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Research Article

A Strong Convergence Algorithm for the Two-Operator Split Common Fixed Point Problem in Hilbert Spaces

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The two-operator split common fixed point problem (two-operator SCFP) with firmly nonexpansive mappings is investigated in this paper. This problem covers the problems of split feasibility, convex feasibility, and equilibrium and can especially be used to model significant image recovery problems such as the intensity-modulated radiation therapy, computed tomography, and the sensor network. An iterative scheme is presented to approximate the minimum norm solution of the two-operator SCFP problem. The performance of the presented algorithm is compared with that of the last algorithm for the two-operator SCFP and the advantage of the presented algorithm is shown through the numerical result.

1. Introduction

Throughout this paper, $\mathscr{H}$ denotes a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$, I the identity mapping on $\mathscr{H}$, $\mathbb{N}$ the set of all natural numbers, $\mathbb{R}$ the set of all real numbers, and P_Ω the metric projection onto set Ω. $\overline{x}$ is the upper bound of sequence $\{x_n\}$, while $\underline{x}$ is the lower bound. For a self-mapping T on $\mathscr{H}$, $\mathrm{Fix}(T)$ denotes the set of all fixed points of T.

It has been an interesting topic of finding zero points of maximal monotone operators. A set-valued map $M:\mathscr{H}\to 2^{\mathscr{H}}$ with domain $\mathscr{D}(M)$ is called monotone if

$$\langle x-y,u-v\rangle\geq 0 \tag{1}$$

for all x, $y\in\mathscr{D}(M)$ and for any $u\in M(x)$ and $v\in M(y)$, where $\mathscr{D}(M)$ is defined to be

$$\mathscr{D}(M)=\{x\in\mathscr{H}:Mx\neq\varnothing\}. \tag{2}$$

M is said to be maximal monotone if its graph $\{(x,u):x\in\mathscr{H},u\in M(x)\}$ is not properly contained in the graph of any other monotone operator. For a positive real number α, we denote by J_α^M the resolvent of a monotone operator M; that is, $J_\alpha^M(x)=(I+\alpha M)^{-1}(x)$ for any $x\in\mathscr{H}$. A point $v\in\mathscr{H}$ is called a zero point of a maximal monotone operator M if $0\in M(v)$. In the sequel, we will denote the set of all zero points of A by $M^{-1}0$, which is equal to $\mathrm{Fix}(J_\alpha^M)$ for any $\alpha>0$. A well-known method to solve this problem is the proximal point algorithm which starts with any initial point $x_1\in\mathscr{H}$ and then generates the sequence $\{x_n\}$ in $\mathscr{H}$ by

$$x_{n+1}=J_{\alpha_n}^A x_n,\quad n\in\mathbb{N}, \tag{3}$$

where $\{\alpha_n\}$ is a sequence of positive real numbers. This algorithm was first introduced by Martinet [1] and then generally studied by Rockafellar [2], who devised the iterative sequence $\{x_n\}$ by

$$x_{n+1}=J_{\alpha_n}^A x_n+e_n,\quad n\in\mathbb{N}, \tag{4}$$

where $\{e_n\}$ is an error sequence in $\mathscr{H}$. Rockafellar showed that the sequence $\{x_n\}$ generated by (4) converges weakly to an element of $A^{-1}0$ provided that $A^{-1}0\neq\varnothing$ and $\liminf_{n\to\infty}\alpha_n>0$. Since then, many authors have conducted research on modifying the sequence in (4) so that the strong convergence is guaranteed; compare [3–12] and the references therein.

On the other hand, let C and Q be nonempty closed convex subsets of two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and let $A : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear mapping. The split feasibility problem (SFP) is the problem of finding a point with the property:

$$x^* \in C, \quad Ax^* \in Q. \tag{5}$$

The SFP was first introduced by Censor and Elfving [13] for modeling inverse problems which arise from phase retrievals and medical image reconstruction. Recently, it has been found that the SFP can also be used to model the intensity-modulated radiation therapy. The most popular algorithm for the SFP is the CQ algorithm introduced by Byrne [14, 15]. The sequence $\{x_n\}$ generated by the CQ algorithm converges weakly to a solution of SFP (5); compare [14–16]. Under the assumption that SFP (5) has a solution, there are many algorithms designed to approximate a solution of SFP; compare [16–23] and the references therein.

Later, Censor and Segal [24] extended the SFP to the split common fixed point problem (SCFP) which is to find a point x^* with the property:

$$x^* \in \bigcap_{i=1}^{p} \mathrm{Fix}(S_i), \quad Ax^* \in \bigcap_{j=1}^{r} \mathrm{Fix}(T_j), \tag{6}$$

where S_i, $i = 1, \ldots, p$, and T_j, $j = 1, \ldots, r$, are directed operators in Hilbert spaces. Censor and Segal [24] gave an algorithm for SCFP (6) in $\mathbb{R}^n$ spaces. Then, Moudafi [25] named SCFP (6) with $p = 1$ the two-operator SCFP and gave an algorithm which generates a sequence weakly converging to the solution of the two-operator SCFP. Till very recently, Cui et al. [26] provided a damped projection algorithm, shown as below, to approach the solution of SCFP (6).

