Research Article

The Existence and Uniqueness Result for a Relativistic Nonlinear Schrödinger Equation

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We study the existence and uniqueness of positive solutions for a class of quasilinear elliptic equations. This model has been proposed in the self-channeling of a high-power ultrashort laser in matter.

1. Introduction

In this paper, we consider the following quasilinear Schrödinger equation:

$$-\Delta u + \omega u - \frac{\kappa}{2}\left[\Delta\left(1 + u^2\right)^{1/2}\right]\frac{u}{\left(1 + u^2\right)^{1/2}}$$
$$= |u|^{p-1}u, \quad x \in \mathbb{R}^N, \tag{1}$$

where $\omega > 0, \kappa > 0, N \geq 3$, and $2 < p+1 < 2^* := 2N/(N-2)$. Solutions of (1) are related to standing waves for the following quasilinear Schrödinger equation:

$$iz_t = -\Delta z + W(x)z - h\left(|z|^2\right)z - \kappa\Delta l\left(|z|^2\right)l'\left(|z|^2\right)z, \tag{2}$$
$$x \in \mathbb{R}^N,$$

where $z : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, κ is real constant, and l and h are real functions. Quasilinear equations such as (2) have been accepted as models of several physical phenomena corresponding to various types of l; see [1–5] for physical backgrounds.

The superfluid film equation in plasma physics has this structure for $l(s) = s$ (see [6]). Putting $z(t, x) = \exp(-iEt)u(x)$, where $E \in \mathbb{R}$ and $u > 0$ is a real function, (2) turns into the following equation:

$$-\Delta u + V(x)u - \Delta\left(|u|^2\right)u = \rho(u), \quad x \in \mathbb{R}^N, \tag{3}$$

where $V(x) = W(x) - E$ is the new potential function and ρ is the new nonlinearity. In this case, the first existence results are due to [7]. In [7], the main existence results are obtained through a constrained minimization argument. Subsequently, a general existence result was derived in [8]. The idea in [8] is to make a change of variables and reduce the quasilinear problem to semilinear one and Orlicz space framework was used to prove the existence of positive solutions via the Mountain pass theorem. The same method of changing of variables was also used in [9] but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was used as the working space. Precisely, since the energy functional associated (3) is not well defined in $H^1(\mathbb{R}^N)$, they first make the changing of unknown variables $v = f^{-1}(u)$, where f is defined by ODE as follows:

$$f'(t) = \frac{1}{\sqrt{1 + 2f^2(t)}}, \quad t \in [0, +\infty), \tag{4}$$

and $f(t) = -f(-t), t \in (-\infty, 0]$. Then, after the changing of variable, to find the solutions of (2), it suffices to study the existence of solutions for the following semilinear equation:

$$-\Delta v = \phi(x, v), \quad x \in \mathbb{R}^N, \tag{5}$$

where

$$\phi(x, v) = \frac{1}{\sqrt{1 + 2f^2(v)}}\left(-V(x)f(v) + \rho(f(v))\right). \tag{6}$$

By using the classical results given by [10], they proved the existence of a spherically symmetric solution. In [11], the authors give a sufficient condition for uniqueness of the ground state solutions by using the same change of variables as [9].

In the case $l(s) = (1+s)^{1/2}$, (2) models the self-channeling of a high-power ultrashort laser in matter (see [12]). In this case, few results are known. In [13], the authors proved global existence and uniqueness of small solutions in transverse space dimensions 2 and 3 and local existence without any smallness condition in transverse space dimension 1. But they did not study the existence of standing waves. But we have to point out that the method of change of variables as (4) cannot be generalized to treat the case $l(s) = (1 + s)^{1/2}$. In [14], the authors made the changing of known variable (see also [15])

$$v = G(u) = \int_0^u \sqrt{1 + 2(sl'(s^2))^2}\, ds \qquad (7)$$

and proved the existence of nontrivial solution with $N \geq 3$ and $\kappa = 1$. In this paper, for $l(s) = (1 + s)^{1/2}$ and $\kappa > 0$, we will show the existence and uniqueness result for (1) by using a change of variables due to [14, 15]. One main difficulty in dealing with this problem seems to be that of obtaining the boundedness of a (PS) sequence for the corresponding functional. We overcome this difficulty by using Jeanjean's result [16].

Our main result is the following.

Theorem 1. *Assume that* $N \geq 3$, $\omega > 0$, $\kappa > 0$, *and* $\max\{4\sqrt{2/(2 + \kappa)} - 1, 2\} < p + 1 < 2^*$. *There exists* $c_0 = c_0(p, \kappa) > 0$ *such that if* $\omega^{1/(p-1)} \geq c_0$, *then the positive solution of* (1) *is unique.*

In this paper, C denotes positive (possibly different) constant, $L^p(\mathbb{R}^N)$ denotes the usual Lebesgue space with norm $\| u \|_p = (\int_{\mathbb{R}^N} |u|^p dx)^{1/p}$, $1 \leq p < \infty$, and $H^1(\mathbb{R}^N)$ denotes the Sobolev space with norm $\| u \| = (\int_{\mathbb{R}^N} (|\nabla u|^2 + \omega u^2) dx)^{1/2}$.

2. Preliminaries

We note that the solutions of (1) are the critical points of the following functional:

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left[1 + \frac{\kappa u^2}{2(1 + u^2)} \right] |\nabla u|^2 dx$$

$$+ \frac{1}{2} \int_{\mathbb{R}^N} \omega u^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx. \qquad (8)$$

Since the functional $I(u)$ may not be well defined in the usual Sobolev spaces $H^1(\mathbb{R}^N)$, we make a change of variables as

$$v = G(u) = \int_0^u g(t)\, dt, \qquad (9)$$

where $g(t) = \sqrt{1 + (\kappa t^2/2(1 + t^2))}$. Since $g(t)$ is monotonous with $|t|$, the inverse function $G^{-1}(t)$ of $G(t)$ exists. Then after the change of variables, $I(u)$ can be written as

$$J(v) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \omega |G^{-1}(v)|^2 dx$$

$$- \frac{1}{p+1} \int_{\mathbb{R}^N} |G^{-1}(v)|^{p+1} dx. \qquad (10)$$

By Lemma 2 listed below, we have $\lim_{t \to 0} G^{-1}(t)/t = 1$ and $\lim_{t \to \infty} |G^{-1}(t)|/t = \sqrt{2/(2 + \kappa)}$, so $J(v)$ is well defined in $H^1(\mathbb{R}^N)$ and $J(v) \in C^1$.

