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Research Article

Refinements of Bounds for Neuman Means

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We present the sharp bounds for the Neuman means S_{HA}, S_{AH}, S_{CA} and S_{AC} in terms of the arithmetic, harmonic, and contraharmonic means. Our results are the refinements or improvements of the results given by Neuman.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Schwab-Borchardt mean $\mathrm{SB}(a,b)$ of a and b is given by

$$\mathrm{SB}(a,b) = \begin{cases} \dfrac{\sqrt{b^2 - a^2}}{\cos^{-1}(a/b)}, & a < b, \\ \dfrac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases} \quad (1)$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

It is well-known that the mean $\mathrm{SB}(a,b)$ is strictly increasing in both a and b, nonsymmetric and homogeneous of degree 1 with respect to a and b. Many symmetric bivariate means are special cases of the Schwab-Borchardt mean; for example,

$$P(a,b) = \frac{a-b}{2\sin^{-1}[(a-b)/(a+b)]} = \mathrm{SB}(G, A)$$

is the first Seiffert mean,

$$T(a,b) = \frac{a-b}{2\tan^{-1}[(a-b)/(a+b)]} = \mathrm{SB}(A, Q)$$

is the second Seiffert mean,

$$M(a,b) = \frac{a-b}{2\sinh^{-1}[(a-b)/(a+b)]} = \mathrm{SB}(Q, A)$$

is the Neuman-Sándor mean,

$$L(a,b) = \frac{a-b}{2\tanh^{-1}[(a-b)/(a+b)]} = \mathrm{SB}(A, G)$$

is the logarithmic mean,

$$(2)$$

where $G(a,b) = \sqrt{ab}$, $A(a,b) = (a+b)/2$, and $Q(a,b) = \sqrt{(a^2+b^2)/2}$ denote the classical geometric mean, arithmetic mean, and quadratic mean of a and b, respectively. The Schwab-Borchardt mean $\mathrm{SB}(a,b)$ was investigated in [1, 2].

Let $H(a,b) = 2ab/(a+b)$ and $C(a,b) = (a^2+b^2)/(a+b)$ be the harmonic and contraharmonic means of two positive numbers a and b, respectively. Then, it is well-known that

$$H(a,b) < G(a,b) < L(a,b) < P(a,b) < A(a,b) < M(a,b) < T(a,b) < Q(a,b) < C(a,b), \quad (3)$$

for $a, b > 0$ with $a \neq b$.

Recently, the Schwab-Borchardt mean and its special cases have been the subject of intensive research. Neuman and Sándor [3, 4] proved that the inequalities

$$
\begin{gathered}
P(a,b) > \frac{2}{\pi} A(a,b), \\
\frac{A(a,b)}{\log(1+\sqrt{2})} > M(a,b) > \frac{\pi}{4\log(1+\sqrt{2})} T(a,b), \\
T(A(a,b), G(a,b)) < P(a,b), \\
T(a,b) > T(A(a,b), Q(a,b)), \\
L(a,b) < L(A(a,b), G(a,b)), \\
M(a,b) < L(A(a,b), Q(a,b)), \\
L(a,b) > H(P(a,b), G(a,b)), \\
P(a,b) > H(L(a,b), A(a,b)), \\
M(a,b) > H(T(a,b), A(a,b)), \\
T(a,b) > H(M(a,b), Q(a,b)), \\
G(a,b) P(a,b) < L^2(a,b) < \frac{G^2(a,b) + P^2(a,b)}{2}, \\
L(a,b) A(a,b) < P^2(a,b) < \frac{L^2(a,b) + A^2(a,b)}{2}, \\
A(a,b) T(a,b) < M^2(a,b) < \frac{A^2(a,b) + T^2(a,b)}{2}, \\
M(a,b) Q(a,b) < T^2(a,b) < \frac{M^2(a,b) + Q^2(a,b)}{2}, \\
Q^{1/3}(a,b) A^{2/3}(a,b) < M(a,b) < \frac{1}{3} Q(a,b) + \frac{2}{3} A(a,b)
\end{gathered} \tag{4}
$$

hold for all $a, b > 0$ with $a \neq b$. In [5], the author proved that the double inequalities

$$
\begin{aligned}
&\alpha Q(a,b) + (1-\alpha) A(a,b) \\
&\quad < M(a,b) < \beta Q(a,b) + (1-\beta) A(a,b), \\
&\lambda C(a,b) + (1-\lambda) A(a,b) \\
&\quad < M(a,b) < \mu C(a,b) + (1-\mu) A(a,b)
\end{aligned} \tag{5}
$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq [1 - \log(1+\sqrt{2})]/[(\sqrt{2}-1)\log(1+\sqrt{2})] = 0.3249\cdots$, $\beta \geq 1/3$, $\lambda \leq [1 - \log(1+\sqrt{2})]/\log(1+\sqrt{2}) = 0.1345\cdots$, and $\mu \geq 1/6$. Chu and Long [6] found that the double inequality

$$
M_p(a,b) < M(a,b) < qI(a,b) \tag{6}
$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq \log 2/\log[2\log(1+\sqrt{2})] = 1.224\cdots$ and $q \geq e/[2\log(1+\sqrt{2})]$, where $M_p(a,b) = [(a^p + b^p)/2]^{1/p} (p \neq 0)$ and $M_0(a,b) = \sqrt{ab}$ is the pth power mean of a and b. Zhao et al. [7] presented the least values α_1, α_2, and α_3 and the greatest values β_1, β_2, and β_3 such that the double inequalities