Assume that the solution set Ω of the SCFP is nonempty. Start with any $x_1 \in \mathscr{H}_1$ and generate a sequence $\{x_n\}$ through the iteration:

$$\begin{aligned} x_{n+1} = {} & (1-b_n)\, x_n \\ & + b_n S_n \left[(1-a_n)\left(x_n - \gamma_n A^* (I - T_n) A x_n\right)\right], \end{aligned} \tag{7}$$

where $\{a_n\} \subset (0,1)$, $\{b_n\} \subset [0,1]$, and $\gamma_n \subset (0,\infty)$ satisfying that

(i) $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;

(ii) $\liminf_{n\to\infty} b_n > 0$;

(iii) $0 < \underline{\gamma} \le \gamma_n \le \overline{\gamma} < 2/\|A\|^2$.

Then, the sequence $\{x_n\}$ converges strongly to $p = P_\Omega 0$.

Inspired by the work of [25, 26], this paper presents another algorithm to find the minimum norm solution of two-operator SCFP. We note that the two-operator SCFP contains the SFP and the zero point problem of maximal monotone operators. Let P_C and P_Q be metric projections onto C and Q, respectively. Putting $S_1 = P_C$ and $T_1 = P_Q$, the two-operator SCFP (6) is reduced to SFP (5). Let M and N be two maximal monotone operators on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Replacing C and Q with $M^{-1}0 = \mathrm{Fix}(J_\alpha^M)$ and $N^{-1}0 = \mathrm{Fix}(J_\beta^N)$, respectively, in (6), the SFP becomes a two-operator SCFP:

$$\text{Find } x^* \in \mathscr{H}_1 \text{ so that } x^* \in \mathrm{Fix}\left(J_\alpha^M\right),\ Ax^* \in \mathrm{Fix}\left(J_\beta^N\right). \tag{8}$$

Putting $A = I$, the above two-operator SCFP is reduced to the common zero point problem of two maximal monotone operators M and N:

$$\text{Find } x^* \in \mathscr{H} \text{ so that } x^* \in M^{-1}0 \cap N^{-1}0. \tag{9}$$

Let S be S_1 in the SCFP (6), and let T be T_1. The target of the two-operator SCFP (6) is to find a fixed point of directed operator S. Since the definition of a directed operator is based on its fixed point set, it may be difficult to show that S is a directed operator before the two-operator SCFP is solved. Therefore, S and T are only considered as firmly nonexpansive mappings in our presented algorithm. The main result in this paper is as follows.

Let S and T be two firmly nonexpansive self-mappings on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Assume that the solution set Ω of the two-operator SCFP is nonempty. For any $u \in \mathscr{H}_1$, start with any $x_1 \in \mathscr{H}_1$ and define the sequence $\{x_n\}$ by

$$\begin{aligned} y_n &= x_n - \gamma A^* (I - T) A x_n, \\ x_{n+1} &= a_n u + (1-a_n)\left[b_n x_n + (1-b_n) S y_n\right], \end{aligned} \tag{10}$$

where $\gamma \in (0, 1/\|A\|^2)$ and $\{a_n\}$ and $\{b_n\}$ are sequences in $(0,1]$ satisfying that

(i) $\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;

(ii) $\liminf_{n\to\infty} b_n(1-b_n) > 0$.

Then the sequence $\{x_n\}$ converges strongly to $p = P_\Omega u$.

The two-operator SCFP covers problems of split feasibility, convex feasibility, and equilibrium as special cases. The presented algorithm can be considered as a unified methodology for solving the aforementioned problems. In Section 4, we use the numerical result to prove that the performance of the presented algorithm is more efficient and more consistent than that of the recent damped projection algorithm [26].

2. Preliminaries

In order to facilitate our investigation in this paper, we recall some basic facts. A mapping $S : \mathscr{H} \to \mathscr{H}$ is said to be

(i) nonexpansive if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in \mathscr{H}; \tag{11}$$

(ii) firmly nonexpansive if

$$\|Sx - Sy\|^2 \le \langle x - y, Sx - Sy \rangle, \quad \forall x, y \in \mathscr{H}; \tag{12}$$

(iii) directed if

$$\langle Tx - x, Tx - q \rangle \le 0, \quad \text{for } x \in \mathscr{H},\ q \in \mathrm{Fix}(T). \tag{13}$$

It is well-known that the fixed point set $\mathrm{Fix}(S)$ of a nonexpansive mapping S is closed and convex; compare [27].

