*Research Article*

# Nontrivial Periodic Solutions to Some Semilinear Sixth-Order Difference Equations

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We establish some new criteria to guarantee nonexistence, existence, and multiplicity of nontrivial periodic solutions of some semilinear sixth-order difference equations by using minmax method, $Z_2$ index theory, and variational technique. Our results only make some assumptions on the period $T$, which are very easy to verify and rather relaxed.

## 1. Introduction

In the present paper, we deal with the following sixth-order difference equation:

$$\Delta^6 u_{t-3} + A\Delta^4 u_{t-2} + B\Delta^2 u_{t-1} + u_t - u_t^3 = 0, \quad t \in \mathbf{Z} \quad (1)$$

with $u_0 = u_T = 0$, where $T \geq 2$ is an integer and $[1, T]$ denotes the discrete interval $\{1, 2, \ldots, T\}$. $\Delta$ is the forward difference operator defined by $\Delta u_t = u_{t+1} - u_t$ and $\Delta^n u_{t-k} = \Delta^{n-1}(\Delta u_{t-k})$. $A$ and $B$ are positive constants satisfying $A^2 < 4B$.

By using $Z_2$ index theory in combination with variational technique, we will prove nonexistence, existence, and multiplicity of nontrivial periodic solutions to (1) under convenient assumptions on $T$. All our results only depend on $A$ and $B$ and are easy to satisfy.

Periodic solution problems for difference equations have been extensively studied (see the monographs of Lakshmikantham and Trigiante [1] and of Agarwal [2]). The classical theory of difference equations employs numerical analysis and features from the linear and nonlinear operator theory, such as fixed point methods; we remark that, usually, the applications of fixed point methods yield existence results only. Recently, although many new results have been established by applying variational methods, we recall here the works of Cai and Yu [3], Guo and Yu [4], and Deng et al. [5].

The variational approach represents an important advance as it allows proving multiplicity results as well.

In general, (1) may be regarded as a discrete analogue of the following sixth-order differential equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^6 u}{\partial x^6} + A\frac{\partial^4 u}{\partial x^4} + B\frac{\partial^2 u}{\partial x^2} + u - u^3. \quad (2)$$

As to (2), it is a model for describing the behavior of phase fronts in materials that are undergoing a transition between the liquid and solid state and be widely studied; one can refer to [6–8] and references therein.

Difference equations, the discrete analogs of differential equations, represent the discrete counterpart of ordinary differential equations and are usually studied in connection with numerical analysis. They occur widely in numerous settings and forms, both in mathematics itself and in its applications to computing, statistics, electrical circuit analysis, biology, dynamical systems, economics, and other fields. For the general background of difference equations, one can refer to monographs [2, 9–13] for details.

Since 2003, critical point theory has been a powerful tool to establish sufficient conditions on the existence of periodic solutions of difference equations and many significant results have been obtained; see, for example, [4, 14, 15]. Compared to first-order or second-order difference equations, the study of

higher order difference equations has received considerably less attention. For example, [16] studied

$$\sum_{i=0}^{n} \Delta^i \left( r_i \left( k - i \right) \Delta^i u \left( k - i \right) \right) = 0 \tag{3}$$

in the context of discrete calculus of variational functional, and Peil and Peterson [17] studied the asymptotic behaviour of solutions of (3) with $r_i(k) \equiv 0$ for $1 \leq i \leq n - 1$. In 2007, based on Linking Theorem, [3] gave some criteria for the existence of periodic solutions of

$$\Delta^n \left( r_{t-n} \right) \Delta^n u_{t-n} + f \left( t, u_t \right) = 0,$$
$$n \in [1, 3], \quad n \in \mathbf{Z} \tag{4}$$

for the case where $f$ grows superlinear at both 0 and $\infty$. Results in [3] made many assumptions on $f$ and they are not easy to verify. The aim of this paper is to apply critical point theory to deal with the periodic solution problems of (1) when it is semilinear under concise and explicit assumptions on the period $T$. The main results of this paper are the following three theorems.

**Theorem 1.** *Let $A > 0$, $A^2 < 4B$, and integer $T$ satisfy*

$$0 < T < T_1, \tag{5}$$

*where $T_1$ is defined as*

$$\sin \frac{\pi}{T_1} = \max \left\{ \left( \frac{1}{4B - A^2} \right)^{1/2}, \frac{1}{2} \left( \frac{4B}{4B - A^2} \right)^{1/6} \right\} \tag{6}$$

*then (1) has only the trivial solution.*

**Theorem 2.** *Let $A > 0$, $A^2 < 4B$, and integer $T$ satisfy*

$$T > T_2, \tag{7}$$

*where $T_2$ is defined as*

$$\sin \frac{\pi}{2T_2} = \frac{1}{2} \left( \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{B^3}{27}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{B^3}{27}}} \right)^{1/2}; \tag{8}$$

*there exists a nontrivial solution of (1).*

**Theorem 3.** *Let $A > 0$, $A^2 < 4B$, and*

$$T > mT_2 \tag{9}$$

*for some $m \in \mathbf{N}$; there exist at least $m$ geometrically distinct nontrivial solutions of (1).*

The remaining of the paper is organized as follows. In Section 2 we establish the variational framework associated with (1) and transfer the problem on the existence of periodic solutions of (1) into the existence of critical points of the corresponding functional. We also state some fundamental lemmas for later use. Then we present the detailed proofs of main results in Section 3. Finally, we exhibit a simple example to illustrate our conclusions.