If u is a nontrivial solution of (1), then for all $\phi \in C_0^\infty(\mathbb{R}^N)$ it should satisfy

$$\int_{\mathbb{R}^N} \left[g^2(u) \nabla u \nabla \phi + g(u) g'(u) |\nabla u|^2 \phi \right.$$

$$\left. + \omega u \phi - u^p \phi \right] dx = 0. \qquad (11)$$

We show that (11) is equivalent to

$$J'(v)\psi$$

$$= \int_{\mathbb{R}^N} \left[\nabla v \nabla \psi + \omega \frac{G^{-1}(v)}{g(G^{-1}(v))} \psi - \frac{|G^{-1}(v)|^p}{g(G^{-1}(v))} \psi \right] dx = 0,$$

$$\forall \psi \in C_0^\infty(\mathbb{R}^N). \qquad (12)$$

Indeed, if we choose $\phi = (1/g(u))\psi$ in (11), then we get (12). On the other hand, since $u = G^{-1}(v)$, if we let $\psi = g(u)\phi$ in (12), we get (11). Therefore, in order to find the nontrivial solutions of (1), it suffices to study the existence of the nontrivial solutions of the following equation:

$$-\Delta v = \frac{|G^{-1}(v)|^p}{g(G^{-1}(v))} - \omega \frac{G^{-1}(v)}{g(G^{-1}(v))}, \quad x \in \mathbb{R}^N. \qquad (13)$$

Before we close this section, we give some properties of the change of variables.

Lemma 2. (1) $\sqrt{2/(2 + \kappa)}\, t \leq |G^{-1}(t)| \leq t$ *for all* $t \geq 0$;
   (2) $|(G^{-1}(t))'| \leq 1$ *for all* $t \in \mathbb{R}$;
   (3) $\lim_{t \to 0}(|G^{-1}(t)|/t) = 1$;
   (4) $\lim_{t \to \infty}(|G^{-1}(t)|/t) = \sqrt{2/(2 + \kappa)}$;
   (5) $tg'(t)/g(t) \leq ((\kappa + 4) - 2\sqrt{2(2 + \kappa)})/\kappa$ *for all* $t \in \mathbb{R}$;
   (6) $\sqrt{2/(2 + \kappa)}\, G^{-1}(t) \leq t(G^{-1}(t))' \leq G^{-1}(t)$ *for all* $t \geq 0$.

Proof. (1) Since $[G^{-1}(t) - (1/g(0))t]' = 1/g(G^{-1}(t)) - 1/g(0) \leq 0$ and $[G^{-1}(t) - (1/g(\infty))t]' = 1/g(G^{-1}(t)) - 1/g(\infty) \geq 0$, so $1/g(\infty)t \leq G^{-1}(t) \leq (1/g(0))t$, for $t \geq 0$; that is, $(1/g(\infty))t = \sqrt{2/(2 + \kappa)}\, t \leq G^{-1}(t) \leq (1/g(0))t = t$, for $t \geq 0$, which proves (1).

Since $\lim_{t \to 0}(G^{-1}(t)/t) = ((G^{-1}(t))'|_{t=0} = 1/g(G^{-1}(0)) = 1$ and $g(t)$ is increasing, so properties (2) and (3) are obvious.

For (4), the result is obvious since $g(t)$ is an increasing bounded function.

Since

$$\frac{t}{g(t)}g'(t) = \frac{\kappa t^2}{2(1+t^2)^2 g^2(t)}$$

$$= \frac{\kappa t^2}{2 + (\kappa + 4)t^2 + (\kappa + 2)t^4}$$

$$= \frac{\kappa}{2/t^2 + (\kappa + 4) + (\kappa + 2)t^2}$$

$$\leq \frac{(\kappa + 4) - 2\sqrt{2(2+\kappa)}}{\kappa}, \qquad (14)$$

which proves (5).

For (6), since g is a increasing function, then $G(t) \leq g(t)t$, which implies that $t(G^{-1}(t))' \leq G^{-1}(t)$. On the other hand, by (1) and $\sqrt{2/(2+\kappa)} \leq (G^{-1}(t))' \leq 1$, we get $\sqrt{2/(2+\kappa)}G^{-1}(t) \leq t(G^{-1}(t))'$. $\qquad \square$

3. Existence

At first, we give two Lemmas.

Lemma 3. *There exist $\rho_0, a_0 > 0$ such that $J(v) \geq a_0$ for all $\| v \| = \rho_0$.*

Proof. Let

$$Q(t) := -\frac{1}{2}\omega\left|G^{-1}(t)\right|^2 + \frac{1}{p+1}\left|G^{-1}(t)\right|^{p+1}. \qquad (15)$$

Then, by Lemma 2 and $2 < p + 1 < 2^*$, we have

$$\lim_{t \to 0}\frac{Q(t)}{t^2} = \lim_{t \to 0}\left[-\frac{1}{2}\omega\left(\frac{G^{-1}(t)}{t}\right)^2\right.$$

$$\left. + \frac{1}{p+1}\left(\frac{G^{-1}(t)}{t}\right)^2\left|G^{-1}(t)\right|^{p-1}\right]$$

$$= -\frac{1}{2}\omega,$$

$$\lim_{t \to \infty}\frac{Q(t)}{t^{2^*}} = \lim_{t \to \infty}\left[-\frac{1}{2}\omega\left(\frac{\left|G^{-1}(t)\right|}{t}\right)^2\frac{1}{t^{2^*-2}}\right.$$

$$\left. + \frac{1}{p+1}\left(\frac{\left|G^{-1}(t)\right|}{t}\right)^{p+1}\frac{1}{t^{2^*-(p+1)}}\right] = 0. \qquad (16)$$

Thus, for $\epsilon > 0$ sufficiently small, there exists a constant $C_\epsilon > 0$ such that

$$Q(t) \leq \left(-\frac{1}{2}\omega + \epsilon\right)t^2 + C_\epsilon|t|^{2^*}. \qquad (17)$$

Then, we have

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N}\omega\left|G^{-1}(v)\right|^2 dx$$

$$- \frac{1}{p+1}\int_{\mathbb{R}^N}\left|G^{-1}(v)\right|^{p+1}dx$$

$$\geq \frac{1}{2}\int_{\mathbb{R}^N}|\nabla v|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N}\omega v^2 dx - \epsilon$$

$$\times \int_{\mathbb{R}^N}v^2 dx - C_\epsilon\int_{\mathbb{R}^N}v^{2^*}dx \qquad (18)$$

$$\geq C\|v\|^2 - C\|v\|^{2^*}.$$

Thus, by choosing ρ_0 small, we get the result when $\| v \| = \rho_0$. $\qquad \square$

Lemma 4. *There exists $v \in H^1(\mathbb{R}^N)$ such that $J(v) < 0$.*

Proof. Given $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ with $\mathrm{supp}\,\phi := \overline{B}_1$, we will prove that $J(s\phi) \to -\infty$ as $s \to \infty$, which will prove the result if we take $v = s\phi$ with s large enough. By Lemma 2, we have $G^{-1}(t) \geq Ct$ as $t \geq 1$, so

$$J(s\phi) \leq \frac{1}{2}s^2\int_{\mathbb{R}^N}|\nabla\phi|^2 dx + \frac{1}{2}s^2\int_{\mathbb{R}^N}\omega\phi^2 dx - s^{p+1}C$$

$$\times \int_{\{|s\phi|\geq 1\}}\phi^{p+1}dx \longrightarrow -\infty, \qquad (19)$$

as $s \to \infty$. Thus, we get the result. $\qquad \square$

We will use the following Theorem which is due to Jeanjean [16].