Let C be a nonempty closed convex subset of $\mathscr{H}$. The metric projection P_C from $\mathscr{H}$ onto C is the mapping that assigns each $x \in \mathscr{H}$ the unique point $P_C x$ in C with the property

$$\|x - P_C x\| = \min_{y \in C} \|y - x\|. \tag{14}$$

It is known that P_C is firmly nonexpansive and characterized by the inequality, for any $x \in \mathscr{H}$,

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \quad \forall y \in C. \tag{15}$$

There is a strongly convergent algorithm for a nonexpansive mapping S with $\mathrm{Fix}(S) \neq \varnothing$, which is related to the iteration scheme in our main result; for any $u \in \mathscr{H}$, choose arbitrarily a point $x_1 \in \mathscr{H}$ and define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = a_n u + (1 - a_n) S x_n, \quad n \in \mathbb{N}, \tag{16}$$

where $\{a_n\}$ is sequence in $[0, 1]$ satisfying

$$\lim_{n\to\infty} a_n = 0, \qquad \sum_{n=1}^{\infty} a_n = \infty, \qquad \sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty. \tag{17}$$

Then, the sequence $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(S)} u$; compare [28, 29].

We need some lemmas that will be quoted in the sequel.

Lemma 1. *For any $x, y \in \mathscr{H}_1$ and $\lambda \in \mathbb{R}$, the following hold:*

(a) $\|\lambda x + (1 - \lambda) y\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$;

(b) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$.

Lemma 2 (see [27], demiclosedness principle). *Suppose that G is a nonexpansive self-mapping on $\mathscr{H}$ and suppose that $\{x_n\}$ is a sequence in $\mathscr{H}$ such that $\{x_n\}$ converges weakly to some $z \in \mathscr{H}$ and $\lim_{n\to\infty}\|x_n - Gx_n\| = 0$. Then, $Gz = z$.*

Lemma 3. *Let M be a maximal monotone operator on $\mathscr{H}$. Then*

(a) *J_α^M is single-valued and firmly nonexpansive;*

(b) *$\mathscr{D}(J_\alpha^A) = \mathscr{H}$ and $\mathrm{Fix}(J_\alpha^A) = A^{-1}0$.*

Lemma 4 (see [12]). *Suppose that $\{z_n\}$ is a sequence of nonnegative real numbers satisfying*

$$z_{n+1} \leq (1 - a_n) z_n + a_n v_n, \quad n \in \mathbb{N}, \tag{18}$$

where $\{a_n\}$ and $\{v_n\}$ verify the following conditions:

(i) $\{a_n\} \subseteq [0, 1]$, $\sum_{n=1}^{\infty} a_n = \infty$;

(ii) $\limsup_{n\to\infty} v_n \leq 0$.

Then $\lim_{n\to\infty} z_n = 0$.

Lemma 5 (see [30]). *Let $\{z_n\}$ be a sequence in $\mathbb{R}$ that does not decrease at infinity in the sense that there exists a subsequence $\{z_{n_i}\}$ such that*

$$z_{n_i} < z_{n_i+1}, \quad \forall i \in \mathbb{N}. \tag{19}$$

For any $k \in \mathbb{N}$, define $m_k = \max\{j \leq k : z_j < z_{j+1}\}$. Then $m_k \to \infty$ as $k \to \infty$ and $\max\{z_{m_k}, z_k\} \leq z_{m_k+1}$, $\forall k \in \mathbb{N}$.

3. Main Theorems

Throughout this section, S and T denote two firmly nonexpansive self-mappings on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and A denotes a bounded linear operator from $\mathscr{H}_1$ to $\mathscr{H}_2$.

Under the assumption that the solution set of two-operator SCFP is nonempty, the following lemma says that the two-operator SCFP is equivalent to the fixed point problem for the operator $S[I - \gamma A^*(I - T)A]$.

Lemma 6 (see [17]). *Let Ω be the solution set of two-operator SCFP (6); that is, $\Omega = \mathrm{Fix}(S) \cap A^{-1}(\mathrm{Fix}(T))$. For any $\gamma \in (0, 2/\|A\|^2)$, let $U = I - \gamma A^*(I - T)A$. Suppose that $\Omega \neq \varnothing$. Then $\mathrm{Fix}(SU) = \mathrm{Fix}(S) \cap \mathrm{Fix}(U) = \Omega$.*

Theorem 7. *Let S and T be two firmly nonexpansive self-mappings on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Assume that the solution set Ω of the two-operator SCFP is nonempty. For any $u \in \mathscr{H}_1$, start with any $x_1 \in \mathscr{H}_1$ and define the sequence $\{x_n\}$ by*

$$\begin{aligned} y_n &= x_n - \gamma A^* (I - T) A x_n \\ x_{n+1} &= a_n u + (1 - a_n) [b_n x_n + (1 - b_n) S y_n], \end{aligned} \tag{20}$$

where $\gamma \in (0, 1/\|A\|^2)$ and $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1]$ satisfying that

(i) *$\lim_{n\to\infty} a_n = 0$ and $\sum_{n=1}^{\infty} a_n = \infty$;*

(ii) *$\liminf_{n\to\infty} b_n(1 - b_n) > 0$.*

Then the sequence $\{x_n\}$ converges strongly to $p = P_\Omega u$.