$$
\begin{aligned}
&\alpha_1 H(a,b) + (1-\alpha_1) Q(a,b) < M(a,b) \\
&\quad < \beta_1 H(a,b) + (1-\beta_1) Q(a,b), \\
&\alpha_2 G(a,b) + (1-\alpha_2) Q(a,b) < M(a,b) \\
&\quad < \beta_2 G(a,b) + (1-\beta_2) Q(a,b), \\
&\alpha_3 H(a,b) + (1-\alpha_3) C(a,b) < M(a,b) \\
&\quad < \beta_3 H(a,b) + (1-\beta_3) C(a,b)
\end{aligned} \tag{7}
$$

hold for all $a, b > 0$ with $a \neq b$.

Very recently, the bivariate means S_{AH}, S_{HA}, S_{CA}, and S_{AC} derived from the Schwab-Borchardt mean are defined by Neuman [8, 9] as follows:

$$
\begin{aligned}
S_{AH} &= SB(A, H), \qquad S_{HA} = SB(H, A), \\
S_{CA} &= SB(C, A), \qquad S_{AC} = SB(A, C).
\end{aligned} \tag{8}
$$

We call the means S_{AH}, S_{HA}, S_{CA}, and S_{AC} given in (8) the Neuman means. Moreover, let $v = (a-b)/(a+b) \in (-1, 1)$; then the following explicit formulas for S_{AH}, S_{HA}, S_{AC}, and S_{CA} are found by Neuman [8]:

$$
S_{AH} = A\frac{\tanh(p)}{p}, \qquad S_{HA} = A\frac{\sin(q)}{q}, \tag{9}
$$

$$
S_{CA} = A\frac{\sinh(r)}{r}, \qquad S_{AC} = A\frac{\tan(s)}{s}, \tag{10}
$$

where p, q, r, and s are defined implicitly as $\operatorname{sech}(p) = 1 - v^2$, $\cos(q) = 1 - v^2$, $\cosh(r) = 1 + v^2$, and $\sec(s) = 1 + v^2$, respectively. Clearly, $p \in (0, \infty)$, $q \in (0, \pi/2)$, $r \in (0, \log(2+\sqrt{3}))$, and $s \in (0, \pi/3)$.

In [8, 9], Neuman proved that the inequalities

$$
\begin{aligned}
&H(a,b) < S_{AH}(a,b) < L(a,b) < S_{HA}(a,b) < P(a,b), \\
&T(a,b) < S_{CA}(a,b) < Q(a,b) < S_{AC}(a,b) < C(a,b),
\end{aligned} \tag{11}
$$

$$
\begin{gathered}
H^{1/3}(a,b) A^{2/3}(a,b) < S_{HA}(a,b) < \frac{1}{3} H(a,b) + \frac{2}{3} A(a,b), \\
C^{1/3}(a,b) A^{2/3}(a,b) < S_{CA}(a,b) < \frac{1}{3} C(a,b) + \frac{2}{3} A(a,b), \\
A^{1/3}(a,b) H^{2/3}(a,b) < S_{AH}(a,b) < \frac{1}{3} A(a,b) + \frac{2}{3} H(a,b), \\
A^{1/3}(a,b) C^{2/3}(a,b) < S_{AC}(a,b) < \frac{1}{3} A(a,b) + \frac{2}{3} C(a,b)
\end{gathered} \tag{12}
$$

hold for $a, b > 0$ with $a \neq b$.

He et al. [10] found the greatest values $\alpha_1, \alpha_2 \in [0,1/2]$, $\alpha_3, \alpha_4 \in [1/2,1]$, and the least values $\beta_1, \beta_2 \in [0,1/2]$, β_3, $\beta_4 \in [1/2,1]$ such that the double inequalities

$$\begin{aligned} &H(\alpha_1 a + (1-\alpha_1)b, \alpha_1 b + (1-\alpha_1)a) < S_{AH}(a,b) \\ &\quad < H(\beta_1 a + (1-\beta_1)b, \beta_1 b + (1-\beta_1)a), \\ &H(\alpha_2 a + (1-\alpha_2)b, \alpha_2 b + (1-\alpha_2)a) < S_{HA}(a,b) \\ &\quad < H(\beta_2 a + (1-\beta_2)b, \beta_2 b + (1-\beta_2)a), \\ &C(\alpha_3 a + (1-\alpha_3)b, \alpha_3 b + (1-\alpha_3)a) < S_{CA}(a,b) \\ &\quad < C(\beta_3 a + (1-\beta_3)b, \beta_3 b + (1-\beta_3)a), \\ &C(\alpha_4 a + (1-\alpha_4)b, \alpha_4 b + (1-\alpha_4)a) < S_{AC}(a,b) \\ &\quad < C(\beta_4 a + (1-\beta_4)b, \beta_4 b + (1-\beta_4)a) \end{aligned} \tag{13}$$

hold for all $a, b > 0$ with $a \neq b$.