Theorem 5. *Let X be a Banach space equipped with the norm $\| \cdot \|$ and let $L \subset \mathbb{R}^+$ be an interval. One considers a family $(I_\lambda)_{\lambda \in L}$ of C^1-functionals on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \forall \lambda \in L, \qquad (20)$$

where $B(u) \geq 0$, for all $u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\| u \| \to \infty$. One assumes that there are two points (v_1, v_2) in X such that setting

$$\Gamma = \{\gamma \in C([0,1], X), \gamma(0) = v_1, \gamma(1) = v_2\}, \qquad (21)$$

there hold, for all $\lambda \in L$,

$$c_\lambda := \inf_{\gamma \in \Gamma}\max_{t \in [0,1]}I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\}. \qquad (22)$$

Then, for almost every $\lambda \in L$, there is a subsequence $\{v_n(\lambda)\} \subset X$ such that

(i) *$\{v_n(\lambda)\}$ is bounded;*

(ii) *$I_\lambda(v_n(\lambda)) \to c_\lambda$;*

(iii) *$I'_\lambda(v_n(\lambda)) \to 0$ in the dual X^{-1} of X.*

We consider the functional

$$J_\lambda(v) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla v|^2 + \omega \left| G^{-1}(v) \right|^2 \right) dx$$
$$- \frac{\lambda}{p+1} \int_{\mathbb{R}^N} \left| G^{-1}(v) \right|^{p+1} dx, \tag{23}$$

where $\lambda \in [1/2, 1]$.

Let $L = [1/2, 1]$. We find that

$$B(v) := \frac{1}{p+1} \int_{\mathbb{R}^N} \left| G^{-1}(v) \right|^{p+1} dx \geq 0 \tag{24}$$

for all $v \in H^1(\mathbb{R}^N)$. On the other hand, if $\| v \| = (\int_{\mathbb{R}^N}(|\nabla v|^2 + \omega v^2)dx)^{1/2} \to +\infty$, then either $\int_{\mathbb{R}^N} |\nabla v|^2 dx \to +\infty$, which implies $A(v) := \int_{\mathbb{R}^N}(|\nabla v|^2 + \omega|G^{-1}(v)|^2)dx \to +\infty$, or $\int_{\mathbb{R}^N} \omega v^2 dx \to +\infty$; in this case, to verify that $A(v) \to +\infty$, we start splitting

$$\int_{\mathbb{R}^N} \omega v^2 dx = \int_{\{x:|v(x)|\geq 1\}} \omega v^2 dx + \int_{\{x:|v(x)|\leq 1\}} \omega v^2 dx, \tag{25}$$

since

$$\int_{\{x:|v(x)|\geq 1\}} \omega v^2 dx \leq C \int_{\{x:|v(x)|\geq 1\}} v^{2^*} dx$$
$$\leq C \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx \right)^{2^*/2}, \tag{26}$$

and by Lemma 2 (6), we have $v = G(G^{-1}(v)) \leq g(G^{-1}(v)) G^{-1}(v)$, so

$$\int_{\{x:|v(x)|\leq 1\}} \omega \left| G^{-1}(v) \right|^2 dx \geq \frac{1}{g^2(1)} \int_{\{x:|v(x)|\leq 1\}} \omega v^2 dx, \tag{27}$$

so $A(v) \to +\infty$.

For $J_\lambda(v)$ defined above with $\lambda \in [1/2, 1]$, using Lemma 4, we get a $v \in H^1(\mathbb{R}^N)$ such that $J_\lambda(v) < 0$. Also from Lemma 2 we know that $B(v) = o(\| v \|^2)$ as $v \to 0$. Thus setting

$$\Gamma = \left\{ \gamma \in C\left([0,1], H^1\left(\mathbb{R}^N\right)\right), \gamma(0) = 0, \gamma(1) = v \right\}, \tag{28}$$

we have, for all $\lambda \in [1/2, 1]$,

$$c_\lambda := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)) > 0 = \max\{J_\lambda(0), J_\lambda(v)\}. \tag{29}$$

Therefore, using Theorem 5, for almost all $\lambda \in [1/2, 1]$, there exists a subsequence $\{v_n(\lambda)\} \subset H^1(\mathbb{R}^N)$ such that

  (i) $\{v_n(\lambda)\}$ is bounded in $H^1(\mathbb{R}^N)$;

  (ii) $J_\lambda(v_n(\lambda)) \to c_\lambda$;

  (iii) $J'_\lambda(v_n(\lambda)) \to 0$ in $H^{-1}(\mathbb{R}^N)$.

Lemma 6. *Assume that* $\{v_n(\lambda)\} \subset H^1(\mathbb{R}^N)$ *is a bounded Palais-Smale sequence of the functional* J_λ *for* $\lambda \in [1/2, 1]$. *Then there exists a nontrivial critical point of* J_λ.

Proof. We first note that $\{v_n(\lambda)\} \subset H^1(\mathbb{R}^N)$ satisfies

$$J_\lambda(v_n(\lambda)) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n(\lambda)|^2 dx$$
$$+ \frac{1}{2} \int_{\mathbb{R}^N} \omega \left| G^{-1}(v_n(\lambda)) \right|^2 dx$$
$$- \frac{\lambda}{p+1} \int_{\mathbb{R}^N} \left| G^{-1}(v_n(\lambda)) \right|^{p+1} dx = c_\lambda + o(1), \tag{30}$$

and, for any $\psi \in C_0^\infty(\mathbb{R}^N)$,

$$J'_\lambda(v_n(\lambda))\psi = \int_{\mathbb{R}^N} \left[\nabla v_n(\lambda) \nabla \psi + \omega \frac{G^{-1}(v_n(\lambda))}{g(G^{-1}(v_n(\lambda)))} \psi \right.$$
$$\left. - \lambda \frac{\left| G^{-1}(v_n(\lambda)) \right|^p}{g(G^{-1}(v_n(\lambda)))} \psi \right] dx = o(1) \|\psi\|. \tag{31}$$

Since $\{v_n(\lambda)\}$ is a bounded Palais-Smale sequence, there exists $v(\lambda) \in H^1(\mathbb{R}^N)$ such that $v_n(\lambda) \rightharpoonup v(\lambda)$ in $H^1(\mathbb{R}^N)$ and $v_n(\lambda) \to v(\lambda)$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ for $p \in [2, 2^*)$. By the Lebesgue Dominated Theorem, we have

$$J'_\lambda(v_n(\lambda))\psi - J'_\lambda(v(\lambda))\psi$$
$$= \int_{\mathbb{R}^N} (\nabla v_n(\lambda) - \nabla v(\lambda)) \nabla \psi \, dx$$
$$+ \omega \int_{\mathbb{R}^N} \left[\frac{G^{-1}(v_n(\lambda))}{g(G^{-1}(v_n(\lambda)))} - \frac{G^{-1}(v)}{g(G^{-1}(v(\lambda)))} \right] \psi \, dx$$
$$- \lambda \int_{\mathbb{R}^N} \left[\frac{\left| G^{-1}(v_n(\lambda)) \right|^p}{g(G^{-1}(v_n(\lambda)))} - \frac{\left| G^{-1}(v(\lambda)) \right|^p}{g(G^{-1}(v(\lambda)))} \right] \psi \, dx$$
$$\longrightarrow 0. \tag{32}$$

Hence, v is a weak solution of (1). If $v(\lambda) \not\equiv 0$, then we get the result.