Proof. Putting $G = S[I - \gamma A^*(I - T)A]$, we see that $Gx_n = Sy_n$, $\forall n \in \mathbb{N}$. By Lemmas 1 and 6, we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \| a_n (u - p) + (1 - a_n) \\ &\quad \times [b_n (x_n - p) + (1 - b_n)(Gx_n - p)] \|^2 \\ &\leq a_n \|u - p\|^2 + (1 - a_n) \\ &\quad \times \left[b_n \|x_n - p\|^2 + (1 - b_n) \|Sy_n - p\|^2 \right. \\ &\quad \left. - b_n (1 - b_n) \|x_n - Gx_n\|^2 \right] \\ &\leq a_n \|u - p\|^2 + (1 - a_n) \\ &\quad \times \left[b_n \|x_n - p\|^2 + (1 - b_n) \|y_n - p\|^2 \right. \\ &\quad \left. - b_n (1 - b_n) \|x_n - Gx_n\|^2 \right]. \end{aligned} \tag{21}$$

In addition,

$$\begin{aligned} \|y_n - p\|^2 &= \|x_n - p - \gamma A^*(I - T)Ax_n\|^2 \\ &= \|x_n - p\|^2 - 2\gamma \langle x_n - p, A^* (I - T) A x_n \rangle \\ &\quad + \gamma^2 \|A^*(I - T)Ax_n\|^2 \\ &\leq \|x_n - p\|^2 - 2\gamma \langle x_n - p, A^* (I - T) A x_n \rangle \\ &\quad + \gamma^2 \|A\|^2 \|(I - T)Ax_n\|^2. \end{aligned} \tag{22}$$

Furthermore, since T is nonexpansive and $Ap \in \mathrm{Fix}(T)$, one has

$$\begin{aligned}\|TAx_n - Ap\|^2 &= \|(Ax_n - Ap) - (I-T)Ax_n\|^2 \\ &= \|Ax_n - Ap\|^2 - 2\langle Ax_n - Ap, (I-T)Ax_n\rangle \\ &\quad + \|(I-T)Ax_n\|^2 \\ &= \|Ax_n - Ap\|^2 - 2\langle x_n - p, A^*(I-T)Ax_n\rangle \\ &\quad + \|(I-T)Ax_n\|^2 \\ &\le \|Ax_n - Ap\|^2,\end{aligned} \tag{23}$$

from which it follows that

$$-2\langle x_n - p, A^*(I-T)Ax_n\rangle \le -\|(I-T)Ax_n\|^2. \tag{24}$$

Therefore, it follows from (21), (22), and (24) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le a_n\|u-p\|^2 + (1-a_n) \\ &\quad \times \left[\|x_n - p\|^2 - (1-b_n)\gamma\left(1-\gamma\|A\|^2\right)\right. \\ &\quad \left.\times \|(I-T)Ax_n\|^2 - b_n(1-b_n)\|Gx_n - x_n\|^2\right] \\ &\le a_n\|u-p\|^2 + (1-a_n)\|x_n - p\|^2.\end{aligned} \tag{25}$$

Hence, by induction, we see that

$$\|x_{n+1} - p\|^2 \le \max\left\{\|u-p\|^2, \|x_1 - p\|^2\right\}. \tag{26}$$

This shows that $\{x_n\}$ is bounded. Now, by Lemma 1 and (22), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|a_n(u-p) + (1-a_n) \\ &\quad \times [b_n(x_n - p) + (1-b_n)(Gx_n - p)]\|^2 \\ &\le (1-a_n) \\ &\quad \times \|b_n(x_n - p) + (1-b_n)(Gx_n - p)\|^2 \\ &\quad + 2a_n\langle u-p, x_{n+1} - p\rangle \\ &= (1-a_n)\left[b_n\|x_n - p\|^2 + (1-b_n)\|Sy_n - p\|^2\right. \\ &\quad \left. - b_n(1-b_n)\|Gx_n - x_n\|^2\right] \\ &\quad + 2a_n\langle u-p, x_{n+1} - p\rangle \\ &\le (1-a_n)\left[b_n\|x_n - p\|^2 + (1-b_n)\|y_n - p\|^2\right. \\ &\quad \left. - b_n(1-b_n)\|Gx_n - x_n\|^2\right] \\ &\quad + 2a_n\langle u-p, x_{n+1} - p\rangle \\ &\le (1-a_n)\left[\|x_n - p\|^2 - (1-b_n)\gamma\left(1-\gamma\|A\|^2\right)\right. \\ &\quad \times \|(I-T)Ax_n\|^2 - b_n(1-b_n) \\ &\quad \left.\times \|Gx_n - x_n\|^2\right] \\ &\quad + 2a_n\langle u-p, x_{n+1} - p\rangle \\ &\le (1-a_n)\left[\|x_n - p\|^2 - b_n(1-b_n)\|Gx_n - x_n\|^2\right] \\ &\quad + 2a_n\langle u-p, x_{n+1} - p\rangle.\end{aligned} \tag{27}$$