Motivated by inequalities (12), it is natural to ask what the greatest values α_1, α_2, α_3, and α_4 and the least values β_1, β_2, β_3, and β_4 are such that the double inequalities

$$\begin{aligned} &\alpha_1\left[\frac{H(a,b)}{3} + \frac{2A(a,b)}{3}\right] + (1-\alpha_1) H^{1/3}(a,b) A^{2/3}(a,b) \\ &\quad < S_{HA}(a,b) < \beta_1\left[\frac{H(a,b)}{3} + \frac{2A(a,b)}{3}\right] \\ &\quad + (1-\beta_1) H^{1/3}(a,b) A^{2/3}(a,b), \\ &\alpha_2\left[\frac{C(a,b)}{3} + \frac{2A(a,b)}{3}\right] + (1-\alpha_2) C^{1/3}(a,b) A^{2/3}(a,b) \\ &\quad < S_{CA}(a,b) < \beta_2\left[\frac{C(a,b)}{3} + \frac{2A(a,b)}{3}\right] \\ &\quad + (1-\beta_2) C^{1/3}(a,b) A^{2/3}(a,b), \\ &\alpha_3\left[\frac{A(a,b)}{3} + \frac{2H(a,b)}{3}\right] + (1-\alpha_3) A^{1/3}(a,b) H^{2/3}(a,b) \\ &\quad < S_{AH}(a,b) < \beta_3\left[\frac{A(a,b)}{3} + \frac{2H(a,b)}{3}\right] \\ &\quad + (1-\beta_3) A^{1/3}(a,b) H^{2/3}(a,b), \\ &\alpha_4\left[\frac{A(a,b)}{3} + \frac{2C(a,b)}{3}\right] + (1-\alpha_4) A^{1/3}(a,b) C^{2/3}(a,b) \\ &\quad < S_{AC}(a,b) < \beta_4\left[\frac{A(a,b)}{3} + \frac{2C(a,b)}{3}\right] \\ &\quad + (1-\beta_4) A^{1/3}(a,b) C^{2/3}(a,b) \end{aligned} \tag{14}$$

hold for all $a, b > 0$ with $a \neq b$.

The purpose of this paper is to answer these questions. All numerical computations are carried out using MATHEMATICA software. Our main results are the following Theorems 1–4.

Theorem 1. *The double inequality*

$$\begin{aligned} &\alpha_1\left[\frac{H(a,b)}{3} + \frac{2A(a,b)}{3}\right] + (1-\alpha_1) H^{1/3}(a,b) A^{2/3}(a,b) \\ &\quad < S_{HA}(a,b) < \beta_1\left[\frac{H(a,b)}{3} + \frac{2A(a,b)}{3}\right] \\ &\quad + (1-\beta_1) H^{1/3}(a,b) A^{2/3}(a,b) \end{aligned} \tag{15}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 4/5$ and $\beta_1 \geq 3/\pi$.

Theorem 2. *The two-sided inequality*

$$\begin{aligned} &\alpha_2\left[\frac{C(a,b)}{3} + \frac{2A(a,b)}{3}\right] + (1-\alpha_2) C^{1/3}(a,b) A^{2/3}(a,b) \\ &\quad < S_{CA}(a,b) < \beta_2\left[\frac{C(a,b)}{3} + \frac{2A(a,b)}{3}\right] \\ &\quad + (1-\beta_2) C^{1/3}(a,b) A^{2/3}(a,b) \end{aligned} \tag{16}$$

holds true for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 3[\sqrt[3]{2}\log(2+\sqrt{3}) - \sqrt{3}]/[(3\sqrt[3]{2}-4)\log(2+\sqrt{3})] = 0.7528\cdots$ and $\beta_2 \geq 4/5$.

Theorem 3. *The double inequality*

$$\begin{aligned} &\alpha_3\left[\frac{A(a,b)}{3} + \frac{2H(a,b)}{3}\right] + (1-\alpha_3) A^{1/3}(a,b) H^{2/3}(a,b) \\ &\quad < S_{AH}(a,b) < \beta_3\left[\frac{A(a,b)}{3} + \frac{2H(a,b)}{3}\right] \\ &\quad + (1-\beta_3) A^{1/3}(a,b) H^{2/3}(a,b) \end{aligned} \tag{17}$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 0$ and $\beta_3 \geq 4/5$.

Theorem 4. *The two-sided inequality*

$$\begin{aligned} &\alpha_4\left[\frac{A(a,b)}{3} + \frac{2C(a,b)}{3}\right] + (1-\alpha_4) A^{1/3}(a,b) C^{2/3}(a,b) \\ &\quad < S_{AC}(a,b) < \beta_4\left[\frac{A(a,b)}{3} + \frac{2C(a,b)}{3}\right] \\ &\quad + (1-\beta_4) A^{1/3}(a,b) C^{2/3}(a,b) \end{aligned} \tag{18}$$

holds true for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leq 4/5$ and $\beta_2 \geq 3(3\sqrt{3} - \sqrt[3]{4}\pi)/[(5 - 3\sqrt[3]{4})\pi] = 0.8400\cdots$.

2. Two Lemmas

In order to prove our main results, we need two lemmas, which we present in this section.

Lemma 5. *Let* $p \in \mathbb{R}$ *and*

$$\begin{aligned} f(x) = {} & p^2x^6 + 2p^2x^5 + 3\left(-p^2+4p-2\right)x^4 \\ & + 2\left(-2p^2+9p-6\right)x^3 + \left(4p^2+6p-9\right)x^2 \\ & + 6\left(p-1\right)x + 3\left(p-1\right). \end{aligned} \quad (19)$$

Then, the following statements are true.

