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Research Article

Lattice-Valued Convergence Spaces: Weaker Regularity and p-Regularity

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By using some lattice-valued Kowalsky's dual diagonal conditions, some weaker regularities for Jäger's generalized stratified L-convergence spaces and those for Boustique et al's stratified L-convergence spaces are defined and studied. Here, the lattice L is a complete Heyting algebra. Some characterizations and properties of weaker regularities are presented. For Jäger's generalized stratified L-convergence spaces, a notion of closures of stratified L-filters is introduced and then a new p-regularity is defined. At last, the relationships between p-regularities and weaker regularities are established.

Dedicated to the first author's father Zonghua Li on the occasion of his 60th birthday

1. Introduction

In 1954, Kowalsky [1] introduced a diagonal condition (the **K**-diagonal condition) to characterize whenever a pretopological convergence space is topological. In 1967, Cook and Fischer [2] defined a stronger diagonal condition (the **F**-diagonal condition) which, as they showed therein, is necessary and sufficient for a convergence space to be topological. Furthermore, a dual version of **F** (the **DF**-diagonal condition) is necessary and sufficient for a convergence space to be regular. Regularity can also be characterized by the requirement that, for each filter $\mathbb{F}$, if $\mathbb{F}$ converges to x then so does $\overline{\mathbb{F}}$ (the closure of $\mathbb{F}$). In [3, 4], by considering a pair of convergence spaces (X, p) and (X, q), Kent and his coauthors introduced a kind of relative topologicalness (resp., regularity) which was called p-topologicalness (resp., p-regularity). They discussed p-topologicalness (resp., p-regularity) both by neighborhood (resp., closure) of filter [5] and generalized **F** (resp., **DF**)-diagonal condition. When $p = q$, p-topologicalness (resp., p-regularity) is precisely topologicalness (resp., regularity). In 1996, Kent and Richardson defined a weaker regularity by using the duality of Kowalsky's diagonal condition. They also proved that weaker regularity, regularity, and p-regularity were distinct notions but closely related to each other [6].

In [7], Jäger investigated a kind of lattice-valued convergence spaces, which were called generalized stratified L-convergence spaces. Later, the theory of these spaces was extensively discussed under different lattice context [8–19]. A supercategory of generalized stratified L-convergence spaces, called levelwise stratified L-convergence spaces in this paper, was researched in [20–24]. Indeed, a generalized stratified L-convergence space is precisely a left-continuous levelwise stratified L-convergence space [22].

Lattice-valued **K**- and **F**-diagonal conditions for generalized stratified L-convergence spaces were studied in [11, 12, 17, 18] and those for levelwise stratified L-convergence spaces were discussed in [18, 23]. Both by lattice-valued **DF**-diagonal condition and α-level closures of stratified L-filters, the lattice-valued regularity for generalized stratified L-convergence spaces was presented in [13] and that for levelwise stratified L-convergence spaces was given in [20, 21]. Later, by α-level closures of stratified L-filters, p-regularity for levelwise generalized stratified L-convergence spaces was studied in [24]. Recently, p-topologicalness and p-regularity for generalized stratified L-convergence spaces and that for level stratified L-convergence spaces were discussed systemically in [25].

In this paper, for generalized stratified L-convergence spaces and levelwise stratified L-convergence spaces, we will discuss some lattice-valued weaker regularities, p-regularities, and their relationships. The content is arranged as follows. Section 2 recalls some basic notions as preliminary. Section 3 presents the definitions, characterizations, and properties of lattice-valued weaker regularities. Section 4 presents a notion of closures of stratified L-filters and a new lattice-valued p-regularity for stratified generalized L-convergence spaces. Also, the relationships between lattice-valued weaker regularities and lattice-valued p-regularities are established.

2. Preliminaries

In this paper, if not otherwise specified, $L = (L, \leq)$ is always a complete lattice with a top element 1 and a bottom element 0, which satisfies the distributive law $\alpha \wedge (\bigvee_{i\in I}\beta_i) = \bigvee_{i\in I}(\alpha\wedge\beta_i)$. A lattice with these conditions is called a complete Heyting algebra or a frame. The operation $\rightarrow: L \times L \rightarrow L$ given by $\alpha \rightarrow \beta = \vee\{\gamma \in L : \alpha \wedge \gamma \leq \beta\}$ is called the residuation with respect to $\wedge$. A complete Heyting algebra L is said to be a complete Boolean algebra if it obeys the *law of double negation:* $\forall \alpha \in L, (\alpha \rightarrow 0) \rightarrow 0 = \alpha$.

For a set X, the set L^X of functions from X to L with the pointwise order becomes a complete lattice. Each element of L^X is called an L-set (or a fuzzy subset) of X. For any $\lambda \in L^X$, $\mathscr{K} \subseteq L^X$, and $\alpha \in L$, we denote by $\alpha \wedge \lambda$, $\alpha \rightarrow \lambda$, $\vee\mathscr{K}$, and $\wedge\mathscr{K}$ the L-sets defined by $(\alpha \wedge \lambda)(x) = \alpha \wedge \lambda(x)$, $(\alpha \rightarrow \lambda)(x) = \alpha \rightarrow \lambda(x)$, $(\vee\mathscr{K})(x) = \bigvee_{\mu\in\mathscr{K}}\mu(x)$, and $(\wedge\mathscr{K})(x) = \bigwedge_{\mu\in\mathscr{K}}\mu(x)$. Also, we make no difference between a constant function and its value since no confusion will arise. For a crisp subset $A \subseteq X$, let 1_A be the characteristic function; that is $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. Clearly, the characteristic function 1_A of a subset $A \subseteq X$ can be regarded as a function from X to L.

Let X be a set. A fuzzy partial order (or an L-partial order) on X [26] is a function $R : X \times X \rightarrow L$ such that (1) $R(a, a) = 1$ for every $a \in X$ (reflexivity); (2) $R(a, b) = R(b, a) = 1$ implies that $a = b$ for all $a, b \in X$ (antisymmetry); (3) $R(a, b) \wedge R(b, c) \leq R(a, c)$ for all $a, b, c \in X$ (transitivity). The pair (X, R) is called an L-partially ordered set.

Let $[L^X] : L^X \times L^X \rightarrow L$ be a function defined by $[L^X](\lambda, \mu) = \bigwedge_{x\in X}(\lambda(x) \rightarrow \mu(x))$; then $[L^X]$ is an L-partial order on L^X. The value $[L^X](\lambda, \mu) \in L$ is interpreted as the degree that λ is contained in μ. In the sequel, we use the symbol $[\lambda, \mu]$ to denote $[L^X](\lambda, \mu)$ for simplicity.

Let $f : X \rightarrow Y$ be an ordinary function. We define $f^{\rightarrow} : L^X \rightarrow L^Y$ and $f^{\leftarrow} : L^Y \rightarrow L^X$ [27] by $f^{\rightarrow}(\lambda)(y) = \bigvee_{f(x)=y}\lambda(x)$ for $\lambda \in L^X$ and $y \in Y$, and $f^{\leftarrow}(\mu) = \mu \circ f$ for $\mu \in L^Y$.

2.1. Stratified L-(Ultra)filters. A stratified L-filter [27] on a set X is a function $\mathscr{F} : L^X \rightarrow L$ such that for each $\lambda, \mu \in L^X$ and each $\alpha \in L$, (F1) $\mathscr{F}(0) = 0$, $\mathscr{F}(1) = 1$; (F2) $\mathscr{F}(\lambda) \wedge \mathscr{F}(\mu) = \mathscr{F}(\lambda \wedge \mu)$; (Fs) $\mathscr{F}(\alpha) \geq \alpha$. A stratified L-filter $\mathscr{F}$ is called tight if $\mathscr{F}(\alpha) = \alpha$ for each $\alpha \in L$ [5]. It is proved in [27] that all stratified L-filters are tight if and only if L is a complete Boolean algebra. It is easily seen that for a stratified L-filter $\mathscr{F}$ on X, we have $\forall \lambda \in L^X$, $\mathscr{F}(\lambda) = \bigvee_{\mu\in L^X}(\mathscr{F}(\mu) \wedge [\mu, \lambda])$.

The set $\mathscr{F}^s_L(X)$ of all stratified L-filters on X is ordered by $\mathscr{F} \leq \mathscr{G} \Leftrightarrow \forall \lambda \in L^X, \mathscr{F}(\lambda) \leq \mathscr{G}(\lambda)$. It is shown in [27] that the partially ordered set $(\mathscr{F}^s_L(X), \leq)$ has maximal elements which are called stratified L-ultrafilters. The set of all stratified L-ultrafilters on X is denoted as $\mathscr{U}^s_L(X)$. Let $\mathscr{F} \in \mathscr{F}^s_L(X)$. Then $\mathscr{F}$ is an L-ultrafilter if and only if for all $\lambda \in L^X$ we have $\mathscr{F}(\lambda) = \mathscr{F}(\lambda \rightarrow 0) \rightarrow 0$. A stratified L-filter $\mathscr{F}$ is called a stratified L-prime filter if $\mathscr{F}(\lambda \vee \mu) = \mathscr{F}(\lambda) \vee \mathscr{F}(\mu)$ for each $\lambda, \mu \in L^X$. And when L is a complete Boolean algebra then $\mathscr{F} = \bigwedge_{\mathscr{F}\leq\mathscr{G}\in\mathscr{U}^s_L(X)}\mathscr{G}$ and $\mathscr{F}$ is prime whenever $\mathscr{F}$ is maximal [27].

For each $\mathscr{F} \in \mathscr{F}^s_L(X)$, it is easily seen that $\mathbb{F}_{\mathscr{F}} = \{A \subseteq X \mid \mathscr{F}(1_A) = 1\}$ is a filter on X. For each $\lambda \in L^X$, take $\iota\lambda = \{x \in X \mid \lambda(x) > 0\}$. Let $\mathbb{F}$ be a filter on X. Then, when L is a linearly order frame or $0 \in L$ is prime ($\alpha\wedge\beta = 0$ implies $\alpha = 0$ or $\beta = 0$), the function $\mathscr{F}_{\mathbb{F}} : L^X \rightarrow L$, defined by $\forall \lambda \in L^X$, $\mathscr{F}_{\mathbb{F}}(\lambda) = 1$ if $\iota\lambda \in \mathbb{F}$ and $\mathscr{F}_{\mathbb{F}}(\lambda) = 0$ if not so, is a stratified L-filter on X [22]. Also, when L is a linearly order frame or $0 \in L$ is prime, a stratified L-ultrafilter takes values in $\{0, 1\}$ only [10].