We now carry on with the proof by considering the following two cases: (I) $\{\|x_n - p\|\}$ is eventually decreasing and (II) $\{\|x_n - p\|\}$ is not eventually decreasing.

Case I. Suppose that $\{\|x_n - p\|\}$ is eventually decreasing; that is, there is $n_0 \in \mathbb{N}$ such that $\{\|x_n - p\|\}_{n\ge n_0}$ is decreasing. In this case, $\lim_{n\to\infty}\|x_n - p\|$ exists in $\mathbb{R}$. From inequality (27), we have

$$\begin{aligned}(1-a_n)b_n(1-b_n)\|Gx_n - x_n\|^2 &\le (1-a_n)\|x_n - p\|^2 \\ &\quad + 2a_n\langle u-p, x_{n+1} - p\rangle \\ &\quad - \|x_{n+1} - p\|^2,\end{aligned} \tag{28}$$

which together with the boundedness of $\{x_n\}$ and conditions (i) and (ii) implies

$$\lim_{n\to\infty}\|Gx_n - x_n\| = 0. \tag{29}$$

Since $\{x_n\}$ is bounded, it has a subsequence $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges weakly to some $z \in \mathscr{H}$ and

$$\begin{aligned}\limsup_{n\to\infty}\langle u-p, x_{n+1} - p\rangle &= \lim_{k\to\infty}\left\langle u-p, x_{n_k} - p\right\rangle \\ &= \langle u-p, z-p\rangle \le 0,\end{aligned} \tag{30}$$

where the last inequality follows from (15) since $z \in \Omega$ by Proposition 8 of [17], (29), and Lemmas 2 and 6. Moreover, from (27), we have

$$\|x_{n+1} - p\|^2 \le (1-a_n)\|x_n - p\|^2 + 2a_n\langle u-p, x_{n+1} - p\rangle. \tag{31}$$

Accordingly, applying Lemma 4 to inequality (31), we conclude that

$$\lim_{n\to\infty} x_n = p. \tag{32}$$

Case II. Suppose that $\{\|x_n - p\|\}$ is not eventually decreasing. In this case, by Lemma 5, there exists a nondecreasing sequence $\{m_k\}$ in $\mathbb{N}$ such that $m_k \to \infty$ and

$$\max\left\{\left\|x_{m_k} - p\right\|, \left\|x_k - p\right\|\right\} \le \left\|x_{m_k+1} - p\right\|, \quad \forall k \in \mathbb{N}. \tag{33}$$

Then it follows from (27) and (33) that

$$\begin{aligned}\left\|x_{m_k} - p\right\|^2 &\le \left\|x_{m_k+1} - p\right\|^2 \\ &\le \left(1 - a_{m_k}\right)\Big[\left\|x_{m_k} - p\right\|^2 \\ &\qquad -b_{m_k}\left(1 - b_{m_k}\right)\left\|Gx_{m_k} - x_{m_k}\right\|^2\Big] \\ &\quad + 2a_{m_k}\left\langle u - p, x_{m_k+1} - p\right\rangle.\end{aligned} \tag{34}$$

Therefore,

$$\begin{aligned}0 &\le \left(1 - a_{m_k}\right) b_{m_k}\left(1 - b_{m_k}\right)\left\|Gx_{m_k} - x_{m_k}\right\|^2 \\ &\le -a_{m_k}\left\|x_{m_k} - p\right\|^2 + 2a_{m_k}\left\langle u - p, x_{m_k+1} - p\right\rangle,\end{aligned} \tag{35}$$

which implies that

$$\lim_{k\to\infty}\left\|x_{m_k} - Gx_{m_k}\right\| = 0, \tag{36}$$

and then it follows that

$$\limsup_{k\to\infty}\left\langle u - p, x_{m_k+1} - p\right\rangle \le 0. \tag{37}$$