## 2. Preliminaries

In order to study the periodic solutions of (1), we will state some basic notations and lemmas, which will be used in the proofs of our main results. Let

$$S = \left\{ u = \{u_t\} \mid u_t \in \mathbf{R}^N, \ t \in \mathbf{Z} \right\}. \tag{10}$$

For a given integer $T \geq 2$, $E_T$ is defined as a subspace of $S$ by

$$E_T = \left\{ u = \{u_t\} \in S \mid u_{t+T} = u_t, \ t \in \mathbf{Z} \right\}, \tag{11}$$

and for $u, v \in E_T$, let

$$\langle u, v \rangle = \sum_{t=1}^{T} \left( \Delta^3 u_{t-1} \Delta^3 v_{t-1} + \Delta^2 u_{t-1} \Delta^2 v_{t-1} \right.$$
$$\left. + \Delta u_{t-1} \Delta v_{t-1} + u_t v_t \right). \tag{12}$$

Then $E_T$ is a finite dimensional Hilbert space with above inner product, and the induced norm is

$$\|u\| = \left( \sum_{t=1}^{T} \left( \left( \Delta^3 u_{t-1} \right)^2 + \left( \Delta^2 u_{t-1} \right)^2 + \left( \Delta u_{t-1} \right)^2 + u_t^2 \right) \right)^{1/2}$$
$$\forall u \in E_T. \tag{13}$$

As usual, for $1 \leq p < +\infty$, let

$$l^p \left( \mathbf{Z}, \mathbf{R}^N \right) = \left\{ u \in S : \sum_{t \in \mathbf{Z}} |u_t|^p < +\infty \right\} \tag{14}$$

and its norm is defined by

$$\|u\|_{l^p} = \left( \sum_{t=1}^{T} |u_t|^p \right)^{1/p}, \quad \forall u \in l^p \left( \mathbf{Z}, \mathbf{R}^N \right). \tag{15}$$

Define the functional $I : E_T \to \mathbf{R}$ as follows:

$$I(u; T) = \sum_{t=1}^{T} \left[ \frac{1}{2} \left( \left( \Delta^3 u_{t-1} \right)^2 - A \left( \Delta^2 u_{t-1} \right)^2 \right. \right.$$
$$\left. \left. + B \left( \Delta u_{t-1} \right)^2 - u_t^2 \right) + \frac{1}{4} u_t^4 \right]. \tag{16}$$

Clearly, $I(u; T) \in \mathbf{C}^1(E_T, \mathbf{R})$ and for any $u, v \in E_T$ one can easily check that

$$\left\langle I'(u; T), v \right\rangle = \sum_{t=1}^{T} \left( \Delta^3 u_{t-1} \Delta^3 v_{t-1} - A \Delta^2 u_{t-1} \Delta^2 v_{t-1} \right.$$
$$\left. + B \Delta u_{t-1} \Delta v_{t-1} - u_t v_t + u_t^3 v_t \right). \tag{17}$$

For any $u = \{u_t\}_{t \in \mathbf{Z}} \in E_T$, by using $u_i = u_{T+i}$ for any $i \in \mathbf{Z}$,

$$\Delta^n u_{t-1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u_{t+n-k-1}. \tag{18}$$

We can compute the partial derivative as

$$\frac{\partial I}{\partial u_t} = \Delta^6 u_{t-3} + A\Delta^4 u_{t-2} + B\Delta^2 u_{t-1} + u_t - u_t^3, \quad t \in [1, T].$$
(19)

Then, $u = \{u_t\}_{t \in \mathbf{Z}}$ is a critical point of $I(u; T)$ on $E_T$; that is, $I'(u; T) = 0$ if and only if

$$\Delta^6 u_{t-3} + A\Delta^4 u_{t-2} + B\Delta^2 u_{t-1} + u_t - u_t^3 = 0, \quad t \in [1, T].$$
(20)

By the periodicity of $u_t$, we have reduced the existence of periodic solutions of (1) to the existence of critical points of $I(u; T)$ on $E_T$. For convenience, we identity $u \in E_T$ with $u = (u_1, u_2, \ldots, u_T)^T$, so we draw a conclusion as follows.

**Lemma 4.** *Suppose that $u_t$ is a critical point of the functional $I(u; T)$; then $u_t$ is a T-periodic solution of* (1).

