-Schauder type.

From Lemma 11, the operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ is continuous and completely continuous.

Let $x \in PC(J, \mathbb{R})$ be such that $x(t) = \lambda(Fx)(t)$ for some $\lambda \in (0, 1)$. Then using the computations in the proof that F maps bounded sets into bounded sets in Lemma 11, we have

$$|x(t)| \leq \|h\|_{L^\infty} \varphi(\|x\|) \left[\frac{T^\alpha}{\Gamma(\alpha + 1)} \left(1 + \frac{|b_1|}{|a_1 + b_1|}\right)\right.$$

$$\left. + \frac{T^\alpha}{\Gamma(\alpha)} \left(\frac{|b_1 b_2|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|}\right)\right]$$

$$+ m\psi(\|x\|) \left(1 + \frac{|b_1|}{|a_1 + b_1|}\right)$$

$$+ \Gamma(2 - \gamma) \psi^*(\|x\|) \left(1 + \frac{|b_1|}{|a_1 + b_1|}\right) \sum_{i=1}^{m} t_i^\gamma$$

$$+ \Gamma(2 - \gamma) \psi^*(\|x\|)$$

$$\times T \left(1 + \frac{|a_2 b_1|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|}\right) \sum_{i=1}^{m} t_i^{\gamma - 1}$$

$$+ \frac{|c_1|}{|a_1 + b_1|} + \frac{|b_1 c_2| T}{|\Delta|} + \frac{|c_2| T}{|a_2 + b_2|}$$

$$\leq \|h\|_{L^\infty} \varphi(\|x\|) P + \psi(\|x\|) Q$$

$$+ \Gamma(2 - \gamma) \psi^*(\|x\|) R + H.$$

$$(48)$$

Consequently, we have

$$\frac{\|x\|}{\|h\|_{L^\infty} \varphi(\|x\|) P + \Gamma(2 - \gamma) \psi^*(\|x\|) R + \psi(\|x\|) Q + H} \leq 1.$$

$$(49)$$

Then, in view of condition (46), there exists $M > 0$ such that $\|x\| \neq M$. Let us set

$$U = \{x \in PC(J, \mathbb{R}) : \|x\| < M\}. \qquad (50)$$

The operator $F : \overline{U} \to PC(J, \mathbb{R})$ is continuous and compact. From the choice of the set U, there is no $x \in \partial U$ such that $x = \lambda Fx$ for some $\lambda \in (0, 1)$. Therefore, by the nonlinear alternative of Leray-Schauder type (see Theorem 9), we deduce that F has a fixed point x in $\overline{U}$ which is a solution of the impulsive fractional boundary value problem (1), (2). The proof is complete.

(H2): there exist $h \in L^\infty(J, \mathbb{R}^+)$ and positive constants H_1 and H_2 such that, for $t \in J$, $x \in \mathbb{R}$, $k = 1, 2, \ldots, m$,

$$\left|f(t, x)\right| \leq h(t), \qquad \left|I_k(x)\right| \leq H_1, \qquad \left|I_k^*(x)\right| \leq H_2.$$

$$(51)$$

$\square$

Theorem 13. *If (H2) holds, then the impulsive fractional boundary value problem (1), (2) has at least one solution on J.*

Proof. Lemma 11 tells us that the operator $F : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ defined by (39) is continuous and compact on each bounded subset B of $PC(J, \mathbb{R})$.

Now let us show that the set $V = \{u \in PC(J, \mathbb{R}) : u = \lambda F u, \ 0 < \lambda < 1\}$ is bounded. Let $x \in V$; then $x = \lambda F x$ for some $0 < \lambda < 1$. For each $t \in J$, using the similar estimations given in Theorem 12, we have

$$
\begin{aligned}
|x(t)| \leq \ &\|h\|_{L^\infty} \left[\frac{T^\alpha}{\Gamma(\alpha+1)} \left(1 + \frac{|b_1|}{|a_1 + b_1|} \right) \right. \\
&\left. + \frac{T^\alpha}{\Gamma(\alpha)} \left(\frac{|b_1 b_2|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|} \right) \right] \\
&+ m H_1 \left(1 + \frac{|b_1|}{|a_1 + b_1|} \right) \\
&+ \Gamma(2 - \gamma) H_2 \left(1 + \frac{|b_1|}{|a_1 + b_1|} \right) \sum_{i=1}^{m} t_i^\gamma \\
&+ \Gamma(2 - \gamma) H_2 T \left(1 + \frac{|a_2 b_1|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|} \right) \sum_{i=1}^{m} t_i^{\gamma-1} \\
&+ \frac{|c_1|}{|a_1 + b_1|} + \frac{|b_1 c_2| T}{|\Delta|} + \frac{|c_2| T}{|a_2 + b_2|}.
\end{aligned}
\tag{52}
$$

