Research Article

Soliton and Breather Solutions for the Mixed Nonlinear Schrödinger Equation via N-Fold Darboux Transformation

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Under investigation in this paper is the mixed nonlinear Schrödinger equation, which describes the propagation of the subpicosecond or femtosecond optical pulse in a monomodal optical fiber. The Darboux transformation is constructed and N-times iterative potential formula is presented. Two-soliton and breather solutions are derived on the vanishing and two types of nonvanishing backgrounds: the continuous wave(cw) background and constant background, respectively. The dynamic features of the solitons and breathers are discussed via analytic solutions and graphical illustration.

1. Introduction

Investigations on the nonlinear Schrödinger equation have attracted certain interest in nonlinear optics [1–6]. In this paper, we mainly investigate the following mixed nonlinear Schrödinger equation (MNLS) [7]:

$$iu_z + \frac{1}{2}u_{tt} + \sigma\frac{1}{4}|u|^2 u - \sigma\frac{3i\delta}{16}\left(|u|^2 u\right)_t + \Gamma u = 0, \qquad (1)$$

where u denotes the slowly varying complex pulse envelope, z and t are the partial derivatives with respect to the longitudinal distance and retarded time, $\sigma = \pm 1$ corresponds to the focusing and defocusing nonlinear Schrödinger (NLS) equation, δ is a real parameter describing the measure of the derivative cubic nonlinearity, and Γ denotes the loss or gain coefficient. Equation (1) usually governs the subpicosecond or femtosecond optical pulse propagation in a monomodal optical fiber.

As a complete integrability model, (1) is equivalent to the following Wadati-Konno-Ichikawa (WKI) spectral system [7, 8]:

$$\Psi_t = U\Psi = \begin{pmatrix} \dfrac{2i\left(\lambda^2 - 1\right)}{3\delta} & \dfrac{i}{2}\lambda u \\[2ex] \dfrac{i}{2}\lambda u^* & \dfrac{2i\left(\lambda^2 - 1\right)}{3\delta} \end{pmatrix}\Psi, \qquad (2a)$$

$$\Psi_z = V\Psi = \begin{pmatrix} -\dfrac{4i}{9\delta^2}\left(\lambda^2 - 1\right)^2 + \dfrac{i}{8}\lambda^2|u|^2 + i\dfrac{\Gamma}{2} & -\dfrac{i\lambda\left(\lambda^2 - 1\right)}{3\delta}u + \dfrac{3i}{32}\delta\lambda|u|^2 u - \dfrac{\lambda}{4}u_t \\[2ex] -\dfrac{i\lambda\left(\lambda^2 - 1\right)}{3\delta}u^* + \dfrac{3i}{32}\delta\lambda|u|^2 u^* + \dfrac{\lambda}{4}u_t^* & \dfrac{4i}{9\delta^2}\left(\lambda^2 - 1\right)^2 - \dfrac{i}{8}\lambda^2|u|^2 - i\dfrac{\Gamma}{2} \end{pmatrix}\Psi, \qquad (2b)$$

where $\Psi = (\psi_1, \psi_2)'$ (' denotes the transpose of a matrix). One can deduce (1) directly from the zero curvature equation $U_z - V_t + UV - VU = 0$.

During the past decades, [8–10] have derived some soliton solutions via the dressing method, the Hirota bilinear method and the technique of determinant calculation, respectively. However, to our knowledge, the soliton and breather solutions of (1) have not been generated through N-fold Darboux transformation and the interaction characters between two breathers have not been analysed. The aim of this paper is mainly to construct N-fold Darboux transformation (DT) in Section 2, derive two-soliton, one- and two-breather solutions via the obtained DT, and discuss dynamic features of those solitons and breathers in Section 3. Finally, our conclusions will be addressed in Section 4.