We provide some lemmas which will be needed in proofs of our main results.

**Lemma 5.** *For any $x(j) > 0$, $y(j) > 0$, $j \in [1, n]$, and $n \in \mathbf{Z}$,*

$$\sum_{j=1}^{n} x(j) y(j) \le \left( \sum_{j=1}^{n} x^r(j) \right)^{1/r} \cdot \left( \sum_{j=1}^{n} y^s(j) \right)^{1/s},$$
(21)

*where $r > 1$, $s > 1$, and $(1/r) + (1/s) = 1$.*

**Lemma 6.** *Let $u \in E_T$ be a critical point of $I(u; T)$; for every $v \in E_T$, there hold*

$$\sum_{t=1}^{T} \Delta u_{t-1} \Delta v_{t-1} = -\sum_{t=1}^{T} \Delta^2 u_{t-1} v_t,$$

$$\sum_{t=1}^{T} \Delta^2 u_{t-1} \Delta^2 v_{t-1} = -\sum_{t=1}^{T} \Delta u_{t-1} \Delta^3 v_{t-1}.$$
(22)

*Proof.* Let $u \in E_T$ be a critical point of $I(u; T)$, according to the definition of $\Delta$ and the periodicity of $u_t$ and $v_t$; then we have

$$\sum_{t=1}^{T} \Delta u_{t-1} \Delta v_{t-1} = \sum_{t=1}^{T} (u_t - u_{t-1})(v_t - v_{t-1})$$

$$= \sum_{t=1}^{T} (u_t v_t - u_{t-1} v_{t-1}) - \sum_{t=1}^{T} (u_t v_{t-1} - u_{t-1} v_{t-1})$$

$$= \sum_{t=1}^{T} \Delta u_{t-1} v_t - \sum_{t=1}^{T} (u_{t+1} v_t - u_t v_t)$$

$$= \sum_{t=1}^{T} (\Delta u_{t-1} - \Delta u_t) v_t$$

$$= -\sum_{t=1}^{T} \Delta^2 u_{t-1} v_t.$$
(23)

Similarly, we get

$$\sum_{t=1}^{T} \Delta^2 u_{t-1} \Delta^2 v_{t-1} = -\sum_{t=1}^{T} \Delta u_{t-1} \Delta^3 v_{t-1}.$$
(24)

Let $X$ be a real Banach space. $\Sigma$ is the subset of $X$, which is closed and symmetric with respect to 0; that is,

$$\Sigma = \{A \subset X \mid A \text{ is closed and } x \in A \text{ iff } -x \in A\}.$$
(25)

For any $A \in \Sigma$, the $Z_2$ geometric index, also called genus, $\gamma$ of $A$ is defined by

$$\gamma(A)$$

$$= \min \left\{ k \in \mathbf{N} \mid \text{there is an odd map } \varphi \in \mathbf{C}\left(X, \mathbf{R}^k \{0\}\right) \right\},$$
(26)

when there exists no such finite $k$, set $\gamma(A) = \infty$. Finally set $\gamma(\emptyset) = 0$. □

The following lemma is trivial.

**Lemma 7** (Chang [18] and Rabinowitz [19]). *Let $X_k$ be a subset of $X$ with dimension $k$ and $S_1$ the unit sphere of $X$ and then let $\gamma(X_k \cap S_1) = k$.*

Next, let us recall the definition of Palais-Smale condition.

Let $X$ be a real Banach space, $\Phi \in \mathbf{C}(X, \mathbf{R})$. $\Phi$ is a continuously Frechet differentiable functional defined on $X$. $\Phi$ is said to be satisfied Palais-Smale condition (P.S. condition for short) if any sequence $\{u_n\} \subset X$ for which $\{\Phi(u_n)\}$ is bounded and $\Phi'(u_n) \to 0$ $(n \to \infty)$ possesses a convergent subsequence in $X$.

**Lemma 8** (see [20]). *Let $E$ be a real Banach space and let $f : E \to \mathbf{R}$ be a $\mathbf{C}^1$ functional and satisfy P.S. condition. If $f$ is bound from below, then*

$$c = \inf_{u \in E} f(u)$$
(27)

*is a critical value of $f$.*

**Lemma 9** (see [19]). *Let $\Phi$ be an even functional defined on $X$ and satisfy P.S. condition. For positive integer $j$, define*

$$c_j = \sup_{\substack{\gamma(A) \ge j \\ A \in \Sigma}} \inf_{u \in A} \Phi(u).$$
(28)

*Then*

$$-\infty \le \cdots \le c_k \le c_{k-1} \le \cdots \le c_2 \le c_1 \le +\infty$$
(29)

*and*

(1) *assume $-\infty < c_j < +\infty$; then $c_j$ is a critical value of $\Phi$;*

(2) *if $c = c_j = c_{j+1} = \cdots = c_{j+l-1}$, then $\gamma(K_c) \ge l$, where $K_c = \{u \in X \mid \Phi(u) = c, \Phi'(u) = 0\}$.*

## 3. Proofs of Main Results

With the above preparations, we will prove our main results in this section. In order to give proofs of theorems, we need the following lemmas.