This implies that there exists some $\overline{C} > 0$ such that $\|x\| \leq \overline{C}$ for all $x \in V$; that is, V is bounded. Thus, by Theorem 10, the operator F has at least one fixed point. Hence the problem (1), (2) has at least one solution. The proof is completed. $\square$

(H3): there exist $h \in L^\infty(J, \mathbb{R}^+)$ and positive constants L and L^* such that, for $t \in J$, $x, y \in \mathbb{R}$, $k = 1, 2, \ldots, m$,

$$
\begin{aligned}
|f(t, x) - f(t, y)| &\leq h(t) |x - y|, \\
|I_k(x) - I_k(y)| &\leq L |x - y|, \\
|I_k^*(x) - I_k^*(y)| &\leq L^* |x - y|.
\end{aligned}
\tag{53}
$$

Theorem 14. *Let (H3) hold. Moreover,*

$$
\begin{aligned}
&\left(\frac{\|h\|_{L^\infty} T^\alpha}{\Gamma(\alpha+1)} + mL + \Gamma(2-\gamma) L^* \sum_{i=1}^{m} t_i^\gamma \right) \left(1 + \frac{|b_1|}{|a_1 + b_1|} \right) \\
&\quad + \Gamma(2 - \gamma) T L^* \\
&\quad \times \left(1 + \frac{|a_2 b_1|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|} \right) \sum_{i=1}^{m} t_i^{\gamma-1} \\
&\quad + \left(\frac{|b_1 b_2|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|} \right) \frac{\|h\|_{L^\infty} T^\alpha}{\Gamma(\alpha)} < 1.
\end{aligned}
\tag{54}
$$

Then the impulsive fractional boundary value problem (1), (2) *has a unique solution on J.*

Proof. Let $x, y \in PC(J, \mathbb{R})$. Then, for each $t \in J$, we have

$$
\begin{aligned}
&|(Fx)(t) - (Fy)(t)| \\
&\leq \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, x(s)) - f(s, y(s))| \, ds + |d_0^x - d_0^y| \\
&\quad + \sum_{i=1}^{m} |I_i(x(t_i^-)) - I_i(y(t_i^-))| \\
&\quad + \Gamma(2 - \gamma) \sum_{i=1}^{m} t_i^\gamma |I_i^*(x(t_i^-)) - I_i^*(y(t_i^-))| \\
&\quad + |e_0^x - e_0^y| T \\
&\quad + \Gamma(2 - \gamma) T \sum_{i=1}^{m} t_i^{\gamma-1} |I_i^*(x(t_i^-)) - I_i^*(y(t_i^-))|.
\end{aligned}
\tag{55}
$$

Since

$$
\begin{aligned}
|d_0^x - d_0^y| \leq \ & \frac{|b_1|}{|a_1 + b_1|} \\
& \times \left(\frac{\|h\|_{L^\infty} T^\alpha}{\Gamma(\alpha+1)} + mL \right. \\
& \quad \left. + \Gamma(2 - \gamma) L^* \sum_{i=1}^{m} t_i^\gamma \right) \|x - y\| \\
& + \left(\frac{|a_2 b_1|}{|\Delta|} \Gamma(2-\gamma) T L^* \sum_{i=1}^{m} t_i^{\gamma-1} \right. \\
& \quad \left. + \frac{|b_1 b_2|}{|\Delta|} \frac{\|h\|_{L^\infty} T^\alpha}{\Gamma(\alpha)} \right) \|x - y\|, \\
|e_0^x - e_0^y| T \leq \ & \frac{|b_2|}{|a_2 + b_2|} \\
& \times \left(\frac{\|h\|_{L^\infty} T^\alpha}{\Gamma(\alpha)} + \Gamma(2-\gamma) T L^* \sum_{i=1}^{m} t_i^{\gamma-1} \right) \|x - y\|,
\end{aligned}
\tag{56}
$$