Lemma 1 (Jäger [28] for $L = [0, 1]$). *Let L be a linearly order frame or let $0 \in L$ be prime. Then, for each $\mathscr{F} \in \mathscr{U}^s_L(X)$, $\mathbb{F}_{\mathscr{F}}$ is an ultrafilter on X and $\mathscr{F} = \mathscr{F}_{\mathbb{F}_{\mathscr{F}}}$.*

Proof. At first, we check that $\mathbb{F}_{\mathscr{F}}$ is an ultrafilter on X. For each $A \subseteq X$, we assume that $A \notin \mathbb{F}_{\mathscr{F}}$; that is, $\mathscr{F}(1_A) = 0$; then $\mathscr{F}(1_{X-A}) = \mathscr{F}(1_{X-A} \rightarrow 0) \rightarrow 0 = \mathscr{F}(1_A) \rightarrow 0 = 1$. That means $X - A \in \mathbb{F}_{\mathscr{F}}$. By the arbitrariness of A we get that $\mathbb{F}_{\mathscr{F}}$ is an ultrafilter on X. At second, we check $\mathscr{F} \leq \mathscr{F}_{\mathbb{F}_{\mathscr{F}}}$. Note that $\mathscr{F}$ takes values in $\{0, 1\}$ only; thus, it suffices to prove that if $\mathscr{F}(\lambda) = 1$; then $\mathscr{F}_{\mathbb{F}_{\mathscr{F}}}(\lambda) = 1$. Indeed, let $\mathscr{F}(\lambda) = 1$; then $\mathscr{F}(1_{\iota\lambda}) \geq \mathscr{F}(\lambda) = 1$; that is, $\iota\lambda \in \mathbb{F}_{\mathscr{F}}$ and so $\mathscr{F}_{\mathbb{F}_{\mathscr{F}}}(\lambda) = 1$. Therefore, $\mathscr{F} \leq \mathscr{F}_{\mathbb{F}_{\mathscr{F}}}$ and it follows that $\mathscr{F} = \mathscr{F}_{\mathbb{F}_{\mathscr{F}}}$ by the maximality of $\mathscr{F}$. □

The following examples belong to the folklore; we list them here because the notations are needed.

Example 2. (1) For each point x in a set X, the function $[x] : L^X \rightarrow L, [x](\lambda) = \lambda(x)$ is a stratified L-filter on X. In general, $[x]$ is not a stratified L-ultrafilter. But when L is a complete Boolean algebra, then it is so.

(2) Let $\{\mathscr{F}_j \mid j \in J\}$ be a family of stratified L-filters on X; then $\bigwedge_{j\in J}\mathscr{F}_j$, in particular, $\mathscr{F}_0 = \wedge\mathscr{F}^s_L(X)$, is a stratified L-filter on X.

(3) Let $f : X \rightarrow Y$ be a function. If $\mathscr{F} \in \mathscr{F}^s_L(X)$, then the function $f^{\Rightarrow}(\mathscr{F}) \in \mathscr{F}^s_L(Y)$, where $f^{\Rightarrow}(\mathscr{F}) : L^Y \rightarrow L$ defined by $\lambda \mapsto \mathscr{F}(\lambda \circ f)$. If $\mathscr{F} \in \mathscr{U}^s_L(X)$, then $f^{\Rightarrow}(\mathscr{F}) \in \mathscr{U}^s_L(Y)$.

There is a natural fuzzy partial order on $\mathscr{F}^s_L(X)$ inherited from $L^{(L^X)}$. Precisely, for all $\mathscr{F}, \mathscr{G} \in \mathscr{F}^s_L(X)$, if we let

$[\mathcal{F}_L^s(X)](\mathcal{F},\mathcal{G}) = [L^{L^X}](\mathcal{F},\mathcal{G}) = \bigwedge_{\lambda\in L^X}(\mathcal{F}(\lambda) \to \mathcal{G}(\lambda))$, then $[\mathcal{F}_L^s(X)]$ is an L-partially order. For simplicity, we use the symbol $[\mathcal{F},\mathcal{G}]$ to denote the value $[\mathcal{F}_L^s(X)](\mathcal{F},\mathcal{G})$ below.

2.2. Lattice-Valued Convergence Spaces

Definition 3. A generalized stratified L-convergence structure [7] on a set X is a function $\lim : \mathcal{F}_L^s(X) \to L^X$ satisfying *(LC1)* $\forall x \in X$, $\lim[x](x) = 1$; and *(LC2)* $\forall \mathcal{F},\mathcal{G} \in \mathcal{F}_L^s(X)$, $\mathcal{F} \le \mathcal{G} \Rightarrow \lim\mathcal{F} \le \lim\mathcal{G}$. The pair $(X,\lim)$ is called a generalized stratified L-convergence space. If lim further satisfies the strong axiom *(LC2′)* $\forall \mathcal{F},\mathcal{G} \in \mathcal{F}_L^s(X), [\mathcal{F},\mathcal{G}] \wedge \lim\mathcal{F} \le \lim\mathcal{G}$, then the pair $(X,\lim)$ is called a strong stratified L-convergence space [8, 15, 16].

A function $f : X \to X'$ between two generalized stratified L-convergence spaces $(X,\lim)$, $(X',\lim')$ is called continuous if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and all $x \in X$ we have $\lim\mathcal{F}(x) \le \lim' f^{\Rightarrow}(\mathcal{F})(f(x))$. The category *SL-GCS* has as objects all generalized stratified L-convergence spaces and as morphisms the continuous functions. This category is topological over *SET* [7, 10]. For a given source $(X \xrightarrow{f_i} (X_i,\lim_i))_{i\in I}$, the initial structure, lim on X is defined by $\forall \mathcal{F} \in \mathcal{F}_L^s(X), \forall x \in X, \lim\mathcal{F}(x) = \bigwedge_{i\in I}\lim_i f_i^{\Rightarrow}(\mathcal{F})(f_i(x))$.

Definition 4. A collection $\overline{q} = (q_\alpha)_{\alpha\in L}$, where $q_\alpha : \mathcal{F}_L^s(X) \to \mathcal{P}(X)$, is called a levelwise stratified L-convergence structure on X [20] if it satisfies the following:

(LL1) $[x] \xrightarrow{q_\alpha} x$ for each $x \in X$;

(LL2) $\mathcal{G} \ge \mathcal{F} \xrightarrow{q_\alpha} x$ implies $\mathcal{G} \xrightarrow{q_\alpha} x$;

(LL3) $\mathcal{F} \xrightarrow{q_\alpha} x$ implies $\mathcal{F} \xrightarrow{q_\beta} x$ whenever $\beta \le \alpha$.

The notation, $\mathcal{F} \xrightarrow{q_\alpha} x$, means that $x \in q_\alpha(\mathcal{F})$. The pair $(X,\overline{q})$ is called a levelwise stratified L-convergence space.

A function $f : X \to X'$ between two levelwise stratified L-convergence spaces $(X,\overline{q})$, $(X',\overline{q'})$ is called continuous if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ all $x \in X$, and all $\alpha \in L$ we have $\mathcal{F} \xrightarrow{q_\alpha} x$ implies $f^{\Rightarrow}(\mathcal{F}) \xrightarrow{q'_\alpha} f(x)$. The category *SL-LCS* has as objects all levelwise stratified L-convergence spaces and as morphisms the continuous functions. This category is topological over *SET* [20, 21]. For a given source $(X \xrightarrow{f_i} (X_i,\overline{q^i}))_{i\in I}$, the initial structure, $\overline{q}$ on X is defined by $\mathcal{F} \xrightarrow{q_\alpha} x \Leftrightarrow \forall i \in I, f_i^{\Rightarrow}(\mathcal{F}) \xrightarrow{q_\alpha^i} f_i(x)$ $(\mathcal{F} \in \mathcal{F}_L^s(X), x \in X, \alpha \in L)$.

3. Lattice-Valued Weaker Regularities

In this section, we will present the definitions, characterizations, and properties of lattice-valued weaker regularities.

Let X be a set; a function $\phi : X \to \mathcal{F}_L^s(X)$ is usually called an L-filter select function on X. We define $\widehat{\phi} : L^X \to L^X$ as $\widehat{\phi}(\lambda) : X \to L, x \mapsto \phi(x)(\lambda)$. Let $\Sigma(X)$ denote the set of all L-filter select functions on X, and let $\Sigma^*(X)$ be the subset consisting of all $\phi \in \Sigma$ such that $\phi(y) \in \mathcal{U}_L^s(X)$ for all $y \in X$.

Let $\phi \in \Sigma(X)$. For all $\mathcal{F} \in \mathcal{F}_L^s(X)$, it can be proved that the function $k_L\phi\mathcal{F} : L^X \to L$, defined by $\forall\lambda \in L^X$, $k_L\phi\mathcal{F}(\lambda) = \mathcal{F}(\widehat{\phi}(\lambda))$, is a stratified L-filter, which is called the L-diagonal filter of $(\phi,\mathcal{F})$ [11, 17]. Then we have the following obvious lemma. It may have appeared in some other places.

Lemma 5. *Let $\phi,\sigma \in \Sigma(X)$ or $\Sigma^*(X)$. Then*

(1) $\widehat{\phi}(0) = 0$, $\widehat{\phi}(1) = 1$;

(2) *for each* $\lambda,\mu \in L^X$, $\widehat{\phi}(\lambda\wedge\mu) = \widehat{\phi}(\lambda) \wedge \widehat{\phi}(\mu)$;

(3) $\sigma \le \phi$ *implies* $\widehat{\sigma} \le \widehat{\phi}$;

(4) *for all* $\mathcal{F},\mathcal{G} \in \mathcal{F}_L^s(X)$, *then* $[\mathcal{F},\mathcal{G}] \le [k_L\phi\mathcal{F}, k_L\phi\mathcal{G}]$. *In particular, if* $\mathcal{F} \le \mathcal{G}$ *then* $k_L\phi\mathcal{F} \le k_L\phi\mathcal{G}$.

3.1. For Generalized Stratified L-Convergence Spaces. Let $(X,\lim)$ be a generalized stratified L-convergence space. We consider the following axioms.

DLK. For each $\phi \in \Sigma(X)$, we have

$$\forall \mathcal{F} \in \mathcal{F}_L^s(X), \quad \bigwedge_{y\in X} \lim\phi(y)(y) \le [\lim k_L\phi\mathcal{F}, \lim\mathcal{F}]. \tag{1}$$

DLK′. Taking ϕ as $\forall y \in X, \lim\phi(y)(y) = 1$ in *DLK*.

Replacing $\mathcal{F}_L^s(X)$ by $\mathcal{U}_L^s(X)$ in *DLK* (resp., *DLK′*), we obtain a weaker axiom in symbol *DLK** (resp., *DLK′**).

Remark 6. The axiom *DLK* is the dual axiom of *LK* which appeared in [11], and the axiom *DLK′* is the dual axiom of *LK′* which appeared in [17].

Definition 7. Let $(X,\lim)$ be a generalized stratified L-convergence space. Then $(X,\lim)$ is called k-regular (resp., k'-regular, k^*-regular, and k'^*-regular) if it satisfies the axiom *DLK* (resp., *DLK′*, *DLK**, and *DLK′**).

Lemma 8 (Li and Jin [25]). *Let $\phi \in \Sigma(X)$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$. We define $\mathcal{F}^\phi : L^X \to L$ as $\mathcal{F}^\phi(\lambda) = \bigvee_{\mu\in L^X}(\mathcal{F}(\mu)\wedge[\widehat{\phi}(\mu),\lambda])$. Then $\mathcal{F}^\phi$ satisfies (F1), (F2), and (Fs); thus, we say that $\mathcal{F}^\phi$ is nearly a stratified L-filter. If $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$ then $k_L\phi(\mathcal{F}^\phi) \ge \mathcal{F}$.*

Lemma 9. *Let $\phi \in \Sigma(X)$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$. Then $(k_L\phi\mathcal{F})^\phi \in \mathcal{F}_L^s(X)$ and $(k_L\phi\mathcal{F})^\phi \le \mathcal{F}$.*

Proof. For each $\lambda \in L^X$, we have

$$\begin{aligned}(k_L\phi\mathcal{F})^\phi(\lambda) &= \bigvee_{\mu\in L^X}\left(k_L\phi\mathcal{F}(\mu) \wedge \left[\widehat{\phi}(\mu),\lambda\right]\right) \\ &= \bigvee_{\mu\in L^X}\left(\mathcal{F}\left(\widehat{\phi}(\mu)\right) \wedge \left[\widehat{\phi}(\mu),\lambda\right]\right) \le \mathcal{F}(\lambda);\end{aligned} \tag{2}$$

that is, $(k_L\phi\mathcal{F})^\phi \le \mathcal{F}$. It follows that $(k_L\phi\mathcal{F})^\phi(0) = 0$. From the above lemma we have that $(k_L\phi\mathcal{F})^\phi$ is a stratified L-filter on X. □

By the above two lemmas, we get the following characteristic theorem.