(1) *If* $p = 4/5$, *then* $f(x) < 0$ *for all* $x \in (0,1)$ *and* $f(x) > 0$ *for all* $x \in (1, \sqrt[3]{2})$.

(2) *If* $p = 3/\pi$, *then there exists* $\lambda_1 \in (0,1)$ *such that* $f(x) < 0$ *for* $x \in (0, \lambda_1)$ *and* $f(x) > 0$ *for* $x \in (\lambda_1, 1)$.

(3) *If* $p = 3[\sqrt[3]{2}\log(2+\sqrt{3}) - \sqrt{3}]/[(3\sqrt[3]{2}-4)\log(2+\sqrt{3})] = 0.7528\cdots$, *then there exists* $\lambda_2 \in (1, \sqrt[3]{2})$ *such that* $f(x) < 0$ *for* $x \in (1, \lambda_2)$ *and* $f(x) > 0$ *for* $x \in (\lambda_2, \sqrt[3]{2})$.

Proof. For part (1), if $p = 4/5$, then (19) becomes

$$f(x) = \frac{1}{25}(x-1)\left(16x^5 + 48x^4 + 90x^3 + 86x^2 + 45x + 15\right). \quad (20)$$

Therefore, part (1) follows easily from (20).

For part (2), if $p = 3/\pi$, then simple computations lead to

$$-p^2 + 4p - 2 = \frac{-2\pi^2 + 12\pi - 9}{\pi^2} > 0, \quad (21)$$

$$-2p^2 + 9p - 6 = \frac{-6\pi^2 + 27\pi - 18}{\pi^2} > 0, \quad (22)$$

$$4p^2 + 6p - 9 = \frac{-9\pi^2 + 18\pi + 36}{\pi^2} > 0, \quad (23)$$

$$f(0) = -\frac{3(\pi-3)}{\pi} < 0, \quad (24)$$

$$f(1) = \frac{9(15-4\pi)}{\pi} > 0, \quad (25)$$

$$\begin{aligned} f'(x) = {} & 6p^2x^5 + 10p^2x^4 + 12\left(-p^2+4p-2\right)x^3 \\ & + 6\left(-2p^2+9p-6\right)x^2 + 2\left(4p^2+6p-9\right)x \\ & + 6\left(p-1\right), \end{aligned} \quad (26)$$

$$f'(0) = \frac{6(3-\pi)}{\pi} < 0, \quad (27)$$

$$f'(1) = \frac{12(30-7\pi)}{\pi} > 0, \quad (28)$$

$$\begin{aligned} f''(x) = {} & 30p^2x^4 + 40p^2x^3 + 36\left(-p^2+4p-2\right)x^2 \\ & + 12\left(-2p^2+9p-6\right)x + 2\left(4p^2+6p-9\right). \end{aligned} \quad (29)$$

It follows from (21)–(23) and (29) that $f'(x)$ is strictly increasing on $(0,1)$. Then, (27) and (28) lead to the conclusion that there exists $x_0 \in (0,1)$ such that $f(x)$ is strictly decreasing in $(0, x_0]$ and strictly increasing in $[x_0, 1)$.

Therefore, part (2) follows from (24) and (25) together with the piecewise monotonicity of $f(x)$.

For part (3), if $p = 3[\sqrt[3]{2}\log(2+\sqrt{3}) - \sqrt{3}]/[(3\sqrt[3]{2}-4)\log(2+\sqrt{3})] = 0.7528\cdots$, then numerical computations lead to

$$-p^2 + 4p - 2 = 0.444\cdots > 0, \quad (30)$$

$$4p^2 + 6p - 9 = -2.215\cdots < 0, \quad (31)$$

$$6(p-1) = -1.483\cdots < 0, \quad (32)$$

$$f(1) = 9(5p-4) = -2.120\cdots < 0, \quad (33)$$

$$f\left(\sqrt[3]{2}\right) = 1.669\cdots > 0. \quad (34)$$

It follows from (26) and (30)–(32) that

$$\begin{aligned} f'(x) > {} & 6p^2x^2 + 10p^2x^2 + 12\left(-p^2+4p-2\right)x^2 \\ & + 6\left(-2p^2+9p-6\right)x^2 + 2\left(4p^2+6p-9\right)x^2 \\ & + 6\left(p-1\right)x^2 \\ = {} & 12\left(10p-7\right)x^2 > 0 \end{aligned} \quad (35)$$

for $x \in (1, \sqrt[3]{2})$.

Therefore, part (3) follows easily from (33)–(35). □

Lemma 6. *Let* $p \in \mathbb{R}$ *and*

$$\begin{aligned} g(x) = {} & 3(1-p)x^6 + 6(1-p)x^5 + \left(-4p^2-6p+9\right)x^4 \\ & + 2\left(2p^2-9p+6\right)x^3 + 3\left(p^2-4p+2\right)x^2 \\ & - 2p^2x - p^2. \end{aligned} \quad (36)$$

Then, the following statements are true.

(1) *If* $p = 4/5$, *then* $g(x) < 0$ *for all* $x \in (0,1)$ *and* $g(x) > 0$ *for all* $x \in (1, \sqrt[3]{2})$.