**Lemma 10.** *For any $u \in E_T$, if $u_t$ is a critical of $I(u; T)$, we have*

$$\sum_{t=1}^{T} u_t^2 \le \frac{1}{4\sin^2(\pi/T)} \sum_{t=1}^{T} (\Delta u_t)^2, \tag{30}$$

$$\sum_{t=1}^{T} u_t^2 \le \frac{1}{16\sin^4(\pi/T)} \sum_{t=1}^{T} (\Delta^2 u_{t-1})^2, \tag{31}$$

$$\sum_{t=1}^{T} u_t^2 \le \frac{1}{64\sin^6(\pi/T)} \sum_{t=1}^{T} (\Delta^3 u_{t-1})^2. \tag{32}$$

*Proof.* From Agarwal [2], we have inequality (30) and

$$\sum_{t=1}^{T} (\Delta u_{t-1})^2 \le \frac{1}{4\sin^2(\pi/T)} \sum_{t=1}^{T} (\Delta^2 u_{t-1})^2; \tag{33}$$

then

$$\sum_{t=1}^{T} u_t^2 \le \frac{1}{4\sin^2(\pi/T)} \cdot \frac{1}{4\sin^2(\pi/T)} \sum_{t=1}^{T} (\Delta^2 u_{t-1})^2$$

$$= \frac{1}{16\sin^4(\pi/T)} \sum_{t=1}^{T} (\Delta^2 u_{t-1})^2 \tag{34}$$

that is inequality (31).

From Lemmas 5 and 6, there holds

$$\sum_{t=1}^{T} (\Delta^2 u_{t-1})^2$$

$$= -\sum_{t=1}^{T} \Delta u_{t-1} \cdot \Delta^3 u_{t-1}$$

$$\le \left( \sum_{t=1}^{T} (\Delta u_{t-1})^2 \right)^{1/2} \cdot \left( \sum_{t=1}^{T} (\Delta^3 u_{t-1})^2 \right)^{1/2}$$

$$= \left( -\sum_{t=1}^{T} u_t \Delta^2 u_{t-1} \right)^{1/2} \cdot \left( \sum_{t=1}^{T} (\Delta^3 u_{t-1})^2 \right)^{1/2}$$

$$\le \left( \sum_{t=1}^{T} u_t^2 \right)^{1/4} \cdot \left( \sum_{t=1}^{T} (\Delta^2 u_{t-1})^2 \right)^{1/4} \cdot \left( \sum_{t=1}^{T} (\Delta^3 u_{t-1})^2 \right)^{1/2} \tag{35}$$

and it follows

$$\sum_{t=1}^{T} (\Delta^2 u_{t-1})^2 \le \left( \sum_{t=1}^{T} u_t^2 \right)^{1/3} \cdot \left( \sum_{t=1}^{T} (\Delta^3 u_{t-1})^2 \right)^{2/3}. \tag{36}$$

Using inequality (31), we have inequality (32) is true. □

**Lemma 11.** *Suppose $A > 0$ and $A^2 < 4B$, $u \in E_T$ is a critical point of $I(u; T)$, and then*

$$A\sum_{t=1}^{T} (\Delta^2 u_{t-1})^2 \le \frac{A^2}{4} \sum_{t=1}^{T} (\Delta u_{t-1})^2 + \sum_{t=1}^{T} (\Delta^3 u_{t-1})^2, \tag{37}$$

$$A\sum_{t=1}^{T} (\Delta^2 u_{t-1})^2 \le B\sum_{t=1}^{T} (\Delta u_{t-1})^2 + \frac{A^2}{4B} \sum_{t=1}^{T} (\Delta^3 u_{t-1})^2. \tag{38}$$

*Proof.* From Lemmas 5 and 6, we have

$$A\sum_{t=1}^{T} (\Delta^2 u_{t-1})^2$$

$$= -A\sum_{t=1}^{T} \Delta u_{t-1} \Delta^3 u_{t-1}$$

$$= -\sum_{t=1}^{T} \frac{\sqrt{2}}{2} A\Delta u_{t-1} \cdot \sqrt{2}\Delta^3 u_{t-1}$$

$$\le \left( \sum_{t=1}^{T} \left( \frac{\sqrt{2}}{2} A\Delta u_{t-1} \right)^2 \right)^{1/2} \cdot \left( \sum_{t=1}^{T} 2(\Delta^3 u_{t-1})^2 \right)^{1/2} \tag{39}$$

$$\le \frac{1}{2} \left( \sum_{t=1}^{T} \frac{A^2}{2} (\Delta u_{t-1})^2 + \sum_{t=1}^{T} 2(\Delta^3 u_{t-1})^2 \right)$$

$$= \frac{A^2}{4} \sum_{t=1}^{T} (\Delta u_{t-1})^2 + \sum_{t=1}^{T} (\Delta^3 u_{t-1})^2,$$

so inequality (37) holds.