Theorem 10. *Let* $(X, \lim)$ *be a generalized stratified L-convergence space. Then* $(X, \lim)$ *is k-regular (resp.,* k^**-regular) if and only if, for each* $\phi \in \Sigma(X)$ *(resp.,* $\phi \in \Sigma^*(X)$*),* $\bigwedge_{y\in X} \lim \phi(y)(y) \leq [\lim \mathcal{F}, \lim \mathcal{F}^{\phi}]$ *whenever* $\mathcal{F}^{\phi} \in \mathcal{F}_L^s(X)$.

Proof. We prove only for k-regularity. Assume the given condition is satisfied, let $\phi \in \Sigma(X)$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$. By Lemma 9 we have $(k_L\phi\mathcal{F})^{\phi} \in \mathcal{F}_L^s(X)$ and

$$\begin{aligned}\bigwedge_{y\in X} \lim \phi(y)(y) &\leq \left[\lim k_L\phi\mathcal{F}, \lim (k_L\phi\mathcal{F})^{\phi}\right] \\ &\leq [\lim k_L\phi\mathcal{F}, \lim \mathcal{F}],\end{aligned} \tag{3}$$

and so *DLK* holds; that is, $(X, \lim)$ is k-regular.

Conversely, let $\mathcal{F} \in \mathcal{F}_L^s(X)$, $\phi \in \Sigma(X)$ with $\mathcal{F}^{\phi} \in \mathcal{F}_L^s(X)$. By Lemma 8, $k_L\phi(\mathcal{F}^{\phi}) \geq \mathcal{F}$. It follows by *DLK* that

$$\begin{aligned}\left[\lim \mathcal{F}, \lim \mathcal{F}^{\phi}\right] &\geq \left[\lim k_L\phi\left(\mathcal{F}^{\phi}\right), \lim \mathcal{F}^{\phi}\right] \\ &\geq \bigwedge_{y\in X} \lim \phi(y)(y).\end{aligned} \tag{4}$$

Thus, the requirement is satisfied. □

Corollary 11. *A generalized stratified L-convergence space* $(X, \lim)$ *is* k'*-regular (resp.,* ${k'}^*$*-regular) if and only if for each* $\phi \in \Sigma(X)$ *(resp.,* $\phi \in \Sigma^*(X)$*) with* $\lim \phi(y)(y) = 1$ *for all* $y \in X$*, we have* $\lim \mathcal{F} \leq \lim \mathcal{F}^{\phi}$ *whenever* $\mathcal{F}^{\phi} \in \mathcal{F}_L^s(X)$.

The following theorem considers lattice-valued weaker regularities w.r.t. the initial structures.

Theorem 12. *Let* $(X, \lim)$ *be the initial structure relative to the source* $(X \xrightarrow{f_i} (X_i, \lim_i))_{i\in I}$ *with each* $f_i : X \to X_i$ *being injective. Then if each* $(X_i, \lim_i)$ *is k-regular (resp.,* k'*-regular), then the same is true of* $(X, \lim)$.

Proof. We prove only for k-regularity. Let $\phi \in \Sigma(X)$. Fix $i \in I$; define $\phi_i \in \Sigma(X_i)$ as $\phi_i(y) = [y]$ if $y \notin f_i(X)$ and $\phi_i(y) = f_i^{\Rightarrow}(\phi(f_i^{-1}(y)))$ if $y \in f_i(X)$. Then for each $i \in I$, by $\lim[y](y) = 1$ it follows that

$$\begin{aligned}\bigwedge_{y\in X_i} \lim_i\phi_i(y)(y) &= \bigwedge_{y\in f_i(X)} \lim_i\phi_i(y)(y) \\ &= \bigwedge_{x\in X} \lim_i f_i^{\Rightarrow}(\phi(x))(f_i(x)).\end{aligned} \tag{5}$$

(In particular, if $\forall x \in X$, $\lim \phi(x)(x) = 1$, then $\forall y \in X_i$, $\lim_i\phi_i(y)(y) = 1$).

For each $\lambda \in L^{X_i}$ and each $x \in X$, it follows that

$$\begin{aligned}\widehat{\phi}(\lambda \circ f_i)(x) &= \phi(x)(\lambda \circ f_i) = f_i^{\Rightarrow}(\phi(x))(\lambda) \\ &= \phi_i(f_i(x))(\lambda) = \widehat{\phi}_i(\lambda)(f_i(x)).\end{aligned} \tag{6}$$

Hence, $\widehat{\phi}(\lambda \circ f_i) = \widehat{\phi}_i(\lambda) \circ f_i$, and then, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$,

$$\begin{aligned}f_i^{\Rightarrow}(k_L\phi\mathcal{F})(\lambda) &= k_L\phi\mathcal{F}(\lambda \circ f_i) = \mathcal{F}\left(\widehat{\phi}(\lambda \circ f_i)\right) \\ &= \mathcal{F}\left(\widehat{\phi}_i(\lambda) \circ f_i\right) = f_i^{\Rightarrow}(\mathcal{F})\left(\widehat{\phi}_i(\lambda)\right) \\ &= k_L\phi_i\left(f_i^{\Rightarrow}(\mathcal{F})\right)(\lambda).\end{aligned} \tag{7}$$

Therefore, $f_i^{\Rightarrow}(k_L\phi\mathcal{F}) = k_L\phi_i(f_i^{\Rightarrow}(\mathcal{F}))$. Then, for each $x \in X$,

$$\begin{aligned}&\bigwedge_{y\in X} \lim \phi(y)(y) \wedge \lim k_L\phi\mathcal{F}(x) \\ &\quad= \bigwedge_{y\in X} \bigwedge_{i\in I} \lim_i f_i^{\Rightarrow}(\phi(y))(f_i(y)) \\ &\quad\wedge \bigwedge_{i\in I} \lim_i f_i^{\Rightarrow}(k_L\phi\mathcal{F})(f_i(x)) \\ &\quad= \bigwedge_{i\in I} \bigwedge_{z_i\in X_i} \lim_i \phi_i(z_i)(z_i) \wedge \bigwedge_{i\in I} \lim_i k_L\phi_i\left(f_i^{\Rightarrow}(\mathcal{F})\right)(f_i(x)) \\ &\quad\leq \bigwedge_{i\in I}\left(\bigwedge_{z_i\in X_i} \lim_i \phi_i(z_i)(z_i) \wedge \lim_i k_L\phi_i\left(f_i^{\Rightarrow}(\mathcal{F})\right)(f_i(x))\right) \\ &\quad\leq \bigwedge_{i\in I} \lim_i f_i^{\Rightarrow}(\mathcal{F})(f_i(x)) = \lim \mathcal{F}(x).\end{aligned} \tag{8}$$

Here, the last inequality holds because each $(X_i, \lim_i)$ is k-regular. Now, we have proved that $(X, \lim)$ is k-regular. □

The following theorem gives the relationship between types of lattice-valued weaker regularities.

Theorem 13. *Let L be a complete Boolean algebra. Then k-regularity* $\Leftrightarrow$ k^**-regularity and* k'*-regularity* $\Leftrightarrow$ ${k'}^*$*-regularity.*

Proof. We check only the equivalence k-regularity $\Leftrightarrow$ k^*-regularity. The other equivalence is similar. Obviously, k-regularity $\Rightarrow$ k^*-regularity. Conversely, let $(X, \lim)$ be k^*-regular. Note that when L is a complete Boolean algebra, then for every stratified L-filter there exists a stratified L-ultrafilter containing it. Thus, for each $\phi \in \Sigma(X)$, there is some $\phi^* \in \Sigma^*$ such that $\phi(y) \leq \phi^*(y)$ for all $y \in X$. Assume that $\mathcal{F} \in \mathcal{F}_L^s(X)$ with $\mathcal{F}^{\phi} \in \mathcal{F}_L^s(X)$. Then it is easily seen that $\mathcal{F}^{\phi^*} \leq \mathcal{F}^{\phi}$ and $\mathcal{F}^{\phi^*} \in \mathcal{F}_L^s(X)$. By Theorem 10,

$$\begin{aligned}\bigwedge_{y\in X} \lim \phi(y)(y) &\leq \bigwedge_{y\in X} \lim \phi^*(y)(y) \leq \left[\lim \mathcal{F}, \lim \mathcal{F}^{\phi^*}\right] \\ &\leq \left[\lim \mathcal{F}, \lim \mathcal{F}^{\phi}\right].\end{aligned} \tag{9}$$

Thus, $(X, \lim)$ is k-regular. □

As a consequence, we obtain that when L is a complete Boolean algebra, Theorem 12 holds for k^*-regularity and ${k'}^*$-regularity.

Obviously, k-regularity $\Rightarrow$ k'-regularity and k^*-regularity $\Rightarrow$ ${k'}^*$-regularity. The following example shows that the reverse inclusions do not hold generally.

Example 14. Let $X = \{x, y\}$ and $L = \{0, \alpha, \beta, 1\}$ with ordering $0 < \alpha, \beta < 1$ and $\alpha \wedge \beta = 0$, $\alpha \vee \beta = 1$. Then $(L, \wedge)$ becomes a complete Boolean algebra. Obviously, $[x]$ and $[y]$ are all stratified L-ultrafilters on X. Thus, it is easily seen that the function $\lim : \mathcal{F}_L^s(X) \to L^X$ defined by

$$\lim \mathcal{F}(x) = \begin{cases} 1, & \mathcal{F} = [x]; \\ \alpha, & \mathcal{F} = [y]; \\ 0, & \text{otherwise}, \end{cases}$$
$$\lim \mathcal{F}(y) = \begin{cases} 1, & \mathcal{F} = [y]; \\ \beta, & \mathcal{F} = [x]; \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

is a generalized stratified L-convergence structure on X.

(1) $(X, \lim)$ *satisfies* $DLK'(DLK'^*)$. Let $\phi \in \Sigma(X)$ with $\lim \phi(x)(x) = \lim \phi(y)(y) = 1$. Then $\phi(x) = [x]$, $\phi(y) = [y]$. Thus, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$, we have $k_L\phi\mathcal{F} = \mathcal{F}$. Then the axiom DLK', and thus the axiom DLK'^* holds obviously.