2. N-**Fold Darboux Transformation**

Taking the transformation as

$$\Psi' = T\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\Psi$$

$$= \begin{pmatrix} (-1)^N + \sum_{n=1}^{N} a_{2n}\lambda^{2n} & \sum_{n=1}^{N} b_{2n-1}\lambda^{2n-1} \\ \sum_{n=1}^{N} c_{2n-1}\lambda^{2n-1} & (-1)^N + \sum_{n=1}^{N} d_{2n}\lambda^{2n} \end{pmatrix}\Psi, \quad (3)$$

we require that Ψ' must satisfy

$$\Psi'_t = U'\Psi' = (T_t + TU)T^{-1}\Psi', \quad (4a)$$

$$\Psi'_z = V'\Psi' = (T_z + TV)T^{-1}\Psi', \quad (4b)$$

where U', V' have the same form as U, V. From the rules of the Darboux transformation [11–14], we know that $T|_{\lambda=\lambda_1}\Psi = 0$, a_{2n}, b_{2n-1}, c_{2n-1}, and d_{2n} can be calculated by the linear algebraic system:

$$\sum_{n=1}^{N}\left(a_{2n}\lambda_j + \alpha_j b_{2n-1}\right)\lambda_j^{2n-1} = (-1)^N,$$

$$\sum_{n=1}^{N}\left(c_{2n-1} + \alpha_j d_{2n}\lambda_j\right)\lambda_j^{2n-1} = (-1)^{N+1}\alpha_j, \quad (5)$$

with

$$\alpha_j = \frac{\phi_2(\lambda_j) - \gamma_j\psi_2(\lambda_j)}{\phi_1(\lambda_j) - \gamma_j\psi_1(\lambda_j)}, \quad 1 \le j \le 2N, \quad (6)$$

and $\phi = (\phi_1, \phi_2)'$, $\psi = (\psi_1, \psi_2)'$ as two basic solutions of the spectral system (2a)-(2b) and λ_j and γ_j ($\lambda_k \ne \lambda_j$, $\gamma_k \ne \gamma_j$, as $k \ne j$) as parameters suitably chosen such that the determinant of coefficients for (5) is nonzero. Thus, a_{2n}, b_{2n-1}, c_{2n-1}, and d_{2n} are uniquely determined by (5). Equations (3) and (5) show that $\det T(\lambda)$ is a $2N$ polynomial of λ, and

$$\det T(\lambda_j) = A(\lambda_j)D(\lambda_j) - B(\lambda_j)C(\lambda_j),$$

$$A(\lambda_j) = -\alpha_j B(\lambda_j), \quad C(\lambda_j) = -\alpha_j D(\lambda_j). \quad (7)$$

Therefore, one can derive that $\det T(\lambda) = \prod_{j=1}^{2N}(\lambda - \lambda_j) = 0$, so λ_j ($j = 1, 2, \ldots$) are $2N$ roots for the determinant of matrix $T(\lambda)$.

Meanwhile, defining

$$(T_t + TU)T^* = \begin{pmatrix} f_{11}(\lambda) & f_{12}(\lambda) \\ f_{21}(\lambda) & f_{22}(\lambda) \end{pmatrix},$$

$$(T_z + TV)T^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}, \quad (8)$$

one can deduce that $f_{11}(\lambda)$ and $f_{22}(\lambda)$ are $(4N + 2)$, $f_{12}(\lambda)$ and $f_{21}(\lambda)$ are $4N + 1$, $g_{11}(\lambda)$ and $g_{22}(\lambda)$ are $(4N + 4)$, $g_{12}(\lambda)$ and $g_{21}(\lambda)$ are $4N + 3$th-order polynomials of λ, and

$$\alpha_{jt} = \frac{i}{2}\lambda_j u^* - \frac{4i(\lambda_j^2 - 1)}{3\delta}\alpha_j - \frac{i}{2}\lambda_j u\alpha_j^2, \quad (9a)$$

$$\alpha_{jz} = c(\lambda_j) - 2a(\lambda_j)\alpha_j - b(\lambda_j)\alpha_j^2, \quad (9b)$$

where

$$a(\lambda_j) = -\frac{4i}{9\delta^2}(\lambda_j^2 - 1)^2 + \frac{i}{8}\lambda_j^2|u|^2,$$

$$b(\lambda_j) = -\frac{i\lambda_j(\lambda_j^2 - 1)}{3\delta}u + \frac{3i}{32}\delta\lambda_j|u|^2 u - \frac{\lambda_j}{4}u_x, \quad (10)$$

$$c(\lambda_j) = -\frac{i\lambda_j(\lambda_j^2 - 1)}{3\delta}u^* + \frac{3i}{32}\delta\lambda_j|u|^2 u^* + \frac{\lambda_j}{4}u_x^*.$$

By virtue of the above conclusions, one can deduce that all λ_j ($1 \le j \le 2N$) are roots of $f_{kj}(\lambda)$, $g_{kj}(\lambda)$ ($k, j = 1, 2$). Noting (7), one can arrive at

$$\det M \mid f_{kj}(\lambda), \quad \det M \mid g_{kj}(\lambda) \quad (k, j = 1, 2). \quad (11)$$