To prove inequality (38), let

$$A = \frac{A}{\sqrt{2B}} \cdot \sqrt{2B}. \tag{40}$$

Similar to the proof of inequality (37), we get inequality (38). □

Now we will give the proof of Theorem 1.

*Proof of Theorem 1.* Suppose all conditions of Theorem 1 hold. Let $u \in E_T$ be a nontrivial solution of (1); then

$$\sum_{t=1}^{T} \left( (\Delta^3 u_{t-1})^2 - A(\Delta^2 u_{t-1})^2 + B(\Delta u_{t-1})^2 - u_t^2 + u_t^4 \right) = 0. \tag{41}$$

By (37), (30), (5), and (6), we have

$$0 > -\sum_{t=1}^{T} u_t^4$$

$$= \sum_{t=1}^{T} \left( (\Delta^3 u_{t-1})^2 - A(\Delta^2 u_{t-1})^2 + B(\Delta u_{t-1})^2 - u_t^2 \right)$$

$$\ge \sum_{t=1}^{T} \left( (\Delta^3 u_{t-1})^2 - \frac{A^2}{4} (\Delta^2 u_{t-1})^2 \right.$$

$$\left. - (\Delta^3 u_{t-1})^2 + B(\Delta u_{t-1})^2 - u_t^2 \right)$$

$$= \sum_{t=1}^{T} \left( B - \frac{A^2}{4} \right) (\Delta u_{t-1})^2 - \sum_{t=1}^{T} u_t^2$$

$$\geq \left( \left( B - \frac{A^2}{4} \right) \cdot 4\sin^2 \frac{\pi}{T} - 1 \right) \sum_{t=1}^{T} u_t^2$$

$$\geq 0$$

(42)

which is a contradiction. Moreover, by (38), (31), (5), and (6), we have

$$0 > -\sum_{t=1}^{T} u_t^4$$

$$= \sum_{t=1}^{T} \left( (\Delta^3 u_{t-1})^2 - A(\Delta^2 u_{t-1})^2 + B(\Delta u_{t-1})^2 - u_t^2 \right)$$

$$\geq \sum_{t=1}^{T} \left( (\Delta^3 u_{t-1})^2 - B(\Delta^2 u_{t-1})^2 - \frac{A^2}{4B}(\Delta^3 u_{t-1})^2 \right.$$

$$\left. + B(\Delta u_{t-1})^2 - u_t^2 \right)$$

(43)

$$= \sum_{t=1}^{T} \left( \left( 1 - \frac{A^2}{4B} \right) (\Delta^3 u_{t-1})^2 - u_t^2 \right)$$

$$\geq \left( \left( 1 - \frac{A^2}{4B} \right) \cdot 64\sin^6 \frac{\pi}{T} - 1 \right) \sum_{t=1}^{T} u_t^2$$

$$\geq 0$$

which is a contradiction. Thus $u_t$ should be the trivial solution of (1); in other words, (1) has no nontrivial solution when (5) and (6) hold. □

To apply Lemmas 8 and 9 to look for nontrivial solutions for (1), next we prove that $I(u;T)$ satisfies P.S. condition.

**Lemma 12.** *Let $A > 0$ and $A^2 < 4B$, and then the functional $I(u;T)$ is bounded from below on $E_T$ and satisfies P.S. condition.*

*Proof.* Denote $\epsilon = 1 - (A^2/4B)$, and then $\epsilon > 0$. From (38) and the elementary inequality

$$-\frac{1}{2}u^2 + \frac{1}{4}u^4 \geq -\frac{1}{4}$$

(44)

there hold

$$I(u;T)$$

$$= \sum_{t=1}^{T} \left[ \frac{1}{2} \left( (\Delta^3 u_{t-1})^2 - A(\Delta^2 u_{t-1})^2 + B(\Delta u_{t-1})^2 - u_t^2 \right) \right.$$

$$\left. + \frac{1}{4}u_t^4 \right]$$

$$= \sum_{t=1}^{T} \frac{1}{2} \left( (\Delta^3 u_{t-1})^2 - A(\Delta^2 u_{t-1})^2 + B(\Delta u_{t-1})^2 \right)$$

$$+ \sum_{t=1}^{T} \left( \frac{1}{2}u_t^2 + \frac{1}{4}u_t^4 \right)$$

$$\geq \sum_{t=1}^{T} \frac{\epsilon}{2} (\Delta^3 u_{t-1})^2 - \sum_{t=1}^{T} \frac{1}{4}$$

$$\geq -\frac{T}{4}$$

(45)

which means that the functional $I(u;T)$ is bounded from below on $E_T$.