From (35), we obtain

$$\left\|x_{m_k} - p\right\|^2 \le 2\left\langle u - p, x_{m_k+1} - p\right\rangle, \tag{38}$$

and thus, letting $k \to \infty$, we obtain

$$\lim_{k\to\infty}\left\|x_{m_k} - p\right\| = 0. \tag{39}$$

Also, since

$$\begin{aligned}\left\|x_{m_k+1} - x_{m_k}\right\| &\le a_{m_k}\left\|u - x_{m_k}\right\| \\ &\quad + \left(1 - a_{m_k}\right)\left(1 - b_{m_k}\right)\left\|Gx_{m_k} - x_{m_k}\right\|,\end{aligned} \tag{40}$$

which together with (36) and conditions (i) and (ii) implies that $\lim_{k\to\infty}\|x_{m_k+1} - x_{m_k}\| = 0$,

$$\lim_{k\to\infty}\left\|x_{m_k+1} - p\right\| = 0 \tag{41}$$

by virtue of (39). Consequently, we conclude that $\lim_{k\to\infty}\|x_k - p\| = 0$ via (33) and (41). This completes the proof. □

This theorem says that the sequence $\{x_n\}$ converges strongly to a point of Ω which is nearest to u. In particular, if u is taken to be 0, then the limit point p of the sequence $\{x_n\}$ is the unique minimum solution of two-operator SCFP (6).

Corollary 8. *Let C and Q be nonempty closed convex subsets of two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Assume that the solution set Ω of the SFP is nonempty. For any $u \in \mathscr{H}_1$, start with any $x_1 \in \mathscr{H}_1$ and define a sequence $\{x_n\}$ iteratively by*

$$\begin{aligned}y_n &= x_n - \gamma A^*\left(I - P_Q\right) Ax_n \\ x_{n+1} &= a_n u + \left(1 - a_n\right)\left[b_n x_n + \left(1 - b_n\right) P_C y_n\right],\end{aligned} \tag{42}$$

where $\gamma \in (0, 1/\|A\|^2)$ and $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1]$ satisfying that

(i) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;

(ii) $\liminf_{n\to\infty} b_n(1 - b_n) > 0$.

Then the sequence $\{x_n\}$ converges strongly to $p = P_\Omega u$.

Proof. Putting $S = P_C$ and $T = P_Q$ in (20), the conclusion follows from Theorem 7. □

Corollary 9. *Suppose that M and N are two maximal monotone operators on $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively. Assume that the solution set Ω of problem*

$$\text{Find } x^* \in \mathscr{H}_1 \text{ so that } x^* \in M^{-1}0,\ Ax^* \in N^{-1}0 \tag{43}$$

is nonempty. Let $\alpha, \beta \in (0, \infty)$. For any $u \in \mathscr{H}_1$, start with any $x_1 \in \mathscr{H}_1$ and define a sequence $\{x_n\}$ iteratively by

$$\begin{aligned}y_n &= x_n - \gamma A^*\left(I - J_\beta^N\right) Ax_n, \\ x_{n+1} &= a_n u + \left(1 - a_n\right)\left[b_n x_n + \left(1 - b_n\right) J_\alpha^M y_n\right],\end{aligned} \tag{44}$$

where $\gamma \in (0, 1/\|A\|^2)$ and $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1]$ satisfying that

(i) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;

(ii) $\liminf_{n\to\infty} b_n(1 - b_n) > 0$.

Then the sequence $\{x_n\}$ converges strongly to $p = P_\Omega u$.

Proof. By Lemma 3, a resolvent of a maximal monotone operator is firmly nonexpansive. Hence, we may put $S = J_\alpha^M$ and $T = J_\beta^N$ in (20) to get the conclusion which follows from Theorem 7. □

Corollary 10. *Let M be a maximal monotone operator on $\mathscr{H}$ with $M^{-1}0 \ne \varnothing$, and let $\alpha, \beta \in (0, \infty)$. For any $u \in \mathscr{H}_1$, start with any $x_1 \in \mathscr{H}_1$ and define a sequence $\{x_n\}$ iteratively by*

$$\begin{aligned}y_n &= x_n - \gamma\left(I - J_\beta^M\right) x_n \\ x_{n+1} &= a_n u + \left(1 - a_n\right)\left[b_n x_n + \left(1 - b_n\right) J_\alpha^M y_n\right],\end{aligned} \tag{45}$$

where $\gamma \in (0, 1)$ and $\{a_n\}$ and $\{b_n\}$ are sequences in $(0, 1]$ satisfying that

Table 1: Numerical results for Example 11.

x_1	The damped projection method in [26]			The presented method		
	CPU (sec.)	n	x_n	CPU (sec.)	n	x_n
$(0, 0)^\top$	67.4971	157248	$(0.2929, 0.2929)^\top$	34.8173	91018	$(0.2929, 0.2929)^\top$
$(1, 1)^\top$	125.352	328067	$(0.2929, 0.2929)^\top$	35.0441	91018	$(0.2929, 0.2929)^\top$
$(10, 10)^\top$	411.5836	1052792	$(0.2929, 0.2929)^\top$	38.6464	91018	$(0.2929, 0.2929)^\top$