(2) $(X, \lim)$ *does not satisfy* $DLK(DLK^*)$. Let $\phi \in \Sigma(X)$ be defined by $\phi(x) = \phi(y) = [y]$. Then, for each $\lambda \in L^X$, we have $\widehat{\phi}(\lambda) = \lambda(y)$. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$,

$$k_L\phi\mathcal{F}(\lambda) = \mathcal{F}\left(\widehat{\phi}(\lambda)\right) = \mathcal{F}(\lambda(y)) \overset{\text{tight}}{=} \lambda(y) = [y](\lambda); \quad \text{that is,} \quad k_L\phi\mathcal{F} = [y]. \tag{11}$$

Taking $\mathcal{G} = [x] \wedge [y]$, then $\lim \mathcal{G}(x) = \lim \mathcal{G}(y) = 0$, and $\lim k_L\phi\mathcal{G}(x) = \lim[y](x) = \alpha$, $\lim k_L\phi\mathcal{G}(y) = \lim[y](y) = 1$. It follows that

$$\alpha = \bigwedge_{z \in X} \lim \phi(z)(z) \nleq 0 = [\lim k_L\phi\mathcal{G}, \lim \mathcal{G}]. \tag{12}$$

It follows that the axiom DLK^* and thus the axiom DLK does not hold.

3.2. For Levelwise Stratified L-Convergence Spaces. Let $(X, \overline{q})$ be a levelwise stratified L-convergence space. We consider the following axioms:

DLLK. For each $\phi \in \Sigma(X)$ and each $\alpha \in L$ with $\forall z \in X$, $\phi(z) \xrightarrow{q_\alpha} z$. Then $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$, $\forall x \in X$, $\mathcal{F} \xrightarrow{q_\alpha} x$ whenever $k_L\phi\mathcal{F} \xrightarrow{q_\alpha} x$.

Replacing $\mathcal{F}_L^s(X)$ by $\mathcal{U}_L^s(X)$ in *DLLK*, we obtain a weaker axiom in symbol $DLLK^*$.

Remark 15. The axiom *DLLK* is a special case of the regular axiom (R2) in [23] with $J = X$ and $\psi = id$.

Definition 16. Let $(X, \overline{q})$ be a levelwise stratified L-convergence space. Then $(X, \overline{q})$ is called k-regular (resp., k^*-regular) if it satisfies the axiom *DLLK* (resp., $DLLK^*$).

For k-regularity (k^*-regularity), we have the following characteristic theorem.

Theorem 17. *Let $(X, \overline{q})$ be a levelwise stratified L-convergence space. Then $(X, \overline{q})$ is k-regular (resp., k^*-regular) if and only if for each $\mathcal{F} \in \mathcal{F}_L^s(X)$ and each $\phi \in \Sigma(X)$(resp., $\phi \in \Sigma^*(X)$) and each $\alpha \in L$ with $\forall z \in X$, $\phi(z) \xrightarrow{q_\alpha} z$, we have that $\mathcal{F} \xrightarrow{q_\alpha} x$ implies $\mathcal{F}^\phi \xrightarrow{q_\alpha} x$ whenever $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$.*

Proof. We prove only for k-regularity. Assume the given condition is satisfied; let $\phi \in \Sigma(X)$ satisfy the condition in *DLLK* and $k_L\phi\mathcal{F} \xrightarrow{q_\alpha} x$. By Lemma 9 we have $(k_L\phi\mathcal{F})^\phi \in \mathcal{F}_L^s(X)$ and $(k_L\phi\mathcal{F})^\phi \le \mathcal{F}$. By the given condition, we have $(k_L\phi\mathcal{F})^\phi \xrightarrow{q_\alpha} x$ and then $\mathcal{F} \xrightarrow{q_\alpha} x$. So, the axiom *DLLK* holds; that is, $(X, \overline{q})$ is k-regular. Conversely, Let $\phi \in \Sigma(X)$ and $\alpha \in L$ with $\forall z \in X$, $\phi(z) \xrightarrow{q_\alpha} z$. Suppose that $\mathcal{F} \xrightarrow{q_\alpha} x$ and $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$. By Lemma 8, $k_L\phi(\mathcal{F}^\phi) \ge \mathcal{F}$, so, $k_L\phi(\mathcal{F}^\phi) \xrightarrow{q_\alpha} x$. It follows by *DLLK* that $\mathcal{F}^\phi \xrightarrow{q_\alpha} x$ as desired. □

The following theorem shows that k-regular is an initial property relative to any family of injection functions.

Theorem 18. *Let $(X, \overline{q})$ be the initial structure relative to the source $(X \xrightarrow{f_i} (X_i, \overline{q^i}))_{i \in I}$ with each $f_i : X \to X_i$ being injective. If each $(X_i, \overline{q_i})$ is k-regular, then the same is true of $(X, \overline{q})$.*

Proof. Let $\phi \in \Sigma(X)$ and $\alpha \in L$ satisfy $\phi(x) \xrightarrow{q_\alpha} x$ for all $x \in X$. Fix $i \in I$; define $\phi_i \in \Sigma(X_i)$ as $\phi_i(y) = [y]$ if $y \notin f_i(X)$ and $\phi_i(y) = f_i^{\Rightarrow}(\phi(f_i^{-1}(y)))$ if $y \in f_i(X)$. Then $\phi_i(y) \xrightarrow{q_\alpha} y$ for each $y \in X_i$. Indeed, if $y \notin f_i(X)$, then $\phi_i(y) = [y] \xrightarrow{q_\alpha} y$, and if $y \in f_i(X)$, then there exists an $x \in X$ such that $f_i(x) = y$ and so $\phi_i(y) = f_i^{\Rightarrow}(\phi(x)) \xrightarrow{q_\alpha} f_i(x) = y$. Let $k_L\phi\mathcal{F} \xrightarrow{q_\alpha} x$. Similar to Theorem 12, we have $f_i^{\Rightarrow}(k_L\phi\mathcal{F}) = k_L\phi_i(f_i^{\Rightarrow}(\mathcal{F}))$ for all $i \in I$. Because each f_i is continuous, thus $k_L\phi_i(f_i^{\Rightarrow}(\mathcal{F})) = f_i^{\Rightarrow}(k_L\phi\mathcal{F}) \xrightarrow{q_\alpha^i} f_i(x)$. Then $f_i^{\Rightarrow}(\mathcal{F}) \xrightarrow{q_\alpha^i} f_i(x)$ since each $(X, \overline{q^i})$ is k-regular. It follows that $\mathcal{F} \xrightarrow{q_\alpha} x$ by the definition of initial structure. We have proved that $(X, \overline{q})$ is k-regular. □

Theorem 19. *Let L be a complete Boolean algebra. Then k-regularity ⇔ k^*-regularity.*

Proof. The proof is similar to Theorem 13 and thus it is omitted. □

As a consequence, we obtain that when L is a complete Boolean algebra, then Theorem 18 holds for k^*-regularity.

The last theorem gives the relationship between k-regularity for generalized stratified L-convergence space and k-regularity for levelwise stratified L-convergence space.

Let $(X, \lim)$ be a generalized stratified L-convergence space. It is proved in [22] that the pair $(X, \overline{q^{\lim}})$, where

$\mathcal{F} \xrightarrow{(q^{\lim})_\alpha} x$ if and only if $\lim \mathcal{F}(x) \geq \alpha$, is a levelwise stratified L-convergence space.

Theorem 20. *Let $(X, \lim)$ be a generalized stratified L-convergence space. Then $(X, \lim)$ is k-regular (resp., k^*-regular) if and only if $(X, \overline{q^{\lim}})$ is k-regular (resp., k^*-regular).*

Proof. We prove only for k-regularity. Let $(X, \lim)$ be k-regular. Take $\phi \in \Sigma(X)$ and $\alpha \in L$ with $\forall z \in X, \phi(z) \xrightarrow{(q^{\lim})_\alpha} z$; then we have $\alpha \leq \bigwedge_{y \in X} \lim \phi(y)(y)$. Take $\mathcal{F} \in \mathcal{F}_L^s(X)$ with $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$; then we have $\mathcal{F} \xrightarrow{(q^{\lim})_\alpha} x$; that is, $\lim \mathcal{F}(x) \geq \alpha$. By Theorem 10 we obtain $\alpha \leq \bigwedge_{y \in X} \lim \phi(y)(y) \leq [\lim \mathcal{F}, \lim \mathcal{F}^\phi]$. Then $\lim \mathcal{F}^\phi(x) \geq \alpha$; that is, $\mathcal{F}^\phi \xrightarrow{(q^{\lim})_\alpha} x$. It follows by Theorem 17 that $(X, \overline{q^{\lim}})$ is k-regular.

Conversely, assume that $(X, \overline{q^{\lim}})$ is k-regular. Let us take $\phi \in \Sigma(X)$ with $\bigwedge_{y \in X} \lim \phi(y)(y) = \alpha$ and take $\mathcal{F} \in \mathcal{F}_L^s(X)$ with $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$. Then if $\lim \mathcal{F}(x) = \beta$ for $x \in X$, we have $\phi(y) \xrightarrow{(q^{\lim})_{\alpha\wedge\beta}} y$ and $\mathcal{F} \xrightarrow{(q^{\lim})_{\alpha\wedge\beta}} x$. It follows by Theorem 17 that $\mathcal{F}^\phi \xrightarrow{(q^{\lim})_{\alpha\wedge\beta}} x$; that is, $\lim \mathcal{F}^\phi(x) \geq \alpha \wedge \beta$. By the arbitrariness of x we note that $\bigwedge_{y \in X} \lim \phi(y)(y) = \alpha \leq [\lim \mathcal{F}, \lim \mathcal{F}^\phi]$. It follows by Theorem 10 that $(X, \lim)$ is k-regular. □

4. On the Relationship between Weaker Regularity and p-Regularity

4.1. For Generalized Stratified L-Convergence Spaces. Generally, p-regularity relates to two different generalized stratified L-convergence structures on the same underlying set. Thus, in this section, we add the lowercases p, q as the superscript of lim and use $\lim^p$, $\lim^q$ to denote different generalized stratified L-convergence structures.

At first, we give the notion of closures of stratified L-filters and then introduce a new p-regularity.

Definition 21. Let $(X, \lim^p)$ be a generalized stratified L-convergence space. For each $\lambda \in L^X$, the L-set $\overline{\lambda}_p \in L^X$ defined by

$$\forall x \in X, \quad \overline{\lambda}_p(x) = \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} \left(\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)\right) \tag{13}$$

is called the closure of λ w.r.t $(X, \lim^p)$.

Lemma 22. *Let $(X, \lim^p)$ be a generalized stratified L-convergence space. Then for all $\lambda, \mu \in L^X$ and all $\alpha \in L$ we get the following:*

(1) $\lambda \leq \overline{\lambda}_p$;

(2) $\lambda \leq \mu$ *implies* $\overline{\lambda}_p \leq \overline{\mu}_p$;

(3) $\overline{(\beta \wedge \lambda)}_p \geq \beta \wedge \overline{\lambda}_p$ *and the equality holds if L is a complete Boolean algebra;*

(4) *if L is a complete Boolean algebra, then* $\forall x \in X$, $\overline{\lambda}_p(x) = \bigvee_{\mathcal{F} \in \mathcal{U}_L^s(X)} (\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda))$, *and* $\overline{(\lambda \vee \mu)}_p = \overline{\lambda}_p \vee \overline{\mu}_p$.

Proof. (1) For each $x \in X$, by $\lim^p[x](x) = 1$ we get $\overline{\lambda}_p(x) \geq [x](\lambda) = \lambda(x)$. So, $\lambda \leq \overline{\lambda}_p$. Take $\lambda = 1$ in (1); we obtain $\overline{1}_p = 1$.