Next we will show that $I(u;T)$ satisfies P.S. condition. Suppose $\{u^{(k)}\} \subset E_T$ satisfy that $\{I(u^{(k)};T)\}$ is a bounded sequence from above; that is, there exists a positive constant $C$ such that

$$I\left( u^{(k)};T \right) \leq C, \quad \forall k \in \mathbf{N}.$$

(46)

By (38), we have

$$\frac{\epsilon}{2} \sum_{t=1}^{T} (\Delta^3 u_{t-1}^{(k)})^2$$

$$\leq \sum_{t=1}^{T} \left( \frac{\epsilon}{2} (\Delta^3 u_{t-1}^{(k)})^2 + \frac{1}{4} \left( (u_t^{(k)})^2 - 1 \right)^2 \right)$$

$$= \sum_{t=1}^{T} \left( \frac{1}{2} \left( 1 - \frac{A^2}{4B} \right) (\Delta^3 u_{t-1}^{(k)})^2 + \frac{1}{4} (u_t^{(k)})^4 \right.$$

$$\left. - \frac{1}{2} (u_t^{(k)})^2 + \frac{1}{4} \right)$$

$$\leq \sum_{t=1}^{T} \left[ \frac{1}{2} \left( (\Delta^3 u_{t-1}^{(k)})^2 - A(\Delta^2 u_{t-1}^{(k)})^2 \right. \right.$$

$$\left. + B(\Delta u_{t-1}^{(k)})^2 - (u_t^{(k)})^2 \right)$$

$$\left. + \frac{1}{4} (u_t^{(k)})^4 + \frac{1}{4} \right]$$

$$= I\left( u^{(k)};T \right) + \frac{T}{4}$$

$$\leq C + \frac{T}{4}$$

(47)

and it follows

$$\sum_{t=1}^{T} (\Delta^3 u_{t-1}^{(k)})^2 \leq \frac{2}{\epsilon} \left( C + \frac{T}{4} \right).$$

(48)

And together with (32), there holds

$$\sum_{t=1}^{T} (u_t^{(k)})^2 \leq \frac{2}{64\epsilon\sin^6 (\pi/T)} \left( C + \frac{T}{4} \right).$$

(49)

From (37), it follows

$$\frac{\epsilon}{2}\sum_{t=1}^{T}\left(\Delta^2 u_{t-1}^{(k)}\right)^2$$

$$\leq \sum_{t=1}^{T}\left(\frac{\epsilon}{2}\left(\Delta^2 u_{t-1}^{(k)}\right)^2 + \frac{1}{4}\left(\left(u_t^{(k)}\right)^2 - 1\right)^2\right)$$

$$\leq \sum_{t=1}^{T}\left[\frac{1}{2}\left(\left(\Delta^2 u_{t-1}^{(k)}\right)^2 - A\left(\Delta^2 u_{t-1}^{(k)}\right)^2\right.\right.$$

$$\left.+ B\left(\Delta u_{t-1}^{(k)}\right)^2 - \left(u_t^{(k)}\right)^2\right)$$

$$\left.+ \frac{1}{4}\left(u_t^{(k)}\right)^4 + \frac{1}{4}\right] \tag{50}$$

$$= I\left(u^{(k)};T\right) + \frac{T}{4}$$

$$\leq C + \frac{T}{4}$$

and then

$$\sum_{t=1}^{T}\left(\Delta^2 u_{t-1}^{(k)}\right)^2 \leq \frac{8}{4B - A^2}\left(C + \frac{T}{4}\right); \tag{51}$$

thus

$$\sum_{t=1}^{T}\left(\Delta u_{t-1}^{(k)}\right)^2 \leq \frac{1}{4\sin^2(\pi/T)}\sum_{t=1}^{T}\left(\Delta^2 u_{t-1}^{(k)}\right)^2$$

$$\leq \frac{8}{4\left(4B - A^2\right)\sin^2(\pi/T)}\left(C + \frac{T}{4}\right). \tag{52}$$

Therefore, by (48)–(52), we get that there exists a positive constant $M$ such that

$$\sum_{t=1}^{T}\left(\left(\Delta^3 u_{t-1}^{(k)}\right)^2 + \left(\Delta^2 u_{t-1}^{(k)}\right)^2 + \left(\Delta u_{t-1}^{(k)}\right)^2 + \left(u_t^{(k)}\right)^2\right) \leq M^2; \tag{53}$$

that is,

$$\left\|u^{(k)}\right\| \leq M. \tag{54}$$

As a consequence, $\{u^{(k)}\}$ possesses a convergence subsequence in the finite dimensional Hilbert space $E_T$ and $I(u;T)$ satisfies P.S. condition. This completes the proof of Lemma 12. □

From Lemmas 8 and 12, there exists $c = \inf_{u \in E_T} I(u;T)$ is a critical value of $I(u;T)$, which means that there exists a critical point of $I(u;T)$ on $E_T$. Next, we devote ourselves to verifying the critical point is nontrivial.