(2) *If* $p = 3(3\sqrt{3} - \sqrt[3]{4}\pi)/[(5 - 3\sqrt[3]{4})\pi] = 0.8400\cdots$, *then there exists* $\lambda_3 \in (1, \sqrt[3]{2})$ *such that* $g(x) < 0$ *for* $x \in (1, \lambda_3)$ *and* $g(x) > 0$ *for* $x \in (\lambda_3, \sqrt[3]{2})$.

Proof. For part (1), if $p = 4/5$, then (36) becomes

$$g(x) = \frac{1}{25}(x-1)\left(15x^5 + 45x^4 + 86x^3 + 90x^2 + 48x + 16\right). \quad (37)$$

Therefore, part (1) follows from (37).

For part (2), if $p = 3(3\sqrt{3} - \sqrt[3]{4}\pi)/[(5 - 3\sqrt[3]{4})\pi] = 0.8400\cdots$, then numerical computations lead to

$$-4p^2 - 6p + 9 = 1.137\cdots > 0, \quad (38)$$

$$p^2 - 4p + 2 = -0.654\cdots < 0, \quad (39)$$

$$g(1) = 9(4 - 5p) = -1.801\cdots < 0, \quad (40)$$

$$g(\sqrt[3]{2}) = 1.635\cdots > 0, \quad (41)$$

$$\begin{aligned} g'(x) = {} & 18(1-p)x^5 + 30(1-p)x^4 \\ & + 4(-4p^2 - 6p + 9)x^3 + 6(2p^2 - 9p + 6)x^2 \\ & + 6(p^2 - 4p + 2)x - 2p^2. \end{aligned} \quad (42)$$

From (38) and (39) together with (42), we clearly see that

$$\begin{aligned} g'(x) > {} & 18(1-p)x^2 + 30(1-p)x^2 \\ & + 4(-4p^2 - 6p + 9)x^2 + 6(2p^2 - 9p + 6)x^2 \\ & + 6(p^2 - 4p + 2)x^2 - 2p^2x^2 \\ = {} & 6(22 - 25p)x^2 > 0 \end{aligned} \quad (43)$$

for $x \in (1, \sqrt[3]{2})$.

Therefore, part (2) follows from (40) and (41) together with (43). □

3. Proofs of Theorems 1–4

Proof of Theorem 1. Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\lambda = v\sqrt{2 - v^2}$, $x = \sqrt[6]{1 - \lambda^2}$, and $p \in \{4/5, 3/\pi\}$. Then, $v, \lambda, x \in (0, 1)$,

$$\begin{aligned} & \frac{S_{HA}(a,b) - H^{1/3}(a,b)A^{2/3}(a,b)}{H(a,b)/3 + 2A(a,b)/3 - H^{1/3}(a,b)\ A^{2/3}(a,b)} \\ & = \frac{\lambda/\sin^{-1}(\lambda) - (1-\lambda^2)^{1/6}}{2/3 + (1-\lambda^2)^{1/2}/3 - (1-\lambda^2)^{1/6}}, \end{aligned} \quad (44)$$

$$\begin{aligned} & S_{HA}(a,b) - \left[p\left(\frac{1}{3}H(a,b) + \frac{2A(a,b)}{3}\right) + (1-p)H^{1/3}(a,b)A^{2/3}(a,b)\right] \\ & = A(a,b)\left[\frac{\lambda}{\sin^{-1}(\lambda)} - p\left(\frac{(1-\lambda^2)^{1/2}}{3} + \frac{2}{3}\right) - (1-p)(1-\lambda^2)^{1/6}\right] \\ & = \left(A(a,b)\left[p\left((1-\lambda^2)^{1/2} + 2\right) + 3(1-p)(1-\lambda^2)^{1/6}\right]\right) \\ & \quad \times \left(3\sin^{-1}(\lambda)\right)^{-1} F(x), \end{aligned} \quad (45)$$

where

$$F(x) = \frac{3\sqrt{1-x^6}}{px^3 + 3(1-p)x + 2p} - \sin^{-1}\left(\sqrt{1-x^6}\right), \quad (46)$$

$$F(0) = \frac{3}{2p} - \frac{\pi}{2}, \quad (47)$$

$$F(1) = 0, \quad (48)$$

$$F'(x) = \frac{3(x-1)^2}{\sqrt{1-x^6}[px^3 + 3(1-p)x + 2p]^2} f(x), \quad (49)$$

where $f(x)$ is defined as in Lemma 5.

We divide the proof into two cases.

Case 1 ($p = 4/5$). Then, from Lemma 5(1) and (49), we clearly see that $F(x)$ is strictly decreasing in $(0, 1)$. Therefore,

$$\begin{aligned} S_{HA}(a,b) > {} & \frac{4}{5}\left[\frac{1}{3}H(a,b) + \frac{2}{3}A(a,b)\right] \\ & + \frac{1}{5}H^{1/3}(a,b)A^{2/3}(a,b) \end{aligned} \quad (50)$$

for all $a, b > 0$ with $a \neq b$ follows from (45) and (48) together with the monotonicity of $F(x)$.