According to $TT^* = \det|T|E$, one can transform (8) into

$$T_t + TU = P(\lambda)T, \quad T_z + TV = Q(\lambda)T, \quad (12)$$

where

$$P(\lambda) = \begin{pmatrix} p_{11}^{(2)}\lambda^2 + p_{11}^{(1)}\lambda + p_{11}^{(0)} & p_{12}^{(1)}\lambda + p_{12}^{(0)} \\ p_{21}^{(1)}\lambda + p_{21}^{(0)} & p_{22}^{(2)}\lambda^2 + p_{22}^{(1)}\lambda + p_{22}^{(0)} \end{pmatrix}, \quad (13a)$$

$$Q(\lambda) = \begin{pmatrix} \sum_{j=0}^{4} q_{11}^{(j)}\lambda^j & \sum_{j=0}^{3} q_{12}^{(j)}\lambda^j \\ \sum_{j=0}^{3} q_{21}^{(j)}\lambda^j & \sum_{j=0}^{4} q_{22}^{(j)}\lambda^j \end{pmatrix}. \quad (13b)$$

Comparing the coefficients of λ^{2N+4}, λ^{2N+3}, λ^{2N+2}, λ^{2N+1}, and λ^{2N} in (12), one can obtain

$$p_{11}^{(2)} = -p_{11}^{(0)} = -p_{22}^{(2)} = p_{22}^{(0)} = \frac{2i}{3\delta}, \tag{14a}$$

$$p_{11}^{(1)} = p_{12}^{(0)} = p_{21}^{(0)} = p_{22}^{(1)} = 0,$$

$$p_{12}^{(1)} = \frac{i}{2}\left(u\frac{a_{2N}}{d_{2N}} - \frac{8}{3\delta}\frac{b_{2N-1}}{d_{2N}}\right) = \frac{i}{2}u', \tag{14b}$$

$$p_{21}^{(1)} = \frac{i}{2}\left(u^*\frac{d_{2N}}{a_{2N}} + \frac{8}{3\delta}\frac{c_{2N-1}}{a_{2N}}\right) = \frac{i}{2}u'^*,$$

$$q_{22}^{(4)} = q_{22}^{(0)} = -q_{11}^{(4)} = -q_{11}^{(0)} = \frac{4i}{9\delta^2}, \tag{14c}$$

$$q_{11}^{(3)} = q_{11}^{(1)} = q_{12}^{(2)} = q_{12}^{(0)} = q_{21}^{(2)} = q_{21}^{(0)} = q_{22}^{(3)} = q_{22}^{(1)} = 0, \tag{14d}$$

$$q_{12}^{(3)} = -\frac{i}{3\delta}\left(u\frac{a_{2N}}{d_{2N}} - \frac{8}{3\delta}\frac{b_{2N-1}}{d_{2N}}\right) = -\frac{i}{3\delta}u', \tag{14e}$$

$$q_{21}^{(3)} = -\frac{i}{3\delta}\left(u^*\frac{d_{2N}}{a_{2N}} + \frac{8}{3\delta}\frac{c_{2N-1}}{a_{2N}}\right) = -\frac{i}{3\delta}u'^*, \tag{14f}$$

$$q_{11}^{(2)} = \frac{8i}{9\delta^2} - \frac{i}{3\delta}\frac{b_{2N-1}}{a_{2N}}u^* + \frac{i}{8}uu^* + q_{12}^{(3)}\frac{c_{2N-1}}{a_{2N}}$$
$$= \frac{8i}{9\delta^2} + \frac{i}{8}u'u'^*, \tag{14g}$$

$$q_{22}^{(2)} = \frac{-8i}{9\delta^2} + \frac{i}{3\delta}\frac{c_{2N-1}}{a_{2N}}u^* - \frac{i}{8}uu^* - q_{21}^{(3)}\frac{b_{2N-1}}{a_{2N}}$$
$$= -\frac{8i}{9\delta^2} - \frac{i}{8}u'u'^*, \tag{14h}$$