Set

$$M(T) = \min\{I(u;T) \mid u \in E_T\}. \tag{55}$$

In Theorem 1, we have shown that $M(T) = 0$ if $T \leq T_1$. To complete the proof of Theorem 2, we will show this does not hold for large $T$. We prove that for $T > T_2$, where $T_2$ is an appropriate number defined as (8), it holds $M(T) < 0$ and the corresponding critical point is nontrivial.

*Proof of Theorem 2.* By Lemma 12, $c = \inf_{u \in E_T} I(u;T)$ is nontrivial. So if we can find some critical point $\overline{u}$ such that $I(\overline{u}) < 0$, then $\overline{u}$ is a nontrivial solution of (1). Let us take

$$\overline{u} = \delta \sin\frac{\pi t}{T}, \tag{56}$$

where $\delta$ will be chosen later. By direct computation, it follows

$$I(\overline{u};T)$$

$$= \sum_{t=1}^{T}\left[\frac{1}{2}\left(\left(\Delta^3 \overline{u}_{t-1}\right)^2 - A\left(\Delta^2 \overline{u}_{t-1}\right)^2 + B\left(\Delta \overline{u}_{t-1}\right)^2 - \overline{u}_t^2\right)\right.$$

$$\left.+ \frac{1}{4}\overline{u}_t^4\right]$$

$$= \frac{1}{4}\delta^2 T\left(\left(2\sin\frac{\pi}{2T}\right)^6 - A\left(2\sin\frac{\pi}{2T}\right)^4\right.$$

$$\left.+ B\left(2\sin\frac{\pi}{2T}\right)^2 - 1 + \frac{3}{8}\delta^2\right). \tag{57}$$

Since $A > 0$ if

$$\left(2\sin\frac{\pi}{2T}\right)^6 + B\left(2\sin\frac{\pi}{2T}\right)^2 - 1 < 0 \tag{58}$$

for sufficiently small $\delta$, we have $I(\overline{u}) < 0$. We show that there exists $T_2$, when $T > T_2$, such that (58) is true. Denote $(2\sin(\pi/2T))^2 = p$, inequality (58) is equivalent to

$$p^3 + Bp - 1 < 0. \tag{59}$$

The roots of the polynomial $P_3(p) = p^3 + Bp - 1$ are

$$p_1 = \sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{B^3}{27}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{B^3}{27}}},$$

$$p_2 = \omega\sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{B^3}{27}}} + \overline{\omega}\sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{B^3}{27}}}, \tag{60}$$

$$p_3 = \overline{\omega}\sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{B^3}{27}}} + \omega\sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{B^3}{27}}},$$

where $\omega = (-1 + \sqrt{3}i)/2$ and $\overline{\omega} = (-1 - \sqrt{3}i)/2$. Therefore, (59) holds for $p < p_1$ and (58) holds for $2\sin(\pi/2T) < p_1^{1/2}$; that is, $\sin(\pi/2T) < (1/2)p_1^{1/2}$. Then (1) has a nontrivial solution for $T > T_2$, where $T_2$ is defined as (8). □

We can prove the multiplicity result in Theorem 3 using Lemma 9.

Denote $B_\delta$ the open ball in $E_T$ with radius $\delta$ about 0; $S_\delta$ is the boundary of $B_\delta$.

*Proof of Theorem 3.* For integer $1 \le m \le T$, let us consider the subset $K \subset E_T$,

$$K = \left\{ \lambda_1 \sin \frac{\pi t}{T} + \lambda_2 \sin \frac{2\pi t}{T} + \cdots + \lambda_m \sin \frac{m\pi t}{T} : \right.$$
$$\left. \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_m^2 = \rho^2 \right\}, \tag{61}$$

where $\rho$ is an appropriate positive number. Give an odd continuous mapping $H : K \to S_\rho$:

$$H\left( \lambda_1 \sin \frac{\pi t}{T} + \lambda_2 \sin \frac{2\pi t}{T} + \cdots + \lambda_m \sin \frac{m\pi t}{T} \right)$$
$$= \left( -\frac{\lambda_1}{\rho}, -\frac{\lambda_2}{\rho}, \ldots, -\frac{\lambda_m}{\rho} \right). \tag{62}$$

As defined in (26), by the monotonicity and superinvariance of genus and Lemma 7, for any $\rho > 0$, we have

$$\gamma \left( K \cap S_\rho \right) = m. \tag{63}$$

$K$ is also a subset of the $m$-dimensional space

$$E_m = \text{ span} \left\{ \sin \frac{\pi t}{T}, \sin \frac{2\pi t}{T}, \ldots, \sin \frac{m\pi t}{T} \right\} \tag{64}$$

equipped with the norm

$$\left\| \lambda_1 \sin \frac{\pi t}{T} + \lambda_2 \sin \frac{2\pi t}{T} + \cdots + \lambda_m \sin \frac{m\pi t}{T} \right\|_m^2$$
$$= \lambda_1^2 + \lambda_2^2 + \cdots \lambda_m^2. \tag{65}$$