Case 2 ($p = 3/\pi$). Then, from (47) and (49) and Lemma 5(2), we know that

$$F(0) = 0 \quad (51)$$

and there exists $\lambda_1 \in (0, 1)$ such that $F(x)$ is strictly decreasing in $(0, \lambda_1]$ and strictly increasing in $[\lambda_1, 1)$. Therefore,

$$\begin{aligned} S_{HA}(a,b) < {} & \frac{3}{\pi}\left[\frac{1}{3}H(a,b) + \frac{2}{3}A(a,b)\right] \\ & + \left(1 - \frac{3}{\pi}\right)H^{1/3}(a,b)A^{2/3}(a,b) \end{aligned} \quad (52)$$

for all $a, b > 0$ with $a \neq b$ follows from (45) and (48) together with (51) and the piecewise monotonicity of $F(x)$.

Note that

$$\lim_{\lambda \to 0^+} \frac{\lambda/\sin^{-1}(\lambda) - (1-\lambda^2)^{1/6}}{2/3 + (1-\lambda^2)^{1/2}/3 - (1-\lambda^2)^{1/6}} = \frac{4}{5}, \quad (53)$$

$$\lim_{\lambda \to 1^-} \frac{\lambda/\sin^{-1}(\lambda) - (1-\lambda^2)^{1/6}}{2/3 + (1-\lambda^2)^{1/2}/3 - (1-\lambda^2)^{1/6}} = \frac{3}{\pi}. \quad (54)$$

Therefore, Theorem 1 follows from (50) and (52)–(54) together with the following statements.

(i) If $\alpha > 4/5$, then (44) and (53) imply that there exists small enough $\delta > 0$ such that $S_{HA}(a,b) < \alpha(H(a,b)/3+2A(a,b)/3) + (1-\alpha)H^{1/3}(a,b)A^{2/3}(a,b)$ for all $a > b > 0$ with $b/a \in (0,\delta)$.

(ii) If $\beta < 3/\pi$, then (44) and (54) imply that there exists large enough $M > 1$ such that $S_{HA}(a,b) > \beta(H(a,b)/3+2A(a,b)/3)+(1-\beta)H^{1/3}(a,b)A^{2/3}(a,b)$ for all $a > b > 0$ with $a/b \in (M,+\infty)$. □

Proof of Theorem 2. Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\mu = v\sqrt{2+v^2}$, $x = \sqrt[6]{1+\mu^2}$, and $p \in \{3[\sqrt[3]{2}\log(2+\sqrt{3})-\sqrt{3}]/[(3\sqrt[3]{2}-4)\log(2+\sqrt{3})], 4/5\}$. Then, $v \in (0,1)$, $\mu \in (0,\sqrt{3})$, $x \in (1,\sqrt[3]{2})$,

$$\frac{S_{CA}(a,b) - C^{1/3}(a,b)A^{2/3}(a,b)}{C(a,b)/3 + 2A(a,b)/3 - C^{1/3}(a,b)A^{2/3}(a,b)} = \frac{\mu/\sinh^{-1}(\mu) - (1+\mu^2)^{1/6}}{2/3 + (1+\mu^2)^{1/2}/3 - (1+\mu^2)^{1/6}}, \quad (55)$$

$$\begin{aligned} S_{CA}(a,b) &- \left[p\left(\frac{1}{3}C(a,b) + \frac{2A(a,b)}{3}\right) + (1-p)C^{1/3}(a,b)A^{2/3}(a,b)\right] \\ &= A(a,b)\left[\frac{\mu}{\sinh^{-1}(\mu)} - p\left(\frac{(1+\mu^2)^{1/2}}{3} + \frac{2}{3}\right) - (1-p)(1+\mu^2)^{1/6}\right] \\ &= \left(A(a,b)\left[p\left((1+\mu^2)^{1/2}+2\right)+3(1-p)(1+\mu^2)^{1/6}\right]\right) \\ &\quad \times \left(3\sinh^{-1}(\mu)\right)^{-1}G(x), \end{aligned} \quad (56)$$

where

$$G(x) = \frac{3\sqrt{x^6-1}}{px^3+3(1-p)x+2p} - \sinh^{-1}\left(\sqrt{x^6-1}\right), \quad (57)$$

$$G(1) = 0, \quad (58)$$

$$G\left(\sqrt[3]{2}\right) = \frac{3\sqrt{3}}{(4-3\sqrt[3]{2})p+3\sqrt[3]{2}} - \log\left(1+\sqrt{3}\right), \quad (59)$$

$$G'(x) = -\frac{3(x-1)^2}{\sqrt{x^6-1}[px^3+3(1-p)x+2p]^2}f(x), \quad (60)$$

where $f(x)$ is defined as in Lemma 5.

We divide the proof into two cases.

Case 1 ($p = 3[\sqrt[3]{2}\log(2+\sqrt{3}) - \sqrt{3}]/[(3\sqrt[3]{2}-4)\log(2+\sqrt{3})] = 0.7528\cdots$). Then, from (59) and (60) together with Lemma 5(3), we clearly see that there exists $\lambda_2 \in (1,\sqrt[3]{2})$ such that $G(x)$ is strictly increasing in $(1,\lambda_2]$ and strictly decreasing in $[\lambda_2, \sqrt[3]{3})$, and

$$G\left(\sqrt[3]{2}\right) = 0. \quad (61)$$

Therefore,

$$\begin{aligned} S_{CA}(a,b) &> \frac{3\left(\sqrt[3]{2}\log(2+\sqrt{3})-\sqrt{3}\right)}{\left(3\sqrt[3]{2}-4\right)\log(2+\sqrt{3})}\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right] \\ &+ \left(1 - \frac{3\left(\sqrt[3]{2}\log(2+\sqrt{3})-\sqrt{3}\right)}{\left(3\sqrt[3]{2}-4\right)\log(2+\sqrt{3})}\right) \\ &\times C^{1/3}(a,b)A^{2/3}(a,b) \end{aligned} \quad (62)$$

for all $a,b > 0$ with $a \neq b$ follows easily from (56) and (58) together with (61) and the piecewise monotonicity of $G(x)$.