Otherwise, if $v(\lambda) = 0$, we claim that for all $R > 0$,

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} v_n(\lambda)^2 dx = 0 \tag{33}$$

cannot occur. Suppose by contradiction that (33) occurs, that is, $\{v_n(\lambda)\}$ vanish; then, by the Lions compactness Lemma (see [17, 18]), $v_n \to 0$ in $L^r(\mathbb{R}^N)$ for any $r \in (2, 2^*)$. Since $2 < p+1 < 2^*$, then by the proof of Lemma 2, we get

$$\left| \frac{\left| G^{-1}(t) \right|^p}{g(G^{-1}(t))} t \right| \leq \left| G^{-1}(t) \right|^{p+1} \leq \varepsilon t^2 + C_\varepsilon |t|^{p+1}, \tag{34}$$

which implies that

$$
\begin{aligned}
0 &= \lim_{n \to \infty} J_\lambda'(v_n(\lambda)) v_n(\lambda) \\
&= \lim_{n \to \infty} \int_{\mathbb{R}^N} \left[|\nabla v_n(\lambda)|^2 + \omega \frac{G^{-1}(v_n(\lambda))}{g(G^{-1}(v_n(\lambda)))} v_n(\lambda) \right. \\
&\qquad\qquad \left. - \lambda \frac{|G^{-1}(v_n(\lambda))|^p}{g(G^{-1}(v_n(\lambda)))} v_n(\lambda) \right] dx \\
&\geq \lim_{n \to \infty} \int_{\mathbb{R}^N} \left[|\nabla v_n(\lambda)|^2 + \omega \frac{G^{-1}(v_n(\lambda))}{g(G^{-1}(v_n(\lambda)))} v_n(\lambda) \right. \\
&\qquad\qquad \left. - \lambda \varepsilon v_n(\lambda)^2 - \lambda C_\varepsilon |v_n(\lambda)|^{p+1} \right] dx.
\end{aligned}
\tag{35}
$$

Since $\varepsilon \to 0$ and $p + 1 \in (2, 2^*)$, then

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla v_n(\lambda)|^2 dx = 0,
\tag{36}
$$

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} \omega \frac{G^{-1}(v_n(\lambda))}{g(G^{-1}(v_n(\lambda)))} v_n(\lambda) \, dx = 0.
\tag{37}
$$

On the other hand, note by Lemma 2 (5) that

$$
\begin{aligned}
&\left| \nabla \left(G^{-1}(v_n(\lambda)) \right) g \left(G^{-1}(v_n(\lambda)) \right) \right| \\
&\leq \left[1 + \frac{G^{-1}(v_n(\lambda))}{g(G^{-1}(v_n(\lambda)))} g' \left(G^{-1}(v_n(\lambda)) \right) \right] |\nabla v_n(\lambda)| \\
&\leq \frac{2(\kappa + 2) - 2\sqrt{2(2 + \kappa)}}{\kappa} |\nabla v_n(\lambda)|.
\end{aligned}
\tag{38}
$$

Combing Lemma 2, we have $G^{-1}(v_n(\lambda)) g(G^{-1}(v_n(\lambda))) \in H^1(\mathbb{R}^N)$. In fact, we only need to show that $|G^{-1}(v_n(\lambda)) g(G^{-1}(v_n(\lambda)))| \leq |v_n(\lambda)|$; let $\Phi(t) = g(t)t - G(t)$; then by Lemma 2 (5), we have

$$
\Phi'(t) = g'(t)t - 2g(t) \leq \frac{4 - \kappa - 2\sqrt{2(2 + \kappa)}}{\kappa} g(t) < 0,
$$
$$
\text{for } \kappa > 0,
\tag{39}
$$

so $g(t)t \leq G(t)$ for $t \geq 0$ and $g(t)t \geq G(t)$ for $t < 0$, which implies that $|G^{-1}(v_n(\lambda)) g(G^{-1}(v_n(\lambda)))| \leq |v_n(\lambda)|$. Thus, since $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, by choosing

$$
\psi = G^{-1}(v_n(\lambda)) g \left(G^{-1}(v_n(\lambda)) \right)
\tag{40}
$$

in (31), we deduce that

$$
\begin{aligned}
&o(1) \| v_n(\lambda) \| \\
&= J_\lambda'(v_n(\lambda)) G^{-1}(v_n(\lambda)) g \left(G^{-1}(v_n(\lambda)) \right) \\
&= \int_{\mathbb{R}^N} \left[\left(1 + \frac{G^{-1}(v_n(\lambda))}{g(G^{-1}(v_n(\lambda)))} g' \left(G^{-1}(v_n(\lambda)) \right) \right) \right. \\
&\qquad\qquad \times |\nabla v_n(\lambda)|^2 \\
&\qquad \left. + \omega |G^{-1}(v_n(\lambda))|^2 - \lambda |G^{-1}(v_n(\lambda))|^{p+1} \right] dx.
\end{aligned}
\tag{41}
$$

So

$$
\lim_{n \to \infty} \int_{\mathbb{R}^N} \omega |G^{-1}(v_n(\lambda))|^2 dx = 0.
\tag{42}
$$

Combing (36) and (34), we have

$$
\begin{aligned}
J_\lambda(v_n(\lambda)) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v_n(\lambda)|^2 dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^N} \omega |G^{-1}(v_n(\lambda))|^2 dx \\
&\quad - \frac{\lambda}{p+1} \int_{\mathbb{R}^N} |G^{-1}(v_n(\lambda))|^{p+1} dx \longrightarrow 0, \\
&\qquad\qquad\qquad\qquad\qquad \text{as } n \longrightarrow \infty,
\end{aligned}
\tag{43}
$$

so we get a contradiction since $J_\lambda(v_n(\lambda)) \to c_\lambda > 0$. Thus, $\{v_n(\lambda)\}$ does not vanish and there exist $k, \ R > 0$, and $\{y_n\} \subset \mathbb{R}^N$ such that

$$
\lim_{n \to \infty} \int_{B_R(y_n)} v_n(\lambda)^2 dx \geq k > 0.
\tag{44}
$$

Define $v_n(\lambda) := v_n^\lambda$ and $\widetilde{v}_n^\lambda(x) = v_n^\lambda(x + y_n)$. Since $\{v_n^\lambda\}$ is a Palais-Smale sequence for J_λ, $\{\widetilde{v}_n^\lambda\}$ is also a Palais-Smale sequence for J_λ with $J_\lambda'(\widetilde{v}^\lambda) = 0$ if $\widetilde{v}_n^\lambda \to \widetilde{v}^\lambda$ in $H^1(\mathbb{R}^N)$. Since $\{\widetilde{v}_n^\lambda\}$ does not vanish, we have that $\widetilde{v}^\lambda \neq 0$ is a nontrivial solution of (1). $\qquad \square$