Then for every $v \in E_m$ there exist positive constants $k_1$ and $k_2$ such that

$$k_1 \|v\|_m \le \|v\|_{l^4} \le k_2 \|v\|_m. \tag{66}$$

In particular,

$$\|v\|_{l^4} \le k_2 \rho, \quad \forall v \in K \cap S_\rho. \tag{67}$$

Since $T > mT_2 > kT_2$ for $k = 1, 2, \ldots, m$, we have

$$0 < \frac{k}{T} \le \frac{m}{T} < \frac{1}{T_2}, \tag{68}$$

$$\left( 2 \sin \frac{k\pi}{2T} \right)^6 + B\left( 2 \sin \frac{k\pi}{2T} \right)^2 - 1$$
$$\le \left( 2 \sin \frac{m\pi}{2T} \right)^6 + B\left( 2 \sin \frac{m\pi}{2T} \right)^2 - 1$$
$$< \left( 2 \sin \frac{\pi}{2T_2} \right)^6 + B\left( 2 \sin \frac{\pi}{2T_2} \right)^2 - 1$$
$$= 0. \tag{69}$$

For $v \in K \cap S_\rho$, $v_t = \sum_{i=1}^m \lambda_i \sin(i\pi t/T)$. Using (67) and (69), there holds

$$I(v; T)$$
$$= \sum_{t=1}^T \left[ \frac{1}{2} \left( (\Delta^3 v_{t-1})^2 - A(\Delta^2 v_{t-1})^2 + B(\Delta v_{t-1})^2 - v_t^2 \right) \right.$$
$$\left. + \frac{1}{4} v_t^4 \right]$$
$$\le \sum_{t=1}^T \left[ \frac{1}{2} \left( (\Delta^3 v_{t-1})^2 + B(\Delta v_{t-1})^2 - v_t^2 \right) + \frac{1}{4} v_t^4 \right]$$
$$= \sum_{t=1}^T \left[ \frac{1}{2} \left( \left( \sum_{i=1}^m \lambda_i \left( 2 \sin \frac{i\pi}{2T} \right)^3 \cos \frac{i\pi(2t+1)}{2T} \right)^2 \right. \right.$$
$$+ B \left( \sum_{i=1}^m \lambda_i \left( 2 \sin \frac{i\pi}{2T} \right) \cos \frac{i\pi(2t-1)}{2T} \right)^2$$
$$\left. \left. - \left( \sum_{i=1}^m \lambda_i \sin \frac{i\pi t}{T} \right)^2 \right) + \frac{1}{4} v_t^4 \right]$$
$$= \frac{T}{4} \left( \sum_{i=1}^m \lambda_i^2 \left( 2 \sin \frac{i\pi}{2T} \right)^6 + B \sum_{i=1}^m \lambda_i^2 \left( 2 \sin \frac{i\pi}{2T} \right)^2 - 1 \right)$$
$$+ \sum_{t=1}^T \frac{1}{4} v_t^4$$
$$\le \frac{\rho^2}{4} \left[ T \left( \left( 2 \sin \frac{m\pi}{2T} \right)^6 + B\left( 2 \sin \frac{m\pi}{2T} \right)^2 - 1 \right) \right] + k_2^4 \rho^2$$
$$< \frac{\rho^2}{4} \left[ T \left( \left( 2 \sin \frac{\pi}{2T_2} \right)^6 + B\left( 2 \sin \frac{\pi}{2T_2} \right)^2 - 1 \right) \right] + k_2^4 \rho^2$$
$$< 0 \tag{70}$$

for sufficiently small $\rho$. It follows

$$c_1 < 0, \tag{71}$$

where $c_1$ is defined in Lemma 9.

According to Lemma 9, $I(u; T)$ has at least $m$ geometrically distinct critical points. Furthermore, $T > mT_2 > T_2$; similar to the proof of Theorem 2, we draw a conclusion that all $m$ distinct critical points we have obtained are all nontrivial. And the proof of Theorem 3 is completed. $\square$

Finally, we exhibit a simple example to illustrate our conclusions.

*Example.* Consider system (1) with $A = \sqrt{10}$ and $B = 3$.

*Solution.* Here $A = \sqrt{10} > 0$, $B = 3 > 0$, $A^2 < 4B$, and

$$\sin \frac{\pi}{2T_2} = \frac{1}{2}\left(\sqrt[3]{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{B^3}{27}}} + \sqrt[3]{\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{B^3}{27}}}\right)^{1/2} = \frac{1}{2} \tag{72}$$

and it follows we can choose $T_2 = 3/2$. When $T > 3/2$, (1) has a nontrivial solution. If one let $T > (3/2)m$, where $1 \le m$ is a integer, then (1) has $m$ distinct nontrivial solution.

*Remark.* From the given example, one can find our results only depend on coefficients $A$ and $B$ which are very easy to verify and rather relaxed.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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