Case 2 ($p = 4/5$). Then, Lemma 5(1) and (60) lead to the conclusion that $G(x)$ is strictly decreasing in $(1,\sqrt[3]{2})$. Therefore,

$$\begin{aligned} S_{CA}(a,b) < \frac{4}{5}\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right] \\ + \frac{1}{5}C^{1/3}(a,b)A^{2/3}(a,b) \end{aligned} \quad (63)$$

for all $a,b > 0$ with $a \neq b$ follows from (56) and (58) together with the monotonicity of $G(x)$.

Note that

$$\lim_{\mu\to 0^+}\frac{\mu/\sinh^{-1}(\mu)-(1+\mu^2)^{1/6}}{2/3+(1+\mu^2)^{1/2}/3-(1+\mu^2)^{1/6}} = \frac{4}{5}, \quad (64)$$

$$\begin{aligned} &\lim_{\mu\to\sqrt{3}^-}\frac{\mu/\sinh^{-1}(\mu)-(1+\mu^2)^{1/6}}{2/3+(1+\mu^2)^{1/2}/3-(1+\mu^2)^{1/6}} \\ &= \frac{3\left(\sqrt[3]{2}\log(2+\sqrt{3})-\sqrt{3}\right)}{\left(3\sqrt[3]{2}-4\right)\log(2+\sqrt{3})}. \end{aligned} \quad (65)$$

Therefore, Theorem 2 follows from (55) and (62)–(65). □

Proof of Theorem 3. Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\lambda = v\sqrt{2-v^2}$, $x = \sqrt[6]{1-\lambda^2}$, and $p \in \{4/5, 0\}$. Then, $v, \lambda, x \in (0,1)$ and (9) leads to

$$S_{AH}(a,b) = A(a,b)\frac{\lambda}{\tanh^{-1}(\lambda)}. \quad (66)$$

It follows from (66) that

$$\frac{S_{AH}(a,b) - A^{1/3}(a,b)H^{2/3}(a,b)}{A(a,b)/3 + 2H(a,b)/3 - A^{1/3}(a,b)H^{2/3}(a,b)} = \frac{\lambda/\tanh^{-1}(\lambda) - (1-\lambda^2)^{1/3}}{1/3 + 2(1-\lambda^2)^{1/2}/3 - (1-\lambda^2)^{1/3}}, \tag{67}$$

$$\begin{aligned} S_{AH}(a,b) &- \left[p\left(\frac{1}{3}A(a,b) + \frac{2H(a,b)}{3}\right) + (1-p)A^{1/3}(a,b)H^{2/3}(a,b)\right] \\ &= A(a,b)\left[\frac{\lambda}{\tanh^{-1}(\lambda)} - p\left(\frac{2(1-\lambda^2)^{1/2}}{3} + \frac{1}{3}\right) - (1-p)(1-\lambda^2)^{1/3}\right] \\ &= \frac{A(a,b)\left[p\left(2(1-\lambda^2)^{1/2}+1\right)+3(1-p)(1-\lambda^2)^{1/3}\right]}{3\tanh^{-1}(\lambda)} \times H(x), \end{aligned} \tag{68}$$

where

$$H(x) = \frac{3\sqrt{1-x^6}}{2px^3 + 3(1-p)x^2 + p} - \tanh^{-1}\left(\sqrt{1-x^6}\right) \tag{69}$$

$$H(1) = 0, \tag{70}$$

$$H'(x) = -\frac{3(1-x)^2}{x\sqrt{1-x^6}[2px^3 + 3(1-p)x^2 + p]^2}g(x), \tag{71}$$

where $g(x)$ is defined as in Lemma 6.

If $p = 4/5$, then Lemma 6(1) and (71) lead to the conclusion that $H(x)$ is strictly increasing in $(0,1)$. Therefore,

$$S_{AH}(a,b) < \frac{4}{5}\left(\frac{1}{3}A(a,b) + \frac{2H(a,b)}{3}\right) + \frac{1}{5}A^{1/3}(a,b)H^{2/3}(a,b) \tag{72}$$

for all $a,b>0$ with $a \neq b$ follows from (68) and (70) together with the monotonicity of $H(x)$.