(2) It follows from the property (F2) of stratified L-filters.

(3) For each $x \in X$ we have

$$\begin{aligned}
\overline{(\beta \wedge \lambda)}_p(x) &= \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} \left(\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\beta \wedge \lambda)\right) \\
&= \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} \left(\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\beta) \wedge \mathcal{F}(\lambda)\right) \\
&\geq \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} \left(\lim^p \mathcal{F}(x) \wedge \beta \wedge \mathcal{F}(\lambda)\right) \\
&= \beta \wedge \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} \left(\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)\right) \\
&= \beta \wedge \overline{\lambda}_p(x).
\end{aligned} \tag{14}$$

When L is a complete Boolean algebra, then $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$, $\mathcal{F}(\beta) = \beta$. So, the "≥" in the above inequality can be replaced by "=". Thus, $\overline{(\beta \wedge \lambda)}_p = \beta \wedge \overline{\lambda}_p$.

(5) Let L be a complete Boolean algebra. That $\overline{\lambda}_p(x) = \bigvee_{\mathcal{F} \in \mathcal{U}_L^s(X)} (\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda))$ follows because, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$, there exists an L-ultrafilter $\mathcal{G}$ such that $\mathcal{F} \leq \mathcal{G}$. To prove $\overline{(\lambda \vee \mu)}_p = \overline{\lambda}_p \vee \overline{\mu}_p$, it suffices to check that $\overline{(\lambda \vee \mu)}_p \leq \overline{\lambda}_p \vee \overline{\mu}_p$ since the reverse inequality holds by (2). Indeed, because each stratified L-ultrafilter is prime we have

$$\begin{aligned}
&\overline{\lambda}_p(x) \vee \overline{\mu}_p(x) \\
&= \left(\bigvee_{\mathcal{F} \in \mathcal{U}_L^s(X)} \left(\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)\right)\right) \\
&\quad \vee \left(\bigvee_{\mathcal{G} \in \mathcal{U}_L^s(X)} \left(\lim^p \mathcal{G}(x) \wedge \mathcal{G}(\mu)\right)\right) \\
&= \bigvee_{\mathcal{F}, \mathcal{G} \in \mathcal{U}_L^s(X)} \left(\left(\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)\right) \right. \\
&\qquad \left. \vee \left(\lim^p \mathcal{G}(x) \wedge \mathcal{G}(\mu)\right)\right) \\
&\geq \bigvee_{\mathcal{F} \in \mathcal{U}_L^s(X)} \left(\left(\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)\right) \right. \\
&\qquad \left. \vee \left(\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\mu)\right)\right) \\
&= \bigvee_{\mathcal{F} \in \mathcal{U}_L^s(X)} \left(\lim^p \mathcal{F}(x) \wedge \left(\mathcal{F}(\lambda) \vee \mathcal{F}(\mu)\right)\right) \\
&= \bigvee_{\mathcal{F} \in \mathcal{U}_L^s(X)} \left(\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda \vee \mu)\right) = \overline{(\lambda \vee \mu)}_p(x).
\end{aligned} \tag{15}$$

□

Theorem 23. *Let $(X, \lim^p)$ be a generalized stratified L-convergence space. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$, the function $\overline{\mathcal{F}}_p : L^X \to L$ defined by*

$$\forall \lambda \in L^X, \quad \overline{\mathcal{F}}_p(\lambda) = \bigvee_{\mu \in L^X} \left(\mathcal{F}(\mu) \wedge \left[\overline{\mu}_p, \lambda \right] \right) \tag{16}$$

is a stratified L-filter, called the closure of $\mathcal{F}$.

Proof. (F1) That $\overline{\mathcal{F}}_p(1) = 1$ is obvious. By Lemma 22(1) we have

$$\begin{aligned} \overline{\mathcal{F}}_p(\lambda) &= \bigvee_{\mu \in L^X} \left(\mathcal{F}(\mu) \wedge \left[\overline{\mu}_p, \lambda \right] \right) \\ &\leq \bigvee_{\mu \in L^X} \left(\mathcal{F}(\mu) \wedge [\mu, \lambda] \right) \leq \mathcal{F}(\lambda). \end{aligned} \tag{17}$$

Thus, $\overline{\mathcal{F}}_p(0) = 0$. □

(F2) Firstly, note that $\overline{\mathcal{F}}_p(\lambda) \leq \overline{\mathcal{F}}_p(\mu)$ whenever $\lambda \leq \mu$. It follows that $\overline{\mathcal{F}}_p(\lambda \wedge \mu) \leq \overline{\mathcal{F}}_p(\lambda) \wedge \overline{\mathcal{F}}_p(\mu)$. Conversely,

$$\begin{aligned} &\overline{\mathcal{F}}_p(\lambda) \wedge \overline{\mathcal{F}}_p(\mu) \\ &\quad = \bigvee_{a \in L^X} \left(\mathcal{F}(a) \wedge \left[\overline{a}_p, \lambda \right] \right) \wedge \bigvee_{b \in L^X} \left(\mathcal{F}(b) \wedge \left[\overline{b}_p, \mu \right] \right) \\ &\quad = \bigvee_{a, b \in L^X} \left(\mathcal{F}(a) \wedge \mathcal{F}(b) \wedge \left[\overline{a}_p, \lambda \right] \wedge \left[\overline{b}_p, \mu \right] \right) \\ &\quad \leq \bigvee_{a, b \in L^X} \left(\mathcal{F}(a \wedge b) \wedge \left[\overline{(a \wedge b)}_p, \lambda \wedge \mu \right] \right) \\ &\quad \leq \bigvee_{c \in L^X} \left(\mathcal{F}(c) \wedge \left[\overline{c}_p, \lambda \wedge \mu \right] \right) = \overline{\mathcal{F}}_p(\lambda \wedge \mu). \end{aligned} \tag{18}$$

(Fs) For all $\beta \in L$, it follows that $\overline{\mathcal{F}}_p(\beta) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\overline{\mu}_p, \beta]) \geq \mathcal{F}(1) \wedge \beta = \beta$ by $\overline{1}_p = 1$.

It is easily seen that the following lemma holds. We omit the routine proof.

Lemma 24. *Let $(X, \lim^p)$ be a generalized stratified L-convergence space. Then, for each $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, $[\mathcal{F}, \mathcal{G}] \leq [\overline{\mathcal{F}}_p, \overline{\mathcal{G}}_p]$.*

Definition 25. Let $(X, \lim^p, \lim^q)$ be a pair of generalized stratified L-convergence spaces. Then $(X, \lim^q)$ is called p-regular if and only if, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$, we have $\lim^q \mathcal{F} \leq \lim^q \overline{\mathcal{F}}_p$.

Remark 26. When $L = \{0, 1\}$, a generalized stratified L-convergence space reduces to a convergence space. It is easily seen that $\overline{\mathcal{F}}_p$ is precisely the filter generated by $\{\overline{A} : A \in \mathbb{F}\}$ as a filterbasis [29]. And the p-regularity reduces to the corresponding crisp notion in [3].

The following theorem shows that p-regularity is preserved under initial constructions.

Theorem 27. *Let $\{(X_i, \lim^{q_i}, \lim^{p_i})\}_{i \in I}$ be pairs of generalized stratified L-convergence spaces with each $\lim^{q_i}$ being p_i-regular. If $\lim^q$ (resp., $\lim^p$) is the initial structure on X relative to the source $(X \xrightarrow{f_i} (X_i, \lim^{q_i}))_{i \in I}$ (resp., $(X \xrightarrow{f_i} (X_i, \lim^{p_i}))_{i \in I}$), then $(X, \lim^q)$ is p-regular.*

Proof. At first, we check below that for each $i \in I$ and each $\lambda_i \in L^{X_i}$ we have $\overline{(f_i^{\leftarrow}(\lambda_i))}_p \leq f_i^{\leftarrow}((\overline{\lambda_i})_{p_i})$. Indeed, for each $x \in X$,

$$\begin{aligned} &\overline{(f_i^{\leftarrow}(\lambda_i))}_p(x) \\ &\quad = \bigvee_{\mathcal{G} \in \mathcal{F}_L^s(X)} \left(\lim^p \mathcal{G}(x) \wedge \mathcal{G}\left(f_i^{\leftarrow}(\lambda_i) \right) \right) \\ &\quad = \bigvee_{\mathcal{G} \in \mathcal{F}_L^s(X)} \left(\left(\bigwedge_{j \in I} \lim^{p_j} f_j^{\Rightarrow}(\mathcal{G})\left(f_j(x) \right) \right) \wedge \mathcal{G}\left(f_i^{\leftarrow}(\lambda_i) \right) \right) \\ &\quad \leq \bigvee_{\mathcal{G} \in \mathcal{F}_L^s(X)} \left(\lim^{p_i} f_i^{\Rightarrow}(\mathcal{G})\left(f_i(x) \right) \wedge f_i^{\Rightarrow}(\mathcal{G})(\lambda_i) \right) \\ &\quad \leq \bigvee_{\mathcal{G}_i \in \mathcal{F}_L^s(X_i)} \left(\lim^{p_i} \mathcal{G}_i\left(f_i(x) \right) \wedge \mathcal{G}_i(\lambda_i) \right) \\ &\quad = f_i^{\leftarrow}\left(\overline{(\lambda_i)}_{p_i} \right)(x). \end{aligned} \tag{19}$$

It follow that, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$ and each $\lambda_i \in L^{X_i}$,

$$\begin{aligned} &f_i^{\Rightarrow}\left(\overline{\mathcal{F}}_p \right)(\lambda_i) \\ &\quad = \overline{\mathcal{F}}_p\left(f_i^{\leftarrow}(\lambda) \right) = \bigvee_{\mu \in L^X} \left(\left[\overline{\mu}_p, f_i^{\leftarrow}(\lambda_i) \right] \wedge \mathcal{F}(\mu) \right) \\ &\quad \geq \bigvee_{\mu_i \in L^{X_i}} \left(\left[\overline{(f_i^{\leftarrow}(\mu_i))}_p, f_i^{\leftarrow}(\lambda_i) \right] \wedge \mathcal{F}\left(f_i^{\leftarrow}(\mu_i) \right) \right) \\ &\quad \geq \bigvee_{\mu_i \in L^{X_i}} \left(\left[f_i^{\leftarrow}\left(\overline{(\mu_i)}_{p_i} \right), f_i^{\leftarrow}(\lambda_i) \right] \wedge \mathcal{F}\left(f_i^{\leftarrow}(\mu_i) \right) \right) \\ &\quad \geq \bigvee_{\mu_i \in L^{X_i}} \left(\left[\overline{(\mu_i)}_{p_i}, \lambda_i \right] \wedge f_i^{\Rightarrow}(\mathcal{F})(\mu_i) \right) \\ &\quad = \overline{(f_i^{\Rightarrow}(\mathcal{F}))}_{p_i}(\lambda_i). \end{aligned} \tag{20}$$

Thus, $f_i^{\Rightarrow}(\overline{\mathcal{F}}_p) \geq \overline{(f_i^{\Rightarrow}(\mathcal{F}))}_{p_i}$ for all $i \in I$. It follows by each $(X_i, \lim^{q_i})$ being p_i-regular that

$$\begin{aligned} \lim^q \overline{\mathcal{F}}_p(x) &= \bigwedge_{i \in I} \lim^{q_i} f_i^{\Rightarrow}\left(\overline{\mathcal{F}}_p \right)\left(f_i(x) \right) \\ &\geq \bigwedge_{i \in I} \lim^{q_i} \overline{(f_i^{\Rightarrow}(\mathcal{F}))}_{p_i}\left(f_i(x) \right) \\ &\geq \bigwedge_{i \in I} \lim^{q_i} f_i^{\Rightarrow}(\mathcal{F})\left(f_i(x) \right) = \lim^q \mathcal{F}(x). \end{aligned} \tag{21}$$

Thus, $(X, \lim^q)$ is p-regular. □

When $L = \{0,1\}$, Kent and Richardson [6] studied the relationships between weaker regularities and p-regularity. Now we discuss them for the general case.