$$q_{12}^{(1)} = \frac{1}{d_{2N}}\left(-\frac{8i}{9\delta^2}b_{2N-1} + \frac{8i}{9\delta^2}b_{2N-3} - q_{11}^{(2)}b_{2N-1}\right.$$
$$\left. - q_{12}^{(3)}d_{2N-2} + \frac{iu}{3\delta}a_{2N} - \frac{iu}{3\delta}a_{2N-2} - \frac{i|u|^2}{8}b_{2N-1}\right.$$
$$\left. + \frac{3i\delta}{32}a_{2N}|u|^2u - \frac{u_t}{4}a_{2N}\right), \tag{14i}$$

$$q_{21}^{(1)} = \frac{1}{a_{2N}}\left(\frac{8i}{9\delta^2}c_{2N-1} - \frac{8i}{9\delta^2}c_{2N-3} - q_{22}^{(2)}c_{2N-1} - q_{21}^{(3)}a_{2N-2}\right.$$
$$\left. + \frac{iu^*}{3\delta}d_{2N} - \frac{iu^*}{3\delta}d_{2N-2} + \frac{i|u|^2}{8}c_{2N-1}\right.$$
$$\left. + \frac{3i\delta}{32}d_{2N}|u|^2u^* + \frac{u_t^*}{4}d_{2N}\right). \tag{14j}$$

Substituting $u' = u(a_{2N}/d_{2N}) - (8/3\delta)(b_{2N-1}/d_{2N})$, $u'^* = u^*(d_{2N}/a_{2N}) + (8/3\delta)(c_{2N-1}/a_{2N})$ into (14i) and (14j), one can calculate that

$$q_{12}^{(1)} = \frac{iu'}{3\delta} + \frac{3i\delta}{32}\left|u'\right|^2u' - \frac{1}{4}u'_x,$$
$$q_{21}^{(1)} = \frac{iu'^*}{3\delta} + \frac{3i\delta}{32}\left|u'\right|^2u'^* - \frac{1}{4}u'^*_x. \tag{15}$$

Summarying the above conclusions, one can arrive at the following theorm.

Theorem 1. *The solution q for* (1) *is mapped into the new solution q' under the Darboux transformation:*

$$u' = u\frac{a_{2N}}{d_{2N}} - \frac{8}{3\delta}\frac{b_{2N-1}}{d_{2N}}, \qquad u'^* = u^*\frac{d_{2N}}{a_{2N}} + \frac{8}{3\delta}\frac{c_{2N-1}}{a_{2N}}. \tag{16}$$

3. Soliton and Breather Solutions for (1)

In this section, we will construct soliton solutions for (1). Choosing $\phi(\lambda) = (\phi_1(\lambda), \phi_2(\lambda))'$ and $\psi(\lambda) = (-\psi_2^*(\lambda^*), \psi_1^*(\lambda^*))'$ as two solutions of the spectral system (2a)-(2b), one can derive that $\lambda_{2j} = \lambda_{2j-1}^*$ and $\gamma_{2j} = \gamma_{2j-1}^*$, so $\alpha_{2j-1} = [\phi_2(\lambda_{2j-1}) - \gamma_{2j-1}\psi_2(\lambda_{2j-1})]/[\phi_1(\lambda_{2j-1}) - \gamma_{2j-1}\psi_1(\lambda_{2j-1})]$.

For $N = 1$, from (5), one can obtain that

$$a_2 = \frac{\lambda_1|\phi_2|^2 + \lambda_1^*|\phi_1|^2}{|\lambda_1|^2\lambda_1|\phi_1|^2 + \lambda_1^*|\phi_2|^2},$$

$$b_1 = \frac{\left(\lambda_1^{*2} - \lambda_1^2\right)\phi_1\phi_2^*}{|\lambda_1|^2\lambda_1|\phi_1|^2 + \lambda_1^*|\phi_2|^2}, \tag{17}$$

and $d_2 = a_2^*$, $c_1 = -b_1^*$. Then from (16), one can obtain

$$u^{(1)} = u\left(\frac{\Delta^*}{\Delta}\right)^2 + \frac{8}{3\delta\Delta^2}\left(\lambda_1^2 - \lambda_1^{*2}\right)\phi_1\phi_2^*\Delta^*, \tag{18}$$

with $\Delta = \lambda_1|\phi_1|^2 + \lambda_1^*|\phi_2|^2$.