From Lemma 6, we see that, for almost all $\lambda \in [1/2, 1]$, there exists a solution $v(\lambda)$ to the following Schrödinger equation:

$$
-\Delta v = \widetilde{h}(x, v), \quad x \in \mathbb{R}^N,
\tag{45}
$$

where

$$
\widetilde{h}(x, v) = -\omega \frac{G^{-1}(v)}{g(G^{-1}(v))} + \lambda \frac{|G^{-1}(v)|^p}{g(G^{-1}(v))}.
\tag{46}
$$

Therefore, we can choose $\{\lambda_j\} \subset [1/2, 1]$ such that $\lambda_j \to 1$. Setting $v_j := v(\lambda_j)$, we have $J_{\lambda_j}'(v_j) = 0$. We can deduce that v is a solution to (13) if we show that $J'(v) = 0$. To prove this, in view of Lemma 6, we first check that $\{v_j\}$ is bounded in $H^1(\mathbb{R}^N)$.

Notice that the Pohozaev identity implies that the solutions of (45) satisfy

$$
\frac{N-2}{2N} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{\omega}{2} \int_{\mathbb{R}^N} \left| G^{-1}(v) \right|^2 dx
$$

$$
= \frac{\lambda}{p+1} \int_{\mathbb{R}^N} \left| G^{-1}(v) \right|^{p+1} dx. \tag{47}
$$

Lemma 7. *The sequence $\{v_j\}$ is bounded.*

Proof. Since v_j is a solution to (45) with $\lambda = \lambda_j$, by (47), we have

$$
\begin{aligned}
c_{1/2} &\geq c_{\lambda_j} \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \left| \nabla v_j \right|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \omega \left| G^{-1}(v_j) \right|^2 dx \\
&\quad - \frac{\lambda_j}{p+1} \int_{\mathbb{R}^N} \left| G^{-1}(v_j) \right|^{p+1} dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \left| \nabla v_j \right|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \omega \left| G^{-1}(v_j) \right|^2 dx \\
&\quad - \left[\frac{N-2}{2N} \int_{\mathbb{R}^N} \left| \nabla v_j \right|^2 dx + \frac{\omega}{2} \int_{\mathbb{R}^N} \left| G^{-1}(v_j) \right|^2 dx \right] \\
&\geq \frac{1}{N} \int_{\mathbb{R}^N} \left| \nabla v_j \right|^2 dx,
\end{aligned} \tag{48}
$$

which implies that $\int_{\mathbb{R}^N} |\nabla v_j|^2 dx$ is bounded. On the other hand, together with (41), we have

$$
\begin{aligned}
(p+1) c_{\lambda_j} - 0 &= (p+1) J_{\lambda_j}(v_j) \\
&\quad - J_{\lambda_j}'(v_j) G^{-1}(v_j) g\left(G^{-1}(v_j)\right) \\
&\geq \left(\frac{p+1}{2} - \frac{2(\kappa+2) - 2\sqrt{2(2+\kappa)}}{\kappa} \right) \\
&\quad \times \int_{\mathbb{R}^N} \left| \nabla v_j \right|^2 dx \\
&\quad + \frac{p-1}{2} \int_{\mathbb{R}^N} \omega \left| G^{-1}(v_j) \right|^2 dx.
\end{aligned} \tag{49}
$$

Since $\int_{\mathbb{R}^N} |\nabla v_j|^2 dx$ is bounded, so $\int_{\mathbb{R}^N} \omega |G^{-1}(v_j)|^2 dx$ is bounded. To verify that $\{v_j\}$ is bounded in $H^1(\mathbb{R}^N)$, we start splitting

$$
\int_{\mathbb{R}^N} \omega v_j^2 dx = \int_{\{x:|v_j|\geq 1\}} \omega v_j^2 dx + \int_{\{x:|v_j|\leq 1\}} \omega v_j^2 dx, \tag{50}
$$

since

$$
\int_{\{x:|v_j|\geq 1\}} \omega v_j^2 dx \leq C \int_{\{x:|v_j|\geq 1\}} v_j^{2^*} dx \leq C \left(\int_{\mathbb{R}^N} \left| \nabla v_j \right|^2 dx \right)^{2^*/2},
$$

$$
\int_{\{x:|v_j|\leq 1\}} \omega \left| G^{-1}(v_j) \right|^2 dx \geq \frac{1}{g^2(1)} \int_{\{x:|v_j|\leq 1\}} \omega v_j^2 dx, \tag{51}
$$

so $\| v_j \| = (\int_{\mathbb{R}^N} (|\nabla v_j|^2 + \omega v_j^2) dx)^{1/2}$ is bounded. $\square$

Lemma 8. *Assume that $N \geq 3$, $\omega > 0$, and $2 < p+1 < 2^*$. Then (1) has a nontrivial solution.*

Proof. The boundedness of $\{v_j\}$ in $H^1(\mathbb{R}^N)$ follows from Lemma 7; we have that $\{G^{-1}(v_j)\}$ is bounded in $L^s(\mathbb{R}^N)$ for $2 \leq s \leq 2^*$. Then for any $\psi \in C_0^\infty(\mathbb{R}^N)$, we have

$$
\begin{aligned}
& J_{\lambda_j}'(v_j)\psi - J'(v_j)\psi \\
&= (1 - \lambda_j) \int_{\mathbb{R}^N} \frac{\left| G^{-1}(v_j) \right|^p}{g\left(G^{-1}(v_j)\right)} \psi \, dx \\
&\leq (1 - \lambda_j) \int_{\mathbb{R}^N} \left| G^{-1}(v_j) \right|^p \psi \, dx \\
&\leq (1 - \lambda_j) \left(\int_{\mathbb{R}^N} \left| G^{-1}(v_j) \right|^{p \times ((p+1)/p)} \right)^{p/(p+1)} \\
&\quad \times \left(\int_{\mathbb{R}^N} \psi^{p+1} \right)^{1/(p+1)},
\end{aligned} \tag{52}
$$

since

$$
\left(\int_{\mathbb{R}^N} \left| G^{-1}(v_j) \right|^{p \times ((p+1)/p)} \right)^{p/(p+1)} \left(\int_{\mathbb{R}^N} \psi^{p+1} \right)^{1/(p+1)} < \infty, \tag{53}
$$

so $J_{\lambda_j}'(v_j)\psi - J'(v_j)\psi \to 0$ as $j \to \infty$; thus we have $J'(v_j) \to 0$ as $j \to \infty$. By knowing that

$$
\begin{aligned}
J(v_j) &= J_{\lambda_j}(v_j) - \frac{(1 - \lambda_j)}{p+1} \\
&\quad \times \int_{\mathbb{R}^N} \left| G^{-1}(v_j) \right|^{p+1} dx,
\end{aligned} \tag{54}
$$

since

$$
\frac{(1 - \lambda_j)}{p+1} \iint_{\mathbb{R}^N} \left| G^{-1}(v_j) \right|^{p+1} dx \longrightarrow 0, \tag{55}
$$

so $\limsup_{j\to\infty} J(v_j) = \limsup_{j\to\infty} J_{\lambda_j}(v_j)$, and we distinguish two cases. Either $\limsup_{j\to\infty} J_{\lambda_j}(v_j) > 0$ or $\limsup_{j\to\infty} J_{\lambda_j}(v_j) \leq 0$. In the first case, we get $\limsup_{j\to\infty} J(v_j) := c > 0$ and the result follows from Lemma 6.