Note that

$$\lim_{\lambda\to 0^+}\frac{\lambda/\tanh^{-1}(\lambda) - (1-\lambda^2)^{1/3}}{1/3 + 2(1-\lambda^2)^{1/2}/3 - (1-\lambda^2)^{1/3}} = \frac{4}{5}, \tag{73}$$

$$\lim_{\lambda\to 1^-}\frac{\lambda/\tanh^{-1}(\lambda) - (1-\lambda^2)^{1/3}}{1/3 + 2(1-\lambda^2)^{1/2}/3 - (1-\lambda^2)^{1/3}} = 0. \tag{74}$$

Therefore, Theorem 3 follows from (12) and (67) together with (72)–(74). □

Proof of Theorem 4. Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\mu = v\sqrt{2+v^2}$, $x = \sqrt[6]{1+\mu^2}$, and $p \in \{3(3\sqrt{3} - \sqrt[3]{4}\pi)/[(5-3\sqrt[3]{4})\pi], 4/5\}$. Then, $v \in (0,1)$, $\mu \in (0,\sqrt{3})$, and $x \in (1, \sqrt[3]{2})$ and (10) leads to

$$S_{AC}(a,b) = A(a,b)\frac{\mu}{\tan^{-1}(\mu)}. \tag{75}$$

It follows from (75) that

$$\frac{S_{AC}(a,b) - A^{1/3}(a,b)C^{2/3}(a,b)}{A(a,b)/3 + 2C(a,b)/3 - A^{1/3}(a,b)C^{2/3}(a,b)} = \frac{\mu/\tan^{-1}(\mu) - (1+\mu^2)^{1/3}}{1/3 + 2(1+\mu^2)^{1/2}/3 - (1+\mu^2)^{1/3}}, \tag{76}$$

$$\begin{aligned} S_{AC}(a,b) &- \left[p\left(\frac{1}{3}A(a,b) + \frac{2C(a,b)}{3}\right) + (1-p)A^{1/3}(a,b)C^{2/3}(a,b)\right] \\ &= A(a,b)\left[\frac{\mu}{\tan^{-1}(\mu)} - p\left(\frac{2(1+\mu^2)^{1/2}}{3} + \frac{1}{3}\right) - (1-p)(1+\mu^2)^{1/3}\right] \\ &= \frac{A(a,b)\left[p\left(2(1+\mu^2)^{1/2}+1\right)+3(1-p)(1+\mu^2)^{1/3}\right]}{3\tan^{-1}(\mu)} \times J(x), \end{aligned} \tag{77}$$

where

$$J(x) = \frac{3\sqrt{x^6-1}}{2px^3 + 3(1-p)x^2 + p} - \tan^{-1}\left(\sqrt{x^6-1}\right), \tag{78}$$

$$J(1) = 0, \tag{79}$$

$$J\left(\sqrt[3]{2}\right) = \frac{3\sqrt{3}}{\left(5-3\sqrt[3]{4}\right)p + 3\sqrt[3]{4}} - \frac{\pi}{3}, \tag{80}$$

$$J'(x) = \frac{3(x-1)^2}{\sqrt{x^6-1}[2px^3 + 3(1-p)x^2 + p]^2}g(x), \tag{81}$$

where $g(x)$ is defined as in Lemma 6.

We divide the proof into two cases.

Case 1 ($p = 4/5$). Then, (81) and Lemma 6(1) lead to the conclusion that $J(x)$ is strictly increasing in $(1, \sqrt[3]{2})$. Therefore,

$$S_{AC}(a,b) > \frac{4}{5}\left(\frac{1}{3}A(a,b) + \frac{2C(a,b)}{3}\right) + \frac{1}{5}A^{1/3}(a,b)C^{2/3}(a,b) \tag{82}$$

for all $a, b > 0$ with $a \neq b$ follows easily from (77) and (79) together with the monotonicity of $J(x)$.

Case 2 ($p = 3(3\sqrt{3} - \sqrt[3]{4}\pi)/(5 - 3\sqrt[3]{4})\pi$). Then, (80) and (81) together with Lemma 6(2) lead to the conclusion that there exists $\lambda_3 \in (1, \sqrt[3]{2})$ such that $J(x)$ is strictly decreasing in $(1, \lambda_3]$ and strictly increasing in $[\lambda_3, \sqrt[3]{2})$, and

$$J\left(\sqrt[3]{2}\right) = 0. \tag{83}$$

Therefore,

$$S_{AC}(a,b) < \frac{3\left(3\sqrt{3} - \sqrt[3]{4}\pi\right)}{\left(5 - 3\sqrt[3]{4}\right)\pi}\left(\frac{1}{3}A(a,b) + \frac{2C(a,b)}{3}\right) + \left(1 - \frac{3\left(3\sqrt{3} - \sqrt[3]{4}\pi\right)}{\left(5 - 3\sqrt[3]{4}\right)\pi}\right)A^{1/3}(a,b)\,C^{2/3}(a,b) \tag{84}$$

for all $a, b > 0$ with $a \neq b$ follows easily from (77) and (79) together with (83) and the piecewise monotonicity of $J(x)$.

Note that

$$\lim_{\mu \to 0^+} \frac{\mu/\tan^{-1}(\mu) - \left(1 + \mu^2\right)^{1/3}}{1/3 + 2\left(1 + \mu^2\right)^{1/2}/3 - \left(1 + \mu^2\right)^{1/3}} = \frac{4}{5}, \tag{85}$$

$$\lim_{\mu \to \sqrt{3}^-} \frac{\mu/\tan^{-1}(\mu) - \left(1 + \mu^2\right)^{1/3}}{1/3 + 2\left(1 + \mu^2\right)^{1/2}/3 - \left(1 + \mu^2\right)^{1/3}} = \frac{3\left(3\sqrt{3} - \sqrt[3]{4}\pi\right)}{\left(5 - 3\sqrt[3]{4}\right)\pi}. \tag{86}$$

Therefore, Theorem 4 follows from (76) and (82) together with (84)–(86). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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