For $N = 2$, from $d_4 = a_4^*$, $c_3 = -b_3^*$, and (5), one can obtain that

$$\sum_{n=1}^{2}\left(a_{2n}\lambda_{2j-1} + \alpha_{2j-1}b_{2n-1}\right)\lambda_{2j-1}^{2n-1} = 1, \tag{19a}$$

$$\sum_{n=1}^{2}\left(a_{2n}\lambda_{2j-1}^*\alpha_{2j-1}^* - b_{2n-1}\right)\lambda_{2j-1}^{*(2n-1)} = -\alpha_{2j-1}, \tag{19b}$$

with $j = 1, 2$, so (19a)-(19b) result:

$$a_4 = \frac{\Omega_{a_4}}{\Omega}, \qquad b_3 = \frac{\Omega_{b_3}}{\Omega}, \tag{20}$$

where

$$\Omega = \begin{vmatrix} \lambda_1^2 & \lambda_1\alpha_1 & \lambda_1^4 & \alpha_1\lambda_1^3 \\ \lambda_3^2 & \lambda_3\alpha_3 & \lambda_3^4 & \alpha_3\lambda_3^3 \\ \lambda_1^{*2}\alpha_1^* & -\lambda_1^* & \lambda_1^{*4}\alpha_1^* & -\lambda_1^{*3} \\ \lambda_1^{*2}\alpha_3^* & -\lambda_3^* & \lambda_3^{*4}\alpha_3^* & -\lambda_3^{*3} \end{vmatrix}. \tag{21}$$

Ω_{a_4} and Ω_{b_3} are produced from Ω by replacing its third and fourth columns with $(1, 1, -\alpha_1^*, -\alpha_3^*)$, respectively. Then the two-soliton solutions for (1) can be derived as

$$u^{(2)} = u\frac{a_4}{d_4} - \frac{8}{3\delta}\frac{b_3}{d_4}. \tag{22}$$

Then we we mainly discuss soliton solutions from three different seeds.

Case 1. Considering the vanishing background of $u = 0$, we can generate elastic interactions and bound states between two solitons for (1) as shown in Figure 1.

Figure 1(a) depicts the interactions of two solitons on the vanishing background. One can find that main features of the interaction are that the shapes, amplitudes, and pulse widths all remain invariant except for slightly visible phase shifts, so the interaction is elastic. Figure 1(b) shows that when suitable parameters are chosen, two bound solitons with the same amplitude propagate in parallel without any effect on each other even if the propagation distance grows long enough. From Figure 1, we can conclude that the parallel bound solitons will form when $|\lambda_1| \rightarrow |\lambda_2|$, while if the value of $||\lambda_1| - |\lambda_2||$ increases, the elastic interactions between two solitons will happen.

Case 2. In the case of the cw background as $u = u_c \exp i(\kappa t + \omega z)$, we can derive the nonlinear dispersion relation $\omega = (4u_c^2 - 8\kappa^2 + 3u_c^2\kappa\delta + 16\Gamma)/16$ for (1). By the method of separation of variables and the superposition principle, we derive

$$\phi_1 = \left(c_1 e^{\theta_1} + c_2 e^{\theta_2}\right) \exp i\left(\kappa t + \omega z\right), \qquad \phi_2 = c_3 e^{\theta_1} + c_4 e^{\theta_2}, \tag{23}$$

where

$$\theta_1 = \frac{t}{2}\left(-i\kappa - \frac{1}{3\delta}\sqrt{B^2 - A^2}\right) + \frac{z}{2}\left(-i\omega - \frac{1}{3\delta}\sqrt{B^2 - A^2}\Theta\right),$$

$$\theta_2 = \frac{t}{2}\left(-i\kappa + \frac{1}{3\delta}\sqrt{B^2 - A^2}\right) + \frac{z}{2}\left(-i\omega + \frac{1}{3\delta}\sqrt{B^2 - A^2}\Theta\right), \tag{24}$$

with

$$A = 4\left(\lambda_1^2 - 1\right), \qquad B = 3iu_c\delta\lambda_1,$$

$$\Theta = \frac{9\delta^2 u_c^2 - 32\left(\lambda_1^2 - 1\right)}{48\delta}. \tag{25}$$

For simplicity, setting $c_4 = c_1$ and through direct computations, we can generate that

$$c_3 = \frac{\theta_{1t} + i\kappa - \gamma}{(i/2)\lambda_1 u_c} c_1, \qquad c_2 = \frac{(i/2)\lambda_1 u_c}{\theta_{1t} + i\kappa - \gamma} c_4, \tag{26}$$

with $\gamma = 2i(\lambda_1^2 - 1)/3\delta$. Substituting (27) into (18) and taking $\text{Re}(\lambda_1) = \text{Im}(\lambda_1)$, $\kappa = 0$, we can obtain Akhmediev-breather solutions on the nonvanishing background for (1) as shown in Figure 2(a). One can observe that the main feature is propagation of the Akhmediev breather which is periodic in the space coordinate and aperiodic in the time coordinate. Therefore, it is considered as a modulation instability (MI) process in which a cw beam becomes unstable [15]. MI was predicted to occur in optical fibers and was experimentally observed [16].