In the second case, we define the sequence $\{z_j\} \subset H^1(\mathbb{R}^N)$ by $z_j = t_j v_j$ with $t_j \in [0,1]$ satisfying

$$
J_{\lambda_j}(z_j) = \max_{t \in [0,1]} J_{\lambda_j}(tv_j) \tag{56}
$$

(if for a $j \in \mathbb{N}$, t_j defined by (56) is not unique, we choose the smaller possible value). By construction $\{z_j\} \subset H^1(\mathbb{R}^N)$ is bounded. Moreover by the definition of (56), we have

$$
\begin{aligned}
0 &= \left. \frac{dJ_{\lambda_j}(tv_j)}{dt} \right|_{t=t_j} \\
&= \frac{1}{t_j} \left[\int_{\mathbb{R}^N} |\nabla z_j|^2 \, dx + \int_{\mathbb{R}^N} \omega \frac{G^{-1}(z_j)}{g(G^{-1}(z_j))} z_j \, dx \right. \\
&\qquad \left. - \lambda_j \frac{|G^{-1}(z_j)|^p}{g(G^{-1}(z_j))} z_j \, dx \right],
\end{aligned}
\tag{57}
$$

so $J'_{\lambda_j}(z_j)z_j = 0$. Then following the proof above, we have $J'(z_j) \to 0$ and $\liminf_{j\to\infty} J(z_j) = \liminf_{j\to\infty} J_{\lambda_j}(z_j)$. On the other hand, by the proof of Lemmas 3 and 2 (6), there exists a constant $C > 0$ such that $J'_{\lambda_j}(v)v \geq C \|v\|^2 + o(\|v\|^2)$ as $v \to 0$, uniformly in $j \in \mathbb{N}$. Thus, since $J'_{\lambda_j}(v_j) = 0$, there is $\alpha > 0$ such that $\|v_j\| \geq \alpha$, for all $j \in \mathbb{N}$. Similarly, following the proof of Lemma 3, we have $J_{\lambda_j}(v) \geq C\|v\|^2 + o(\|v\|^2)$ with $C > 0$ as $v \to 0$. Then recording that $\limsup_{j\to\infty} J_{\lambda_j}(v_j) \leq 0$, we obtain from (56) that

$$
\liminf_{j\to\infty} J(z_j) = \liminf_{j\to\infty} J_{\lambda_j}(z_j) := c' > 0.
\tag{58}
$$

Using Lemma 6 again, we complete the proof of Lemma 8 which implies that $u = G^{-1}(v)$ is a solution for (1). $\square$

Remark 9. In [14], the authors considered the existence of solutions for the following quasilinear Schrödinger equation:

$$
-\operatorname{div}\left(g^2(u)\nabla u\right) + g(u)g'(u)|\nabla u|^2 + V(x)u = h(u),
$$
$$
x \in \mathbb{R}^N,
\tag{59}
$$

where the nonlinearity h is Hölder continuous and satisfies the following conditions:

(h_0) $h(s) = 0$ if $s \leq 0$;

(h_1) $h(s) = o(s)$ as $s \to 0^+$;

(h_2) there exists $2 < p < 2^* := 2N/(N-2)$ such that $|h(s)| \leq C(1 + g(s)|G(s)|^{p-1})$;

(h_3) there exists $\mu > 2$ such that for any $s > 0$, there holds $0 < \mu g(s)H(s) \leq G(s)h(s)$.

If we take $g^2(t) = 1 + \kappa t^2/2(1+t^2)$, $V(x) \equiv \omega$, and $h(u) = |u|^{p-1}u$, (59) turns into (1) with $\kappa = 1$. We point out that the existence result in [14] does not cover our result.

Now, we show that (h_3) is not satisfied for $h(u) = |u|^{p-1}u$ if $2 < p + 1 < 2^*$. In fact, $0 < \mu g(s)H(s) \leq G(s)h(s)$ if and only if

$$
\frac{\mu H(s)}{sh(s)} \leq \frac{G(s)}{sg(s)}.
\tag{60}
$$

By Lemma 2 (5), we have

$$
\frac{G(s)}{sg(s)} \geq \frac{\kappa}{2\kappa + 4 - 2\sqrt{2(2+\kappa)}}.
\tag{61}
$$

Thus, we only need to show

$$
\frac{\mu H(s)}{sh(s)} \leq \frac{\kappa}{2\kappa + 4 - 2\sqrt{2(2+\kappa)}};
\tag{62}
$$

that is

$$
\frac{2\kappa + 4 - 2\sqrt{2(2+\kappa)}}{\kappa}\mu \leq p + 1,
\tag{63}
$$

under the hypothesis $h(s) = s^p$. Then, by (63), we have

$$
\begin{aligned}
2 < 2\left(6 - 2\sqrt{6}\right) < p + 1 < 2^*, \quad \kappa = 1, \\
4 < p + 1 < 2^*, \quad \kappa \longrightarrow \infty.
\end{aligned}
\tag{64}
$$

Remark 10. In [14], (h_3) is used to prove the boundedness of (PS) sequence. In this paper, since (h_3) does not satisfy our condition, we obtain the boundedness of (PS) sequence by using Jeanjean's result [16].

4. Uniqueness

In this section, we study the uniqueness of the positive radial solution of (13). We put

$$
f(s) := \frac{|G^{-1}(s)|^p}{g(G^{-1}(s))} - \omega \frac{G^{-1}(s)}{g(G^{-1}(s))},
$$
$$
\text{for } s \geq 0, \quad K_f(s) := \frac{sf'(s)}{f(s)}.
\tag{65}
$$

We apply the following uniqueness result due to Serrin and Tang [19].

Theorem 11. *Suppose that there exists $b > 0$ such that*

(1) *f is continuous on $(0,\infty)$, $f(s) \leq 0$ on $(0,b]$, and $f(s) > 0$ for $s > b$;*

(2) *$f \in C^1(b,\infty)$ and $K'_f(s) \leq 0$ on (b,∞).*

Then the semilinear problem

$$
-\Delta u = f(u) \text{ in } \mathbb{R}^N, \quad u > 0,
$$
$$
u \longrightarrow 0 \text{ as } |x| \longrightarrow \infty, \quad u(0) = \max u(x),
\tag{66}
$$

has at most one positive radial solution.