Table 2: Numerical results for Example 12.

x_1	The damped projection method in [26]			The presented method		
	CPU (sec.)	n	x_n	CPU (sec.)	n	x_n
$(0, 0)^\top$	31.3902	84818	$(0.2929, 0.2929)^\top$	35.5024	91018	$(0.2929, 0.2929)^\top$
$(1, 1)^\top$	142.6763	362480	$(0.2929, 0.2929)^\top$	37.0838	91018	$(0.2929, 0.2929)^\top$
$(10, 10)^\top$	448.3774	1042364	$(0.2928, 0.2930)^\top$	33.8532	91031	$(0.2929, 0.2929)^\top$

(i) $\lim_{n\to\infty} a_n = 0$ *and* $\sum_{n=1}^{\infty} a_n = \infty$;

(ii) $\liminf_{n\to\infty} b_n(1 - b_n) > 0$.

Then, the sequence $\{x_n\}$ *converges strongly to* $p = P_{M^{-1}0}(u)$.

Proof. Putting $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$, $A = I$, $M = N$, and $S = J_\alpha^M$, $T = J_\beta^M$ in Corollary 9, the result follows immediately. □

4. Numerical Results

There are four examples in this section provided to demonstrate our presented algorithm. The first three examples are the SFP, while the fourth example is the common zero point problem of two maximal monotone operators. The performance of the presented algorithm to solve the three examples of SFP is compared with that of the recent damped projection method [26]. The result shows that the presented algorithm is more efficient and more consistent than the damped algorithm. In the first three examples, we assign the parameters in both algorithms to be $u = (0,0)^\top$, $a_n = 1/(n+1)$, $b_n = 0.5$, and $\gamma_n = \gamma = 0.01$. Let $\|x_{n+1} - x_n\| \le 10^{-10}$ be their stop criterion. All codes were written in Matlab R2011a and ran on laptop ASUS ZenbookUX31E with i7-2677M CPU.

Example 11. Let $C = \{(x, y)^\top \mid (x-1)^2 + (y-1)^2 \le 1\}$, $Q = \{(x, y, z)^\top \mid (x-1)^2 + (y-1)^2 + (z-1)^2 \le 9\}$, and

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}. \tag{46}$$

The metric projections for C and Q are

$$P_C(x, y)^\top = \begin{cases} (x, y)^\top, & \text{if } (x, y)^\top \in C; \\ \dfrac{(x-1, y-1)^\top}{\sqrt{(x-1)^2 + (y-1)^2}} + (1, 1)^\top, & \text{if } (x, y)^\top \notin C, \end{cases}$$

$$P_Q(x, y, z)^\top = \begin{cases} (x, y, z)^\top, & \text{if } (x, y, z)^\top \in Q; \\ \dfrac{3(x-1, y-1, z-1)^\top}{\sqrt{(x-1)^2 + (y-1)^2 + (z-1)^2}} \\ \quad +(1, 1, 1)^\top, & \text{if } (x, y, z)^\top \notin Q. \end{cases} \tag{47}$$

Then, we can use both the presented algorithm and the damped projection algorithm to approach a point such that

$$x^* \in \operatorname{Fix}(P_C), \quad Ax^* \in \operatorname{Fix}(P_Q). \tag{48}$$

From Table 1, we observe that the presented algorithm is more efficient than the damped projection algorithm.

Example 12. Let all conditions be the same with those in Example 11 except to

$$A = \begin{bmatrix} 2 & -1 \\ 4 & 2 \\ 2 & 0 \end{bmatrix}. \tag{49}$$

The result for solving Example 12 is shown in Table 2. We observe that the presented algorithm is still more efficient than the damped algorithm. From the columns for the runtime (CPU) and the approximate solution (x_n), the result of the presented algorithm is consistent although it starts from different initial points.

Example 13. In this example, we use A in Example 11 but change its C and Q. Let $C = \{(x, y)^\top \mid (x-1)^2 + (y-3)^2 \le 9\}$ and $Q = \{(x, y, z)^\top \mid (x-6)^2 + (y-15)^2 + (z-22)^2 \le 9\}$. The metric projections for C and Q are

$$P_C(x, y)^\top = \begin{cases} (x, y)^\top, & \text{if } (x, y)^\top \in C; \\ \dfrac{3(x-1, y-3)^\top}{\sqrt{(x-1)^2 + (y-3)^2}} + (1, 3)^\top, & \text{if } (x, y)^\top \notin C, \end{cases}$$

Table 3: Numerical results for Example 13.