Definition 28. A generalized (strong) stratified L-convergence space $(X, \lim^q)$ is called

(i) a (strong) L-Kent convergence space [10] if $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$, $\forall x \in X$, $\lim^q \mathcal{F}(x) \le \lim^q(\mathcal{F} \wedge [x])(x)$;

(ii) pretopological [11] if $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$, $\forall x \in X$, $\lim^q \mathcal{F}(x) = [\mathcal{U}_q(x), \mathcal{F}]$, where $\mathcal{U}_q(x)$, defined by $\forall \lambda \in L^X$, $\mathcal{U}_q(x)(\lambda) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)}(\lim^q \mathcal{F}(x) \to \mathcal{F}(\lambda))$, is called the stratified neighborhood L-filter of x w.r.t. $\lim^q$, and when $(X, \lim^q)$ is a strong stratified L-convergence space, then $(X, \lim^q)$ is pretopological if and only if it satisfies $\lim^q \mathcal{U}_q(x)(x) = 1$ for all $x \in X$ [17];

(iii) ultrapretopological if it is pretopological and for each $x \in X$, there exists a stratified L-ultrafilter $\mathcal{F}_x$ such that $\mathcal{U}_q(x) = [x] \wedge \mathcal{F}_x$;

(iv) topological [11] if there exists a stratified L-topology $\mathcal{T}$ such that $\forall \lambda \in L^X$, $\forall x \in X$, we have $\mathcal{U}_q(x)(\lambda) = \text{int}(\lambda)(x)$, where $\text{int}(\lambda) = \bigvee_{\mu \in \mathcal{T}}(\mu \wedge [\mu, \lambda])$ is called the interior of λ w.r.t. $\mathcal{T}$ [11, 30].

Proposition 29. *Let $(X, \lim^q)$ be a strong stratified L-Kent convergence space which is p-regular relative to every ultrapretopological generalized stratified L-convergence structure $\lim^p \le \lim^q$. Then $(X, \lim^q)$ is k'^*-regular.*

Proof. Let $\phi \in \Sigma^*(X)$ with $\forall y \in X$, $\lim^q \phi(y)(y) = 1$. Let $\lim^p$ be the ultrapretopological generalized stratified L-convergence structure defined by $\forall y \in X$, $\mathcal{U}_p(y) = \phi(y) \wedge [y]$. From $\phi(y) \ge \mathcal{U}_p(y)$ we have $\lim^p \phi(y)(y) = 1$. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$ with $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$, it follows that for each $\lambda \in L^X$, $\overline{\lambda}^p(y) = \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)}(\lim^p \mathcal{F}(y) \wedge \mathcal{F}(\lambda)) \ge \phi(y)(\lambda)$, which means $\overline{\lambda}_p \ge \widehat{\phi}(\lambda)$. Thus,

$$\begin{aligned} \overline{\mathcal{F}}_p(\lambda) &= \bigvee_{\mu \in L^X}\left(\mathcal{F}(\mu) \wedge \left[\overline{\mu}_p, \lambda\right]\right) \\ &\le \bigvee_{\mu \in L^X}\left(\mathcal{F}(\mu) \wedge \left[\widehat{\phi}(\mu), \lambda\right]\right) = \mathcal{F}^\phi(\lambda); \end{aligned} \tag{22}$$

that is, $\overline{\mathcal{F}}_p \le \mathcal{F}^\phi$. Because $(X, \lim^q)$ is a strong L-Kent convergence space, then it follows that $\lim^q \mathcal{U}_p(y) = \lim^q(\phi(y) \wedge [y])(y) \ge \lim^q \phi(y)(y) = 1$, and so

$$\begin{aligned} &\forall \mathcal{G} \in \mathcal{F}_L^s(X), \ \forall y \in X, \\ &\lim^p \mathcal{G}(y) = \left[\mathcal{U}_p(y), \mathcal{G}\right] = \lim^q \mathcal{U}_p(y) \wedge \left[\mathcal{U}_p(y), \mathcal{G}\right] \\ &\qquad \overset{(LC2')}{\le} \lim^q \mathcal{G}(y). \end{aligned} \tag{23}$$

That is, $\lim^p \le \lim^q$. It follows by the assumption that $(X, \lim^q)$ is p-regular. Thus $\lim^q \mathcal{F}^\phi(x) \ge \lim^q \overline{\mathcal{F}}_p(x) \ge \lim^q \mathcal{F}(x)$. By Theorem 17 we know that $(X, \lim^q)$ is k'^*-regular. □

It is easily seen that when L is a complete Boolean algebra, then the above proposition holds for k'-regularity.

Lemma 30. *Let $(X, \lim^q)$ be a topological generalized stratified L-convergence space and let $\mathcal{T}$ be the stratified L-topology corresponding to $\lim^q$. Then $\mathcal{F} \ge \mathcal{U}_q(x)$ if and only if $\mathcal{F}(\mu) \ge \mathcal{U}_q(x)(\mu)$ for all $\mu \in \mathcal{T}$.*

Proof. We need only to check the sufficiency. Note that to for each $\mu \in L^X$, $\mathcal{U}_q(x)(\mu) = \text{int}(\mu)(x)$ and $\mathcal{U}_q(x)(\mu) = \text{int}(\mu)(x) = \mu(x)$ if $\mu \in \mathcal{T}$ [11, 30]. It follows that, for each $\lambda \in L^X$,

$$\begin{aligned} \mathcal{F}(\lambda) &= \bigvee_{\mu \in L^X}(\mathcal{F}(\mu) \wedge [\mu, \lambda]) \\ &\ge \bigvee_{\mu \in \mathcal{T}}(\mathcal{F}(\mu) \wedge [\mu, \lambda]) \ge \bigvee_{\mu \in \mathcal{T}}\left(\mathcal{U}_q(x)(\mu) \wedge [\mu, \lambda]\right) \\ &= \bigvee_{\mu \in \mathcal{T}}(\mu(x) \wedge [\mu, \lambda]) = \text{int}(\lambda)(x) = \mathcal{U}_q(x)(\lambda). \end{aligned} \tag{24}$$

□

Theorem 31. *Let L be a linearly order frame or let $0 \in L$ be prime. A topological generalized stratified L-convergence space $(X, \lim^q)$ is k'^*-regular if and only if it is p-regular for every ultrapretopological generalized stratified L-convergence structure $\lim^p \le \lim^q$.*

Proof. Note that a topological generalized stratified L-convergence space is natural a strong stratified L-Kent convergence space [17]. Then the sufficiency follows by Proposition 29. Thus, we prove only the necessity. Let $(X, \lim^q)$ be k'^*-regular and let $\lim^p$ be an arbitrary ultrapretopological generalized stratified L-convergence structure with $\lim^p \le \lim^q$. Then, for each $y \in X$, there exists a $\mathcal{H}_y \in \mathcal{U}_L^s(X)$ such that $\mathcal{U}_p(y) = \mathcal{H}_y \wedge [y]$. Obviously, $\lim^p \mathcal{H}_y(y) \ge \lim^p \mathcal{U}_p(y)(y) = 1$ and then $\lim^q \mathcal{H}_y(y) = 1$ by $\lim^p \le \lim^q$.

Let $\phi \in \Sigma^*(X)$ be defined by $\phi(y) = \mathcal{H}_y$, for all $y \in X$. Then $\lim^q \phi(y)(y) = 1$ for each $y \in X$. For each $\lambda \in \mathcal{T}$, we check below $[\overline{\lambda}_p, \widehat{\phi}(\lambda)] = 1$. Here, $\mathcal{T}$ is the stratified L-topology corresponding to $\lim^q$. For each $\phi(y) \in \mathcal{U}_L^s(X)$, it follows by Lemma 1 that $\phi(y)_{\mathbb{F}_{\phi(y)}} = \phi(y)$; that is,

$$\widehat{\phi}(\lambda)(y) = \phi(y)(\lambda) = \begin{cases} 1, & \iota\lambda \in \mathbb{F}_{\phi(y)}; \\ 0, & \iota\lambda \notin \mathbb{F}_{\phi(y)}. \end{cases} \tag{25}$$

Note that $[\overline{\lambda}_p, \widehat{\phi}(\lambda)] = \bigwedge_{y \in \iota(\overline{\lambda}_p)}(\overline{\lambda}_p(y) \to \phi(y)(\lambda))$. For each $y \in \iota(\overline{\lambda}_p)$, it follows that $\overline{\lambda}_p(y) = \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)}(\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)) > 0$, which means that there exists an $\mathcal{F}_y \in \mathcal{F}_L^s(X)$ such that $\lim^p \mathcal{F}_y(y) > 0$ and $\mathcal{F}_y(\lambda) > 0$. Thus, $\mathcal{F}_y(1_{\iota\lambda}) \ge \mathcal{F}_y(\lambda) > 0$. Fix $y \in \iota(\overline{\lambda}_p)$; we have $y \in \iota\lambda$ or $y \in X - \iota\lambda$.

Case 1. $y \in \iota\lambda$; that is, $\lambda(y) > 0$. Because $(X, \lim^q)$ is topological, then $\lambda(y) = \mathcal{U}_q(y)(\lambda) > 0$. From $\lim^q \phi(y)(y) = 1$, we get $\phi(y) \geq \mathcal{U}_q(y)$ and then $\phi(y)(\lambda) > 0$; indeed, $\phi(y)(\lambda) = 1$ since $\phi(y) \in \mathcal{U}_L^s(X)$ takes values in $\{0, 1\}$.

Case 2. $y \in X - \iota\lambda$; that is, $\lambda(y) = 0$. We assume that $\phi(y)(\lambda) \neq 1$; it follows by equality (25) that $\iota\lambda \notin \mathbb{F}_{\phi(y)}$. Because $\mathbb{F}_{\phi(y)}$ is an ultrafilter on X, then $X - \iota(\lambda) \in \mathbb{F}_{\phi(y)}$ and so $\phi(y)(1_{X-\iota\lambda}) = 1$. As we have known $\lim^p \mathcal{F}_y(y) > 0$ and $(X, \lim^p)$ is ultrapretopological; hence, $\lim^p \mathcal{F}_y(y) = [\mathcal{U}_p(y), \mathcal{F}_y] > 0$, then by $\mathcal{U}_p(y)(1_{X-\iota\lambda}) = \phi(y)(1_{X-\iota\lambda}) \wedge [y](1_{X-\iota\lambda}) = 1$ it follows that $\mathcal{F}_y(1_{X-\iota\lambda}) > 0$. Now,

$$0 = \mathcal{F}_y(1_{\iota\lambda} \wedge 1_{X-\iota\lambda}) \geq \mathcal{F}_y(1_{\iota\lambda}) \wedge \mathcal{F}_y(1_{X-\iota\lambda}) > 0. \quad (26)$$

A contradiction! Thus, if $y \in X - \iota\lambda$, then $\phi(y)(\lambda) = 1$.