Now we can see that f defined in (65) is of the class $C^1[0,\infty)$. Moreover, by the proof of Lemma 2, we have that $G^{-1}(s)$ is increasing and $G^{-1}(0) = 0$; then $G^{-1}(s) \geq 0$.

So $-\omega + |G^{-1}(s)|^{p-1} \geq -\omega$; then there exists a unique b such that $-\omega + |G^{-1}(b)|^{p-1} = 0$, and

$$f(s) = \frac{G^{-1}(s)}{g(G^{-1}(s))}\left(-\omega + |G^{-1}(s)|^{p-1}\right) \leq 0,$$

$$\text{for } s \in (0, b],$$

$$f(s) = \frac{G^{-1}(s)}{g(G^{-1}(s))}\left(-\omega + |G^{-1}(s)|^{p-1}\right) > 0,$$

$$\text{for } s \in (b, +\infty).$$
(67)

So (1) of Theorem 11 holds. From $-\omega + |G^{-1}(b)|^{p-1} = 0$, we can also observe that $b = G(\omega^{1/(p-1)})$. Since $G(s)$ is increasing and $\lim\limits_{s \to \infty}(G(s)/s) = \sqrt{(2 + \kappa)/2}$, this implies that

$$b \longrightarrow \infty \quad \text{iff } \omega^{1/(p-1)} \longrightarrow \infty.$$
(68)

Lemma 12. *Suppose $N \geq 3$ and $p > 4\sqrt{2/(2 + \kappa)} - 2$. Then there exists $c_0 = c_0(p, \kappa) > 0$ such that if $\omega^{1/(p-1)} \geq c_0$, then f satisfies (2) of Theorem 11.*

Proof. We observe that

$$K_f'(s) = \frac{1}{f(s)^2}\left(sf(s)f''(s) + f(s)f'(s) - sf'(s)^2\right).$$
(69)

Thus we have only to show that $sff'' + ff' - sf'^2 < 0$, for $s > b$. Since

$$f(s) = \frac{(G^{-1}(s))^p - \omega G^{-1}(s)}{g(G^{-1}(s))},$$
(70)

so

$$f'(s) = \frac{1}{g^2(G^{-1}(s))}\left[p(G^{-1}(s))^{p-1} - \omega\right] - \frac{g'(G^{-1}(s))}{g^3(G^{-1}(s))}$$

$$\times \left[(G^{-1}(s))^p - \omega G^{-1}(s)\right],$$

$$f''(s) = \frac{1}{g^3(G^{-1}(s))}p(p-1)(G^{-1}(s))^{p-2}$$

$$- \frac{3g'(G^{-1}(s))}{g^4(G^{-1}(s))}\left[p(G^{-1}(s))^{p-1} - \omega\right]$$

$$- \left[\frac{g''(G^{-1}(s))}{g^4(G^{-1}(s))} - \frac{3(g'(G^{-1}(s)))^2}{g^5(G^{-1}(s))}\right]$$

$$\times \left[(G^{-1}(s))^p - \omega G^{-1}(s)\right].$$
(71)

Then by complicated computations, we have

$$sf(s)f''(s) + f(s)f'(s) - s(f'(s))^2$$

$$= \frac{1}{g^3(G^{-1}(s))}\left[(G^{-1}(s))^p - \omega G^{-1}(s)\right]$$

$$\times \left(\frac{p(p-1)s(G^{-1}(s))^{p-2}}{g(G^{-1}(s))} + \left[p(G^{-1}(s))^{p-1} - \omega\right]\right.$$

$$- \frac{s}{g(G^{-1}(s))}\frac{\left[p(G^{-1}(s))^{p-1} - \omega\right]^2}{(G^{-1}(s))^p - \omega G^{-1}(s)}$$

$$- \frac{sg''(G^{-1}(s))}{g^2(G^{-1}(s))}\left[(G^{-1}(s))^p - \omega G^{-1}(s)\right]$$

$$- \frac{sg'(G^{-1}(s))}{g^2(G^{-1}(s))}\left[p(G^{-1}(s))^{p-1} - \omega\right]$$

$$- \frac{g'(G^{-1}(s))}{g(G^{-1}(s))}\left[(G^{-1}(s))^p - \omega G^{-1}(s)\right]$$

$$\left.+ \frac{2s(g'(G^{-1}(s)))^2}{g^3(G^{-1}(s))}\left[(G^{-1}(s))^p - \omega G^{-1}(s)\right]\right)$$

$$= \frac{1}{g^3(G^{-1}(s))}\left[(G^{-1}(s))^p - \omega G^{-1}(s)\right](H_1(s) + H_2(s)),$$
(72)

where

$$H_1(s) := \frac{p(p-1)s(G^{-1}(s))^{p-2}}{g(G^{-1}(s))} + \left[p(G^{-1}(s))^{p-1} - \omega\right]$$

$$- \frac{s}{g(G^{-1}(s))}\frac{\left[p(G^{-1}(s))^{p-1} - \omega\right]^2}{(G^{-1}(s))^p - \omega G^{-1}(s)},$$

$$H_2(s) := - \frac{sg''(G^{-1}(s))}{g^2(G^{-1}(s))}\left[(G^{-1}(s))^p - \omega G^{-1}(s)\right]$$

$$- \frac{sg'(G^{-1}(s))}{g^2(G^{-1}(s))}\left[p(G^{-1}(s))^{p-1} - \omega\right]$$

$$- \frac{g'(G^{-1}(s))}{g(G^{-1}(s))}\left[(G^{-1}(s))^p - \omega G^{-1}(s)\right]$$

$$+ \frac{2s(g'(G^{-1}(s)))^2}{g^3(G^{-1}(s))}\left[(G^{-1}(s))^p - \omega G^{-1}(s)\right].$$
(73)

For $s > b$, it follows that $(1/g^3(G^{-1}(s)))[(G^{-1}(s))^p - \omega G^{-1}(s)] > 0$. Thus it suffices to show that $H_1(s) + H_2(s) < 0$, for $s > b$, in order to prove that $sf(s)f''(s) + f(s)f'(s) - s(f'(s))^2 < 0$.

By (4) of Lemma 2 and $\lim_{t\to\infty} g(t) = \sqrt{(2+\kappa)/2}$, we have

$$
\begin{aligned}
\lim_{s\to\infty} \frac{H_1(s)}{s^{p-1}} &= \lim_{s\to\infty} \frac{p(p-1)}{g(G^{-1}(s))} \frac{\left(G^{-1}(s)\right)^{p-2}}{s^{p-2}} \\
&\quad + \lim_{s\to\infty} p \frac{\left(G^{-1}(s)\right)^{p-1} - \omega}{s^{p-1}} \\
&\quad - \lim_{s\to\infty} \frac{1}{g(G^{-1}(s))} \\
&\quad \times \frac{\left[p\left(G^{-1}(s)\right)^{p-1} - \omega\right]^2}{s^{p-2}\left[\left(G^{-1}(s)\right)^p - \omega G^{-1}(s)\right]} \\
&= \left(\sqrt{\frac{2}{2+\kappa}}\right)^{p-1}\left(p(p-1) + p - p^2\right) = 0.
\end{aligned}
\tag{74}
$$

Thus, for sufficiently large b, we obtain $H_1(s) + H_2(s) < 0$ if and only if $H_2(s) < 0$ for $s > b$.