then combining these two estimations with (55), we obtain

$$
\begin{aligned}
&\|Fx - Fy\| \\
&\leq \left[\left(\frac{\|h\|_{L^\infty} T^\alpha}{\Gamma(\alpha+1)} + mL + \Gamma(2-\gamma) L^* \sum_{i=1}^{m} t_i^\gamma \right) \right. \\
&\quad \times \left(1 + \frac{|b_1|}{|a_1 + b_1|} \right) \\
&\quad + \Gamma(2 - \gamma) T L^* \left(1 + \frac{|a_2 b_1|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|} \right) \sum_{i=1}^{m} t_i^{\gamma-1} \\
&\quad \left. + \left(\frac{|b_1 b_2|}{|\Delta|} + \frac{|b_2|}{|a_2 + b_2|} \right) \frac{\|h\|_{L^\infty} T^\alpha}{\Gamma(\alpha)} \right] \|x - y\|.
\end{aligned}
\tag{57}
$$

Therefore, by (54), the operator F is a contraction mapping on $PC(J, \mathbb{R})$. Then it follows from Banach's fixed point theorem that the problems (1), (2) has a unique solution on J. This completes the proof. $\square$

Next we will state some existence results for the problems (1), (3) without proofs since these are similar to the ones obtained for the problems (1), (2) above.

Theorem 15. *Let (H1) hold and there exists a constant $M > 0$ such that*

$$
\frac{M}{\|h\|_{L^\infty}\varphi(M) P_1 + \Gamma(2-\gamma)\psi^*(M) R_1 + \psi(M) Q_1 + H_1} > 1,
\tag{58}
$$

where

$$
P_1 = \left(1 + \frac{|b_1|}{|a_1 + b_1|}\right)\left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{\Gamma(2-\gamma) T^\alpha}{\Gamma(\alpha-\gamma+1)}\right),
$$

$$
Q_1 = m\left(1 + \frac{|b_1|}{|a_1 + b_1|}\right),
$$

$$
R_1 = \left(1 + \frac{|b_1|}{|a_1 + b_1|}\right)\sum_{i=1}^{m} t_i^\gamma + 2T\sum_{i=1}^{m} t_i^{\gamma-1},
\tag{59}
$$

$$
H_1 = \frac{|c_1 b_2 - b_1 c_2 T^\gamma \Gamma(2-\gamma)|}{|(a_1 + b_1) b_2|} + \frac{\Gamma(2-\gamma)|c_2| T^\gamma}{|b_2|}.
$$

Then the impulsive fractional boundary value problem (1), (3) has at least one solution on J.

Theorem 16. *Let (H2) hold; then the impulsive fractional boundary value problem (1), (3) has at least one solution on J.*

Theorem 17. *Assume that (H3) holds. Moreover,*

$$
\left(1 + \frac{|b_1|}{|a_1 + b_1|}\right)\left[\frac{\|h\|_{L^\infty} T^\alpha}{\Gamma(\alpha+1)} + \frac{\|h\|_{L^\infty}\Gamma(2-\gamma) T^\alpha}{\Gamma(\alpha-\gamma+1)}\right.
$$

$$
\left. + mL + \Gamma(2-\gamma) L^* \sum_{i=1}^{m} t_i^\gamma \right]
\tag{60}
$$

$$
+ 2\Gamma(2-\gamma) T L^* \sum_{i=1}^{m} t_i^{\gamma-1} < 1.
$$

Then the impulsive fractional boundary value problem (1), (3) has a unique solution on J.

Remark 18. The results obtained in this paper can easily be generalized to the boundary value problems of impulsive fractional differential equations (1) with nonseparated integral boundary conditions

$$
a_1 x(0) + b_1 x(T) = c_1 \int_0^T g(s, x(s))\, ds,
$$

$$
a_2 x'(0) + b_2 x'(T) = c_2 \int_0^T h(s, x(s))\, ds,
$$

$$
a_1 x(0) + b_1 x(T) = c_1 \int_0^T g(s, x(s))\, ds,
\tag{61}
$$

$$
a_2\left({}^c D^\gamma x(0)\right) + b_2\left({}^c D^\gamma x(T)\right) = c_2 \int_0^T h(s, x(s))\, ds,
$$

where g and $h : [0, T] \times \mathbb{R} \to \mathbb{R}$ are given continuous functions.