Combining Cases 1 and 2 we get that if $y \in \iota(\overline{\lambda}_p)$ then $\widehat{\phi}(\lambda)(y) = 1$. It follows immediately that $[\overline{\lambda}_p, \widehat{\phi}(\lambda)] = 1$.

Next we prove that $k_L\phi(\overline{\mathcal{U}_q(x)}_p) \geq \mathcal{U}_q(x)$. By Lemma 30, we need only to check that $k_L\phi(\overline{\mathcal{U}_q(x)}_p)(\lambda) \geq \mathcal{U}_q(x)(\lambda)$ for all $\lambda \in \mathcal{T}$. Indeed,

$$\begin{aligned} k_L\phi\left(\overline{\mathcal{U}_q(x)}_p\right)(\lambda) &= \overline{\mathcal{U}_q(x)}_p\left(\widehat{\phi}(\lambda)\right) \\ &= \bigvee_{\mu \in L^X}\left(\mathcal{U}_q(x)(\mu) \wedge \left[\overline{\mu}_p, \widehat{\phi}(\lambda)\right]\right) \\ &\geq \mathcal{U}_q(x)(\lambda) \wedge \left[\overline{\lambda}_p, \widehat{\phi}(\lambda)\right] \\ &= \mathcal{U}_q(x)(\lambda). \end{aligned} \quad (27)$$

Then, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$,

$$\begin{aligned} \lim{}^q \mathcal{F}(x) &= \left[\mathcal{U}_q(x), \mathcal{F}\right] \leq \left[\overline{\mathcal{U}_q(x)}_p, \overline{\mathcal{F}}_p\right] \\ &\leq \left[k_L\phi\left(\overline{\mathcal{U}_q(x)}_p\right), k_L\phi\left(\overline{\mathcal{F}}_p\right)\right] \\ &\leq \left[\mathcal{U}_q(x), k_L\phi\left(\overline{\mathcal{F}}_p\right)\right] \\ &= \lim{}^q k_L\phi\left(\overline{\mathcal{F}}_p\right)(x) \\ &\leq \lim{}^q \overline{\mathcal{F}}_p(x), \end{aligned} \quad (28)$$

where the first and the second equalities hold by the pretopologicalness of $(X, \lim^q)$, the first inequality holds by Lemma 24, the second inequality holds by Lemma 5(4), and the last inequality holds because $(X, \lim^q)$ is k'^*-regular. Then it follows that $(X, \lim^q)$ is p-regular. □

Remark 32. To prove that Theorem 31 holds for k'-regularity, it seems that L must be a complete Boolean algebra. If we further assume that L is linearly ordered or $0 \in L$ is prime then $L = \{0, 1\}$. Thus, we guess that Theorem 31 holds for k'-regularity only if $L = \{0, 1\}$.

4.2. For Levelwise Stratified L-Convergence Spaces

Definition 33 (see [31]). Let $(X, \overline{p})$ be a levelwise stratified L-convergence space. For each $\lambda \in L^X$, the L-set $\overline{\lambda}_p^\alpha \in L^X$ defined by

$$\begin{gathered} \forall x \in X, \quad \overline{\lambda}_p^\alpha(x) = \bigvee_{\mathcal{F} \in c_p^\alpha(x)} \mathcal{F}(\lambda), \\ c_p^\alpha(x) = \left\{\mathcal{F} \in \mathcal{F}_L^s(X) : \mathcal{F} \xrightarrow{p_\alpha} x\right\} \end{gathered} \quad (29)$$

is called α-level closure of λ w.r.t. $(X, \overline{p})$.

It is easily seen that α-level closures of L-sets have similar properties to closures of L-sets. We do not list them but use them directly.

In [20], Boustique and Richardson modified Jäger's definition [11] and introduced a notion of α-level closures of stratified L-filters. In [25], we give an equivalent characterization of Boustique and Richardson's definition. This characterization seems more simple and more intuitive. Thus, we use it as the definition of α-level closures of stratified L-filters.

Definition 34. Let $(X, \overline{p})$ be a levelwise stratified L-convergence space. For each $\alpha \in L$ and each $\mathcal{F} \in \mathcal{F}_L^s(X)$, it is easily seen that the function $\overline{\mathcal{F}}_p^\alpha : L^X \to L$, defined by $\forall \lambda \in L^X$, $\overline{\mathcal{F}}_p^\alpha(\lambda) = \bigvee_{\mu \in L^X}(\mathcal{F}(\mu) \wedge [\overline{\mu}_p^\alpha, \lambda])$, is a stratified L-filter; then $\overline{\mathcal{F}}_p^\alpha$ is called the α-level closure of $\mathcal{F}$ w.r.t. $(X, \overline{p})$.

Definition 35 (see [24]). Let $(X, \overline{p}, \overline{q})$ be a pair of levelwise stratified L-convergence spaces. Then $(X, \overline{q})$ is called p-regular if, for each $\alpha \in L$ and each $\mathcal{F} \in \mathcal{F}_L^s(X)$, we have $\overline{\mathcal{F}}_p^\alpha \xrightarrow{q_\alpha} x$ whenever $\mathcal{F} \xrightarrow{q_\alpha} x$.

It is proved in [25] that p-regularity is preserved under initial constructions. Now, we look at the relationships between weaker regularities and p-regularity.

Definition 36. A levelwise stratified L-convergence space $(X, \overline{q})$ is called

(i) an L-Kent convergence space if $[x] \wedge \mathcal{F} \xrightarrow{q_\alpha} x$ whenever $\mathcal{F} \xrightarrow{q_\alpha} x$;

(ii) pretopological [23] if $\mathcal{F} \xrightarrow{q_\alpha} x$ if and only if $\mathcal{F} \geq \mathcal{U}_q^\alpha(x) = \wedge\{\mathcal{F} \mid \mathcal{F} \xrightarrow{q_\alpha} x\}$;

(iii) ultrapretopological if, for each $x \in X$ and each $\alpha \in L$, there exists a stratified L-ultrafilter $\mathcal{F}_x$ such that $\mathcal{U}_q^\alpha(x) = [x] \wedge \mathcal{F}_x$;

(iv) topological [23] if there exists a stratified L-topology $\mathcal{T}_\alpha$ for each $\alpha \in L$ such that $\forall \lambda \in L^X, \forall x \in X$, we have $\mathcal{U}_q^\alpha(x)(\lambda) = \mathrm{int}^\alpha(\lambda)(x)$, where $\mathrm{int}^\alpha(\lambda)$ is the interior of λ w.r.t. $\mathcal{T}_\alpha$.

Proposition 37. *Let $(X,\overline{q})$ be a levelwise stratified L-Kent convergence space which is p-regular relative to every ultrapretopological levelwise stratified L-convergence structure $\overline{p} \geq \overline{q}$. Here for $\overline{p} \geq \overline{q}$, we mean that $\mathcal{F} \xrightarrow{p_\alpha} x$ implies $\mathcal{F} \xrightarrow{q_\alpha} x$. Then $(X,\overline{q})$ is k^*-regular.*

Proof. Let $\phi \in \Sigma^*(X)$ and $\alpha \in L$ with $\forall y \in X, \phi(y) \xrightarrow{q_\alpha} y$. Let $\overline{p}$ be the ultrapretopological levelwise stratified L-convergence structure defined by $\forall \alpha \in L, \forall y \in X, \mathcal{U}_p^\alpha(y) = \phi(y) \wedge [y]$. From $\phi(y) \geq \mathcal{U}_p^\alpha(y)$ we have $\phi(y) \xrightarrow{p_\alpha} y$. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$ such that $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$ and $\mathcal{F} \xrightarrow{q_\alpha} x$, it follows that for each $\lambda \in L^X$, $\overline{\lambda}_p^\alpha(y) = \bigvee_{\mathcal{F} \in c_p^\alpha(y)} \mathcal{F}(\lambda) \geq \phi(y)(\lambda)$, which means $\overline{\lambda}_p^\alpha \geq \widehat{\phi}(\lambda)$. Thus,

$$\begin{aligned} \overline{\mathcal{F}}_p^\alpha(\lambda) &= \bigvee_{\mu \in L^X} \left(\mathcal{F}(\mu) \wedge \left[\overline{\mu}_p^\alpha, \lambda\right]\right) \\ &\leq \bigvee_{\mu \in L^X} \left(\mathcal{F}(\mu) \wedge \left[\widehat{\phi}(\mu), \lambda\right]\right) = \mathcal{F}^\phi(\lambda); \end{aligned} \tag{30}$$

that is, $\overline{\mathcal{F}}_p^\alpha \leq \mathcal{F}^\phi$. Because $(X,\overline{q})$ is an L-Kent convergence space, then it follows by $\phi(y) \xrightarrow{q_\alpha} y$ that $\mathcal{U}_p^\alpha(y) = \phi(y) \wedge [y] \xrightarrow{q_\alpha} y$. Thus, $\overline{p} \geq \overline{q}$; then $(X,\overline{q})$ is p-regular by the assumption. It follows that $\overline{\mathcal{F}}_p^\alpha \xrightarrow{q_\alpha} x$ and then $\mathcal{F}^\phi \xrightarrow{q_\alpha} x$ by $\overline{\mathcal{F}}_p^\alpha \leq \mathcal{F}^\phi$. By Theorem 17 we know that $(X,\overline{q})$ is k^*-regular. □

It is easily seen that when L is a complete Boolean algebra, then the above proposition holds for k-regularity.

Lemma 38. *Let $(X,\overline{q})$ be a topological levelwise stratified L-convergence space and let $\mathcal{T}_\alpha(\alpha \in L)$ be the stratified L-topologies corresponding to $\overline{q}$. Then $\mathcal{F} \geq \mathcal{U}_q^\alpha(x)$ if and only if $\mathcal{F}(\mu) \geq \mathcal{U}_q^\alpha(x)(\mu)$ for all $\mu \in \mathcal{T}_\alpha$.*

Proof. The proof is similar to Lemma 30 and thus it is omitted. □

Theorem 39. *Let L be a linearly order frame or let $0 \in L$ be prime. A topological levelwise stratified L-convergence space $(X,\overline{q})$ is k^*-regular if and only if it is p-regular for every ultrapretopological levelwise stratified L-convergence structure $\overline{p} \geq \overline{q}$.*

Proof. The sufficiency follows by Proposition 37. We prove only the necessity. Let $(X,\overline{q})$ be k^*-regular and let $\overline{p}$ be an arbitrary ultrapretopological levelwise stratified L-convergence structure with $\overline{p} \geq \overline{q}$. Fix $\alpha \in L$; then, for each $y \in X$, there exists a $\mathcal{H}_y \in \mathcal{U}_L^s(X)$ such that $\mathcal{U}_p^\alpha(y) = \mathcal{H}_y \wedge [y]$. Obviously, $\mathcal{H}_y \xrightarrow{p_\alpha} y$ and then $\mathcal{H}_y \xrightarrow{q_\alpha} y$ by $\overline{p} \geq \overline{q}$.