x_1	The damped projection method in [26]			The presented method		
	CPU (sec.)	n	x_n	CPU (sec.)	n	x_n
$(0,0)^\top$	382.487	933580	$(1.5845, 2.0122)^\top$	100.475	247651	$(1.5844, 2.0123)^\top$
$(1,1)^\top$	581.5485	1438799	$(1.5846, 2.0121)^\top$	101.4875	247960	$(1.5844, 2.0123)^\top$
$(10,10)^\top$	>1000			100.0661	252832	$(1.5844, 2.0123)^\top$

Table 4: Numerical results for Example 13 with $u = (3,3)^\top$.

x_1	The presented method		
	CPU (sec.)	n	x_n
$(0,0)^\top$	64.0093	159081	$(2.3455, 2.1812)^\top$
$(1,1)^\top$	66.2390	159477	$(2.3455, 2.1812)^\top$
$(10,10)^\top$	66.0244	172465	$(2.3455, 2.1812)^\top$

$$P_Q(x,y,z)^\top = \begin{cases} (x,y,z)^\top, & \text{if } (x,y,z)^\top \in Q; \\ \dfrac{3(x-6, y-15, z-22)^\top}{\sqrt{(x-6)^2+(y-15)^2+(z-22)^2}} \\ \quad +(6,15,22)^\top, & \text{if } (x,y,z)^\top \notin Q. \end{cases} \tag{50}$$

The result is shown in Table 3. We also observe that the presented algorithm is more efficient and more consistent than the damped projection algorithm.

The presented algorithm contains an arbitrary point u and that is an advantage of the algorithm. Knowing any information about the solution of two-operator SCFP of interest, we can choose a better u to enhance the performance of the presented algorithm. For instance, let $u = (3,3)^\top$ which is different with $u = (0,0)^\top$ related to the result in Table 3. From Table 4, we observe that the runtime of the presented algorithm is reduced by one-third.

Example 14. Minimizing a convex function is called a convex minimization problem. This example shows that the presented algorithm can be used to search the common optimal solutions of two convex minimization problems. Let f and g be two functions from $\mathbb{R}^2$ to $\mathbb{R}$ and define $f(x_1,x_2) = x_1^2 + x_2^2 + 1 - 2x_1x_2 - 2x_1 + 2x_2$ and $g(x_1,x_2) = x_1^2 + x_2^2 + 1 + 2x_1x_2 - 2x_1 - 2x_2$. We know that both f and g are convex functions. Now, we would like to search a common minimal point of the two convex functions.

Let $\partial f/\partial x_i$ denote the partial derivative of function f with respect to x_i. Define two operators M and N from $\mathbb{R}^2$ to $\mathbb{R}$ by

$$M = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \end{bmatrix},$$

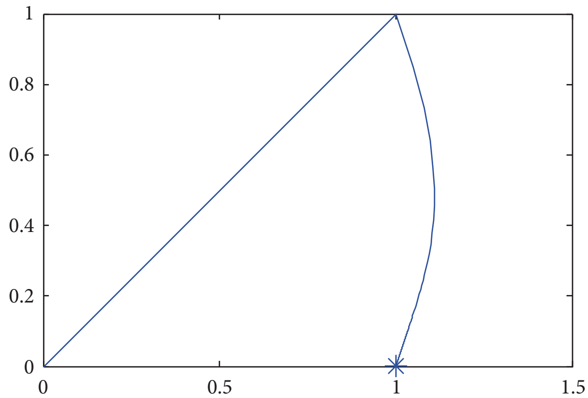

Figure 1: The behavior of our presented algorithm to search the common minimal point of two convex minimization problems. The star sign marks the stop point, $\tilde{x}$, of the algorithm.

$$N = \begin{bmatrix} \dfrac{\partial g}{\partial x_1} \\ \dfrac{\partial g}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix}. \tag{51}$$

Since f and g are convex functions, M and N are maximal monotone operators and any one of their common zero points is the common minimal point of f and g. The resolvents of M and N are

$$\begin{aligned} J_\alpha^M \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 2+\alpha & -2 \\ -2 & 2+\alpha \end{bmatrix}^{-1} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \alpha \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right), \\ J_\alpha^N \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 2+\alpha & 2 \\ 2 & 2+\alpha \end{bmatrix}^{-1} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \alpha \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right). \end{aligned} \tag{52}$$

According to Corollary 9, our presented algorithm can be used to search a common zero point of M and N. Let $\alpha = 1$, $u = (1,1)^\top$, $a_n = 1/(n+1)$, $b_n = 0.5$, and $\gamma = 0.5$ in the algorithm, and let $\|x_{n+1} - x_n\| \le 10^{-6}$ be the stop criterion. We ran the algorithm and started from point $x_1 = (0,0)^\top$. The algorithm stopped at point $\tilde{x} = (1.0006, 0.0019)^\top$ after 1,988 iterations. We know that $M\tilde{x} \approx 0$ and $N\tilde{x} \approx 0$. Finally, we use Figure 1 to show the behavior of sequence $\{x_n\}$ which converges to the common minimal point of f and g.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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