Iterating the DT again, we can obtain the two-breather solution for (1) as shown in Figures 2(b), 3(a), and 3(b). One can observe that the interactions between those breathers are also elastic. Through adjusting the value of λ_1 and λ_2, we can control the directions of those breathers as shown in those figures.

Case 3. Considering the constant background of $u = \kappa_1 + i\kappa_2$, (1) gives the relation $\kappa_1^2 + \kappa_2^2 = -4\Gamma$. Defining $\lambda = A_c + i\kappa_c$, and for simplicity setting $\kappa_c = \sqrt{8(A_c^2 - 1) - 9\Gamma\delta^2}$, we can obtain

$$\phi_1 = c_1 e^\vartheta + c_2 e^{-\vartheta}, \qquad \phi_2 = c_3 e^\vartheta + c_4 e^{-\vartheta}, \tag{27}$$

where

$$\vartheta = 2t\left(\xi_1 + i\eta_1\right) + z\left(\xi_1 + i\eta_1\right)\frac{8\left(\lambda^2 - 1\right) + 9\Gamma\delta^2}{36\delta^2},$$

$$c_3 = \frac{\vartheta_t - (2i/3\delta)\left(\lambda^2 - 1\right)}{(i/2)\lambda\left(\kappa_1 + i\kappa_2\right)} c_1, \tag{28}$$

$$c_2 = \frac{(i/2)\lambda\left(\kappa_1 + i\kappa_2\right)}{-\vartheta_t - (2i/3\delta)\left(\lambda^2 - 1\right)} c_4,$$

with

$$\xi_1 + i\eta_1 = \sqrt{\beta\left(16 + \beta\right)}, \qquad \beta = 9\Gamma A_c^2. \tag{29}$$

Owing to β which is always negative, we discuss two different cases according to the sign of $(16 + \beta)$.

(A) When $(16 + \beta) > 0$, the value of $\beta(16 + \beta)$ will be negative; then we can arrive at $\xi_1 = 0$, $\eta_1 = \sqrt{-\beta(16 + \beta)}$. Substituting these conclusions and (27) into (18), we can derive the Ma-breather solution for (1) as shown in Figure 4(a).

(B) When $(16 + \beta) < 0$, the value of $\beta(16 + \beta)$ will be positive; then we can arrive at $\xi_1 = \sqrt{\beta(16 + \beta)}$, $\eta_1 = 0$. Substituting these conclusions and (27) into (18), we can derive the Akhmediev-breather solution for (1) as shown in Figure 4(b).

One can observe from Figure 4(a) that the breathers time periodically propagate on the constant backgrounds; that is, they are the Ma-breathers. In addition, as $\kappa_1 + i\kappa_2$ approaches zero, the Ma-breather will become the one-soliton solution. Since $\kappa_1^2 + \kappa_2^2 = -4\Gamma$ and Γ denotes the gain or loss coefficient in in (1), we can conclude that the absence of the gain or loss term can become breathers into one-soliton solutions.

From Figure 4(b), one can find that the Akhmediev breather is periodic in the space coordinate and aperiodic in the time coordinate. Generally, the time-aperiodic solution can be regarded as a homoclinic or separatrix trajectory in the infinite-dimension phase space of the solutions for (1) with periodic boundary conditions in space. Through numerical simulation, one can gain the facts that are in Figure 4(b) as follows:

(1) the periods are in inverse proportion to the value of $16 + \beta$, so the group velocities of the Akhmediev breathers are dependent on parameters β;
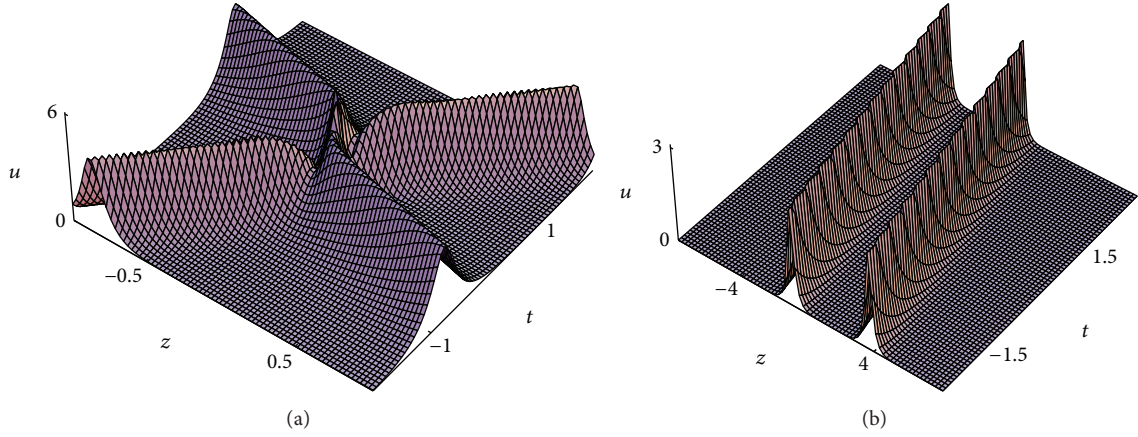
(2) parameters A_c can affect the amplitudes.

(a)

(b)

Figure 1: Evolution of the two-soliton solutions of (1). Parameters are (a) $\lambda_1 = 2 + 2i$, $\lambda_2 = 3 + 2i$, $\delta = 2.5$, $\Gamma = 0.1$ and (b) $\lambda_1 = 1 + 2i$, $\lambda_2 = 1.0001 - 2i$, $\delta = 2.5$, $\Gamma = 0.1$.
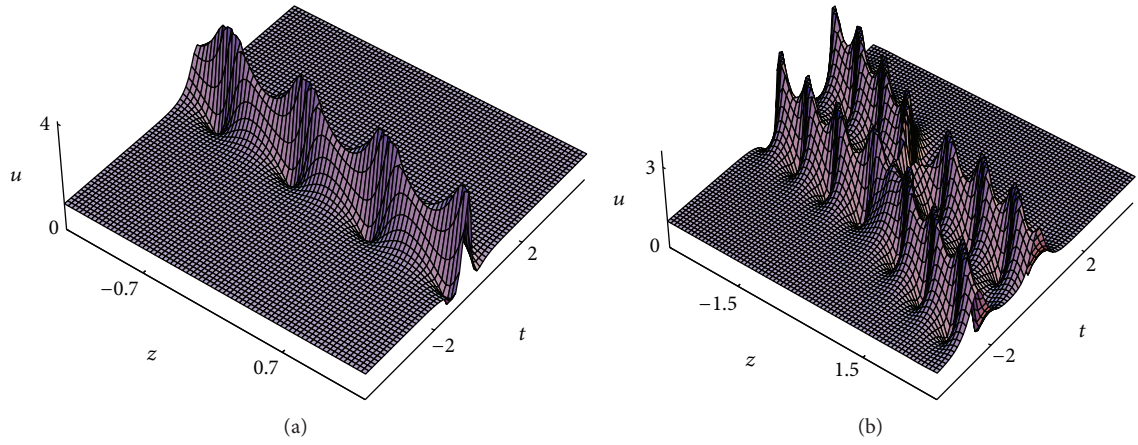

(a)

(b)