Next, we investigate the sign of $H_2(s)$. Firstly, we express g'' in terms of g and g', and since $g(t) = \sqrt{1 + \kappa t^2/2(1 + t^2)}$, so $g'(t) = (1/2g(t))(\kappa t/(1 + t^2)^2)$ and

$$
\begin{aligned}
g''(t) &= -\frac{g'(t)}{2g^2(t)} \frac{\kappa t}{(1+t^2)^2} + \frac{1}{2g(t)} \frac{\kappa}{(1+t^2)^2} \\
&\quad - \frac{2t}{1+t^2} \frac{1}{g(t)} \frac{\kappa t}{(1+t^2)^2} \\
&= -\frac{\left(g'(t)\right)^2}{g(t)} + \frac{g'(t)}{t} - \frac{4tg'(t)}{1+t^2}.
\end{aligned}
\tag{75}
$$

Thus we obtain

$$
\begin{aligned}
H_2(s) &= \left[\frac{3s\left(g'\left(G^{-1}(s)\right)\right)^2}{g^3\left(G^{-1}(s)\right)} - \frac{sg'\left(G^{-1}(s)\right)}{G^{-1}(s)\,g^2\left(G^{-1}(s)\right)}\right. \\
&\quad \left. + \frac{4sG^{-1}(s)\,g'\left(G^{-1}(s)\right)}{\left[1 + \left(G^{-1}(s)\right)^2\right] g^3\left(G^{-1}(s)\right)}\right] \\
&\quad \times \left[\left(G^{-1}(s)\right)^p - \omega G^{-1}(s)\right] \\
&\quad - \frac{sg'\left(G^{-1}(s)\right)}{g^2\left(G^{-1}(s)\right)}\left[p\left(G^{-1}(s)\right)^{p-1} - \omega\right] \\
&\quad - \frac{g'\left(G^{-1}(s)\right)}{g\left(G^{-1}(s)\right)}\left[\left(G^{-1}(s)\right)^p - \omega G^{-1}(s)\right].
\end{aligned}
\tag{76}
$$

We note that

$$
\begin{aligned}
\lim_{s\to\infty} \frac{\left(G^{-1}(s)\right)^p - \omega G^{-1}(s)}{s^p} &= \left(\sqrt{\frac{2}{2+\kappa}}\right)^p, \\
\lim_{s\to\infty} \frac{p\left(G^{-1}(s)\right)^{p-1} - \omega}{s^{p-1}} &= p\left(\sqrt{\frac{2}{2+\kappa}}\right)^{p-1},
\end{aligned}
\tag{77}
$$

so

$$
\begin{aligned}
\lim_{s\to\infty} &\frac{H_2(s)}{s^{p-3}} \\
&= \left(\sqrt{\frac{2}{2+\kappa}}\right)^p \left[\lim_{s\to\infty} \frac{3s^4\left(g'\left(G^{-1}(s)\right)\right)^2}{g^3\left(G^{-1}(s)\right)}\right. \\
&\quad - \lim_{s\to\infty} \frac{s^4 g'\left(G^{-1}(s)\right)}{G^{-1}(s)\,g^2\left(G^{-1}(s)\right)} \\
&\quad + \lim_{s\to\infty} \frac{4s^4 G^{-1}(s)\,g'\left(G^{-1}(s)\right)}{\left[1 + \left(G^{-1}(s)\right)^2\right] g^3\left(G^{-1}(s)\right)} \\
&\quad \left. - \lim_{s\to\infty} \frac{s^3 g'\left(G^{-1}(s)\right)}{g\left(G^{-1}(s)\right)}\right] \\
&\quad - p\left(\sqrt{\frac{2}{2+\kappa}}\right)^{p-1} \lim_{s\to\infty} \frac{s^3 g'\left(G^{-1}(s)\right)}{g^2\left(G^{-1}(s)\right)}.
\end{aligned}
\tag{78}
$$

Moreover, we have $G^{-1}(s) \sim \sqrt{2/(2+\kappa)}\,s$, $g(G^{-1}(s)) \sim \sqrt{(2+\kappa)/2}$, and $g'(G^{-1}(s)) \sim (\kappa(2+\kappa)/4)s^{-3}$ as $s \to \infty$. Then, from $p > 4\sqrt{2/(2+\kappa)} - 2$, we have

$$
\begin{aligned}
\lim_{s\to\infty} \frac{H_2(s)}{s^{p-3}} &= \left(\sqrt{\frac{2}{2+\kappa}}\right)^p \\
&\quad \times \left[-\frac{\kappa}{2}\sqrt{\frac{2+\kappa}{2}} + 2\kappa - \frac{\kappa(\kappa+2)}{4}\sqrt{\frac{2}{2+\kappa}}\right] \\
&\quad - p\left(\sqrt{\frac{2}{2+\kappa}}\right)^{p-1} \frac{\kappa}{2} \\
&= \left(\sqrt{\frac{2}{2+\kappa}}\right)^{p-1} \kappa\left[2\sqrt{\frac{2}{2+\kappa}} - 1 - \frac{p}{2}\right] < 0,
\end{aligned}
\tag{79}
$$

so $H_2(s) < 0$ for $s > b$ if b is sufficiently large; that is, $H_1(s) + H_2(s) < 0$ for $s > b$. From (68), there exists $c_0 = c_0(p, \kappa) > 0$ such that if $\omega^{1/(p-1)} \geq c_0$, then we obtain $sf(s)f''(s) + f(s)f'(s) - s(f'(s))^2 < 0$ for $s > b = G(\omega^{1/(p-1)})$. $\quad\square$

By Lemma 8, we can apply Theorem 11, Hence we obtain the uniqueness of positive radial solutions of (13).

5. Conclusion

By the discussion of Section 3, we have a nontrivial solution of (1). Then using the result of Gidas et al. [20], we know that the nontrivial solution is a positive radial solution with $u \to 0$ as $|x| \to \infty$ and $u(0) = \max u(x)$. Combined with the discussion of Section 4, we complete the proof of Theorem 1. That is, if $N \geq 3$, $\kappa > 0$, $\omega > 0$, and $\max\{4\sqrt{2/(2+\kappa)} - 1, 2\} < p + 1 < 2^*$, there exists $c_0(p, \kappa) > 0$ such that if $\omega^{1/(p-1)} \geq c_0(p, \kappa)$, then the positive solution of (1) is unique.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

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