4. Examples

In this section, we give two simple examples to show the applicability of our results.

Example 1. Consider the following impulsive fractional BVP:

$$
{}^c D^{7/4} x(t) = \frac{\cos t}{(t+6)^2}(x(t) + \arctan x(t)),
$$

$$
t \in [0, 1], \quad t \neq \frac{1}{2},
$$

$$
\Delta x\left(\frac{1}{2}\right) = \frac{|x(1/2^-)|}{(17 + |x(1/2^-)|)},
$$

$$
\Delta\left({}^c D^{1/4} x\left(\frac{1}{2}\right)\right) = \frac{|x(1/2^-)|}{(20 + |x(1/2^-)|)},
\tag{62}
$$

$$
3x(0) + \frac{1}{3}x(1) = 1,
$$

$$
2\left({}^c D^{1/4} x(0)\right) + 3\left({}^c D^{1/4} x(1)\right) = \frac{1}{4}.
$$

Here $\alpha = 7/4$, $\gamma = 1/4$, $T = 1$, $m = 1$, $a_1 = 3$, $b_1 = 1/3$, $c_1 = 1$, $a_2 = 2$, $b_2 = 3$, and $c_2 = 1/4$. Clearly, we can take $h(t) = 2\cos t/(t+6)^2$, $L = 1/17$, and $L^* = 1/20$ such that the relations (53) hold. Moreover,

$$
\left(1 + \frac{|b_1|}{|a_1 + b_1|}\right)\left[\frac{\|h\|_{L^\infty} T^\alpha}{\Gamma(\alpha+1)} + \frac{\|h\|_{L^\infty}\Gamma(2-\gamma) T^\alpha}{\Gamma(\alpha-\gamma+1)}\right.
$$

$$
\left. + mL + \Gamma(2-\gamma) L^* \sum_{i=1}^{m} t_i^\gamma \right]
$$

$$
+ 2\Gamma(2-\gamma) T L^* \sum_{i=1}^{m} t_i^{\gamma-1}
\tag{63}
$$

$$
\approx \frac{11}{10} \times \left(\frac{1.3131}{18} + \frac{1}{17} + 0.0386\right) + 0.1546
$$

$$
= 0.3420 < 1.
$$

Thus, all the assumptions of Theorem 17 are satisfied. Hence, by the conclusion of Theorem 17, the impulsive fractional BVP (62) has a unique solution on $[0, 1]$.

Example 2. Consider the following impulsive fractional BVP:

$$^cD^{3/2}x(t) = 5t^2 + e^{-|x(t)|} + \sin x(t),$$

$$t \in [0, 1], \ t \neq \frac{1}{4},$$

$$\Delta x\left(\frac{1}{4}\right) = \frac{2\left|x(1/4^-)\right|}{\left(1 + \left|x(1/4^-)\right|\right)}, \tag{64}$$

$$\Delta\left(^cD^{1/2}x\left(\frac{1}{4}\right)\right) = \cos x\left(\frac{1}{4}^-\right) + 3,$$

$$3x(0) + 2x(1) = 1, \qquad x'(0) + 3x'(1) = 2.5.$$

Here $\alpha = 3/2$, $\gamma = 1/2$, $T = 1$, $m = 1$, $a_1 = 3$, $b_1 = 2$, $c_1 = 1$, $a_2 = 1$, $b_2 = 3$, and $c_2 = 2.5$. In the context of this problem, we have

$$\left|f(t, x)\right| = \left|5t^2 + e^{-|x|} + \sin x\right| \leq 7, \quad t \in [0, 1], \ x \in \mathbb{R},$$

$$\left|I_k(x)\right| \leq 2, \quad \left|I_k^*(x)\right| \leq 4, \quad x \in \mathbb{R}. \tag{65}$$

Put $h(t) \equiv 7$, $H_1 = 2$, and $H_2 = 4$; we know that the condition (51) holds. Then from Theorem 13, the impulsive fractional BVP (64) has at least one solution on $[0, 1]$.

5. Conclusions

In this paper, we investigated the existence and uniqueness of solutions for a class of fractional differential equations with fractional impulsive conditions and nonseparated boundary conditions. We obtained the results by using Schaefer fixed point theorem, Banach fixed point theorem, and nonlinear alternative of Leray-Schauder type. And two examples are provided in Section 4 to illustrate the results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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