Let $\phi \in \Sigma^*(X)$ be defined by $\phi(y) = \mathcal{H}_y$, for all $y \in X$. For each $\lambda \in \mathcal{T}_\alpha$, we check below $[\overline{\lambda}_p^\alpha, \widehat{\phi}(\lambda)] = 1$. Here, $\mathcal{T}_\alpha(\alpha \in L)$ are the stratified L-topologies corresponding to $\overline{q}$.

Note that $[\overline{\lambda}_p^\alpha, \widehat{\phi}(\lambda)] = \bigwedge_{y \in \iota(\overline{\lambda}_p^\alpha)} (\overline{\lambda}_p^\alpha(y) \rightarrow \phi(y)(\lambda))$. For each $y \in \iota(\overline{\lambda}_p^\alpha)$, it follows that $\overline{\lambda}_p^\alpha(y) = \bigvee_{\mathcal{F} \in c_p^\alpha(y)} \mathcal{F}(\lambda) > 0$, which means that there exists an $\mathcal{F}_y \xrightarrow{p_\alpha} y$ such that $\mathcal{F}_y(\lambda) > 0$. Thus, $\mathcal{F}_y(1_{\iota\lambda}) \geq \mathcal{F}_y(\lambda) > 0$. Fix $y \in \iota(\overline{\lambda}_p^\alpha)$; then $y \in \iota\lambda$ or $y \in X - \iota\lambda$.

Case 1. $y \in \iota\lambda$; that is, $\lambda(y) > 0$. Because $(X,\overline{q})$ is topological, thus $\lambda(y) = \mathcal{U}_q^\alpha(y)(\lambda) = \wedge\{\mathcal{F}(\lambda) \mid \mathcal{F} \xrightarrow{q_\alpha} y\} > 0$. From $\phi(y) \xrightarrow{q_\alpha} y$, we get $\phi(y)(\lambda) > 0$; indeed, $\phi(y)(\lambda) = 1$ since $\phi(y) \in \mathcal{U}_L^s(X)$ takes values in $\{0, 1\}$.

Case 2. $y \in X - \iota\lambda$; that is, $\lambda(y) = 0$. We assume that $\phi(y)(\lambda) \neq 1$; it follows by equality (25) that $\iota\lambda \notin \mathbb{F}_{\phi(y)}$. Because $\mathbb{F}_{\phi(y)}$ is an ultrafilter on X, then $X - \iota(\lambda) \in \mathbb{F}_{\phi(y)}$ and so $\phi(y)(1_{X-\iota\lambda}) = 1$. As we have known $\mathcal{F}_y \xrightarrow{p_\alpha} y$; hence, $\mathcal{F}_y \geq \mathcal{U}_p^\alpha(y) = \phi(y) \wedge [y]$; then $\mathcal{F}_y(1_{X-\iota\lambda}) \geq \phi(y)(1_{X-\iota\lambda}) \wedge 1_{X-\iota\lambda}(y) = 1$. Now,

$$\begin{aligned} 0 &= \mathcal{F}_y\left(1_{\iota\lambda} \wedge 1_{X-\iota\lambda}\right) \\ &\geq \mathcal{F}_y\left(1_{\iota\lambda}\right) \wedge \mathcal{F}_y\left(1_{X-\iota\lambda}\right) = \mathcal{F}_y\left(1_{\iota\lambda}\right) > 0. \end{aligned} \tag{31}$$

A contradiction! Thus, if $y \in X - \iota\lambda$, then $\phi(y)(\lambda) = 1$.

Combining of Cases 1 and 2 we get that if $y \in \iota(\overline{\lambda}_p^\alpha)$ then $\widehat{\phi}(\lambda)(y) = 1$. It follows immediately that $[\overline{\lambda}_p^\alpha, \widehat{\phi}(\lambda)] = 1$. Then similar to Lemma 30 we have $k_L\phi(\overline{\mathcal{U}_q^\alpha(x)}_p^\alpha) \geq \mathcal{U}_q^\alpha(x)$. Let $\mathcal{F} \xrightarrow{q_\alpha} x$; then $\mathcal{F} \geq \mathcal{U}_q^\alpha(x)$ by the topologicalness of $\overline{q}$. Hence, $\overline{\mathcal{F}}_p^\alpha \geq \overline{\mathcal{U}_q^\alpha(x)}_p^\alpha$ and then $k_L\phi(\overline{\mathcal{F}}_p^\alpha) \geq k_L\phi(\overline{\mathcal{U}_q^\alpha(x)}_p^\alpha) \geq \mathcal{U}_q^\alpha(x)$, which means $k_L\phi(\overline{\mathcal{F}}_p^\alpha) \xrightarrow{q_\alpha} x$. Because $(X,\overline{q})$ is k^*-regular, then $\overline{\mathcal{F}}_p^\alpha \xrightarrow{q_\alpha} x$. It follows that $(X,\overline{q})$ is p-regular. □

Remark 40. Similar to Remark 32, we guess that Theorem 39 holds for k-regularity only if $L = \{0, 1\}$.

5. Conclusions

In this paper, we introduce some weaker regularities for levelwise stratified L-convergence spaces and generalized stratified L-convergence spaces and study their characterizations and properties. For generalized stratified L-convergence spaces, we also investigate a notion of closures of stratified L-filters and then define by it a new p-regularity which is different from the p-regularity in [25] defined by the notion of α-level closures of stratified L-filters. At last, we discuss the relationships between weaker regularities and p-regularities. In addition, it seems that the p-regularity (for generalized stratified L-convergence spaces in [25]) has close relationships with k-regularity and k^*-regularity. But we fail to establish those relationships for it is difficult to find an appropriate definition for ultrapretopological generalized stratified L-convergence spaces.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] H. J. Kowalsky, "Limesräume und Komplettierung," *Mathematische Nachrichten*, vol. 12, pp. 301–340, 1954.

[2] C. H. Cook and H. R. Fischer, "Regular convergence spaces," *Mathematische Annalen*, vol. 174, no. 1, pp. 1–7, 1967.

[3] D. C. Kent and G. D. Richardson, "*p*-regular convergence spaces," *Mathematische Nachrichten*, vol. 149, pp. 215–222, 1990.

[4] S. A. Wilde and D. C. Kent, "*p*-topological and *p*-regular: dual notions in convergence theory," *International Journal of Mathematics and Mathematical Sciences*, vol. 22, pp. 1–12, 1999.

[5] W. Gähler, "Monadic topology—a new concept of generalized topology," in *Recent Developments of General Topology*, vol. 67 of *Mathematical Research*, pp. 136–149, Akademie, Berlin, Germany, 1992.

[6] D. C. Kent and G. D. Richardson, "Convergence spaces and diagonal conditions," *Topology and its Applications*, vol. 70, no. 2-3, pp. 167–174, 1996.

[7] G. Jäger, "A category of *L*-fuzzy convergence spaces," *Quaestiones Mathematicae*, vol. 24, pp. 501–517, 2001.

[8] J. M. Fang, "Stratified *L*-ordered convergence structures," *Fuzzy Sets and Systems*, vol. 161, no. 16, pp. 2130–2149, 2010.

[9] J. M. Fang, "Relationships between *L*-ordered convergence structures and strong *L*-topologies," *Fuzzy Sets and Systems*, vol. 161, no. 22, pp. 2923–2944, 2010.

[10] G. Jäger, "Subcategories of lattice-valued convergence spaces," *Fuzzy Sets and Systems*, vol. 156, no. 1, pp. 1–24, 2005.

[11] G. Jäger, "Pretopological and topological lattice-valued convergence spaces," *Fuzzy Sets and Systems*, vol. 158, no. 4, pp. 424–435, 2007.

[12] G. Jäger, "Fischer's diagonal condition for lattice-valued convergence spaces," *Quaestiones Mathematicae*, vol. 31, no. 1, pp. 11–25, 2008.

[13] G. Jäger, "Lattice-valued convergence spaces and regularity," *Fuzzy Sets and Systems*, vol. 159, no. 19, pp. 2488–2502, 2008.

[14] G. Jäger, "Gähler's neighbourhood condition for lattice-valued convergence spaces," *Fuzzy Sets and Systems*, vol. 204, pp. 27–39, 2012.

[15] L. Li, *Many-valued convergence, many-valued topology, and many-valued order structure [Ph.D. thesis]*, Sichuan University, 2008, (Chinese).

[16] L. Li and Q. Jin, "On adjunctions between Lim, S*L*-Top, and S*L*-Lim," *Fuzzy Sets and Systems*, vol. 182, no. 1, pp. 66–78, 2011.

[17] L. Li and Q. Jin, "On stratified *L*-convergence spaces: pretopological axioms and diagonal axioms," *Fuzzy Sets and Systems*, vol. 204, pp. 40–52, 2012.

[18] D. Orpen and G. Jäger, "Lattice-valued convergence spaces: extending the lattice context," *Fuzzy Sets and Systems*, vol. 190, pp. 1–20, 2012.

[19] W. Yao, "On many-valued stratified *L*-fuzzy convergence spaces," *Fuzzy Sets and Systems*, vol. 159, no. 19, pp. 2503–2519, 2008.

[20] H. Boustique and G. Richardson, "A note on regularity," *Fuzzy Sets and Systems*, vol. 162, no. 1, pp. 64–66, 2011.

[21] H. Boustique and G. Richardson, "Regularity: lattice-valued Cauchy spaces," *Fuzzy Sets and Systems*, vol. 190, pp. 94–104, 2012.

[22] P. V. Flores, R. N. Mohapatra, and G. Richardson, "Lattice-valued spaces: fuzzy convergence," *Fuzzy Sets and Systems*, vol. 157, no. 20, pp. 2706–2714, 2006.

[23] P. V. Flores and G. Richardson, "Lattice-valued convergence: diagonal axioms," *Fuzzy Sets and Systems*, vol. 159, no. 19, pp. 2520–2528, 2008.

[24] B. Losert, H. Boustique, and G. Richardson, "Modifications: lattice-valued structures," *Fuzzy Sets and Systems*, vol. 210, pp. 54–62, 2013.

[25] L. Li and Q. Jin, "*p*-Topologicalness and *p*-regularity for lattice-valued convergence spaces," *Fuzzy Sets and Systems*, 2013.

[26] R. Bělohlávek, *Fuzzy Relational Systems: Foundations and Principles*, Kluwer Academic, New York, NY, USA, 2002.

[27] U. Höhle and A. Šostak, "Axiomatic foundations of fixed-basis fuzzy topology," in *Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory*, U. Höhle and S. E. Rodabaugh, Eds., vol. 3 of *The Handbooks of Fuzzy Sets Series*, pp. 123–273, Kluwer Academic, London, UK, 1999.

[28] G. Jäger, "Lowen fuzzy convergence spaces viewed as *L*-fuzzy convergence spaces," *The Journal of Fuzzy Mathematics*, vol. 10, pp. 227–236, 2002.

[29] G. Preuss, *Fundations of Topology*, Kluwer Academic, London, UK, 2002.

[30] D. Zhang, "An enriched category approach to many valued topology," *Fuzzy Sets and Systems*, vol. 158, no. 4, pp. 349–366, 2007.

[31] G. Jäger, "Diagonal conditions for lattice-valued uniform convergence spaces," *Fuzzy Sets and Systems, Fuzzy Sets and Systems*, vol. 210, pp. 39–53, 2013.