Figure 2: Evolution of the breather solutions of (1). Parameters are (a) $u_c = 1$, $\kappa = 0$, $\delta = 3$, $\lambda_1 = 2.05 + 2.05i$, $\Gamma = 0$ and (b) $u_c = 1$, $\kappa = 0$, $\lambda_1 = 2.05 + 2.05i$, $\lambda_2 = 2 + 2i$, $\delta = 3$, $\Gamma = 0$.
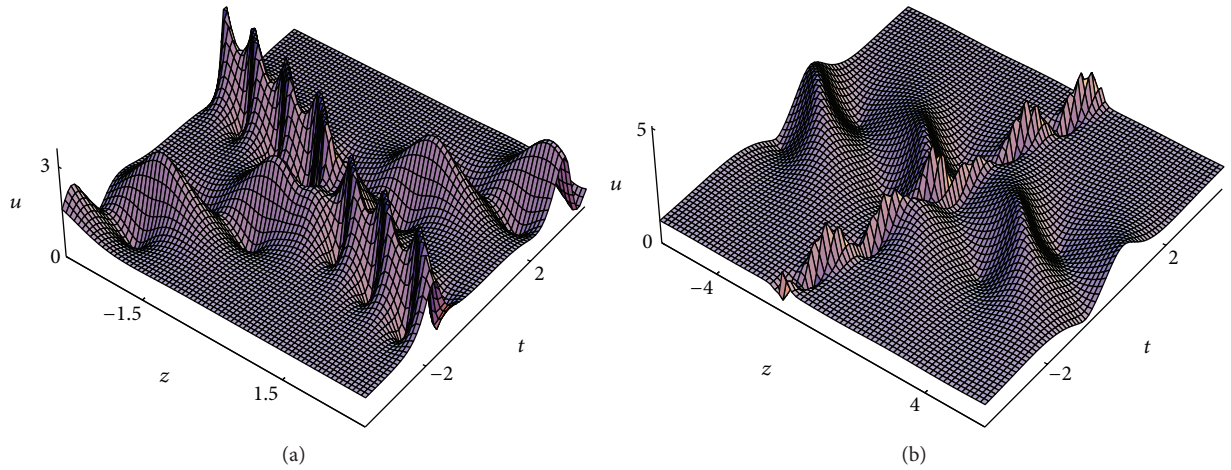

(a)

(b)

Figure 3: Evolution of the breather solutions of (1). Parameters are (a) $u_c = 1$, $\kappa = 0$, $\lambda_1 = 2.05 + 1.05i$, $\lambda_2 = 2 + 2i$, $\delta = 3$, $\Gamma = 0$ and (b) $u_c = 1$, $\kappa = 0$, $\lambda_1 = 1.5 + i$, $\lambda_2 = 3 + 1.1i$, $\delta = 3$, $\Gamma = 0$.
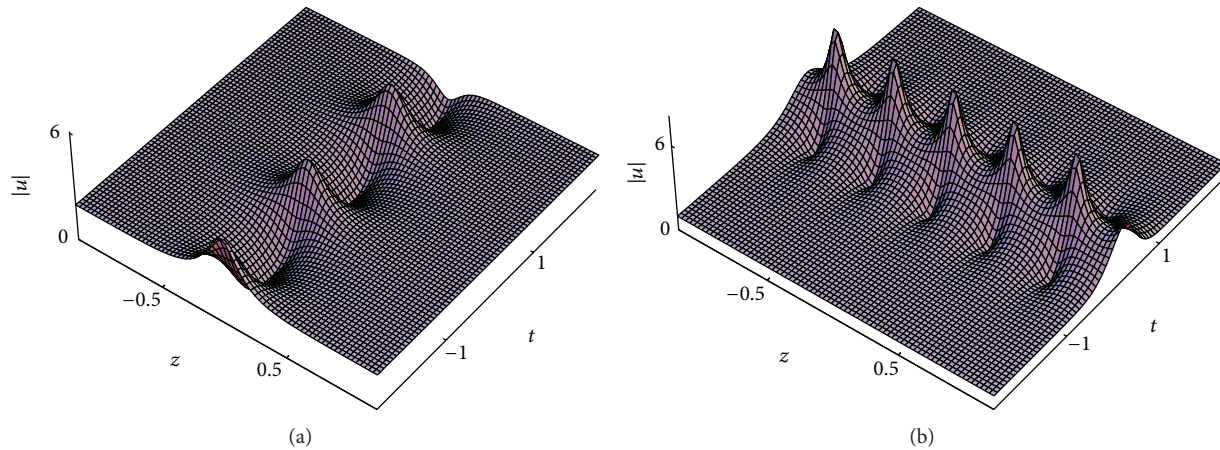
(a)

(b)

FIGURE 4: Evolution of the breather solutions of (1). Parameters are (a) $A_c = 0.5$, $\kappa_1 = 1$, $\kappa_2 = 1$, $\delta = 1$ and (b) $A_c = 0.05$, $\kappa_1 = 1$, $\kappa_2 = 1$, $\delta = 1$.

4. Conclusions

Our attention has been focused on (1) which describes the propagation of the subpicosecond or femtosecond optical pulse in a monomodal optical fiber. With symbolic computation, we have constructed N-fold Darboux transformation and derived two-soliton and breather solutions on the vanishing and two types of nonvanishing backgrounds, respectively. In addition, some figures have been plotted to display the dynamic characteristics of those solitons.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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