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Letter to the Editor

Extension of the GSMW Formula in Weaker Assumptions

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In this note, the generalized Sherman-Morrison-Woodbury (for short GSMW) formula $(A+YGZ^*)^{\odot} = A^{\odot} - A^{\odot}Y(G^{\odot}+Z^*A^{\odot}Y)^{\odot}Z^*A^{\odot}$ is extended under some assumptions weaker than those used by Duan, 2013.

1. Introduction

Denote by $\mathscr{B}(\mathscr{H},\mathscr{K})$ (by $\mathscr{B}(\mathscr{H})$, when $\mathscr{H}=\mathscr{K}$) the set of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$, where $\mathscr{H}$ and $\mathscr{K}$ are complex Hilbert spaces. For $T \in \mathscr{B}(\mathscr{H},\mathscr{K})$, let T^*, $\mathscr{R}(T)$, and $\mathscr{N}(T)$ be the adjoint, the range, and the null space of T, respectively. Recall that the original SMW [1–3] formula (1) is only valid when $A \in \mathscr{B}(\mathscr{H})$, $G \in \mathscr{B}(\mathscr{K})$, and $G^{-1}+Z^*A^{-1}Y$ are invertible and the SMW formula has the form

$$(A+YGZ^*)^{-1} = A^{-1} - A^{-1}Y\left(G^{-1}+Z^*A^{-1}Y\right)^{-1}Z^*A^{-1}, \tag{1}$$

where $Y, Z \in \mathscr{B}(\mathscr{K},\mathscr{H})$.

Let I be the identity in $\in \mathscr{B}(\mathscr{H})$ and let $T \in \mathscr{B}(\mathscr{H})$. Recall that the standard inverse T^{-1} of T must satisfy (I) $TT^{-1}=T^{-1}T=I$, while the generalized inverse S of T need only to satisfy (I) $TST=T$. Note that S is unique if imposed additional conditions as (II) $STS=S$, (III) $(TS)^*=TS$, (IV) $(ST)^*=ST$, (V) $TS=ST$, and (VI) $T^kST=T^k$, where $S \in \mathscr{B}(\mathscr{H})$ satisfying (II) are called $\{2\}$-inverse of T, denoted by $S=T^-$. Similarly, (I, II, V)-inverses are called group inverses, denoted by $S=T^{\#}$. (I, II, III, IV)-inverses are Moore-Penrose inverses, denoted by $S=T^+$. And (II, V, VI)-inverses are called Drazin inverses, denoted by $S=T^D$ (see [4]), where k is the Drazin index of T. Note that the standard inverse, the group inverse, the Moore-Penrose inverse, and the Drazin inverse all belong to the 2-inverse. It is straight that the SMW formula holds for all the inverses if and only if it holds for the $\{2\}$-inverse.

Because of its wide applications in statistics, networks, structural analysis, asymptotic analysis, optimization, and partial differential equations (see [5]), the properties and generalizations of the SMW formula have caught mathematicians attention (see [1–8]). Duan (see [9]) finally generalized the SMW formula to the $\{2\}$-inverse (hence, to all the inverses, uniformly denoted by $T^{\odot}$). Under some sufficient conditions (see [9]), the generalized Sherman-Morrison-Woodbury (for short GSMW) formula has the form

$$(A+YGZ^*)^{\odot} = A^{\odot} - A^{\odot}Y(G^{\odot}+Z^*A^{\odot}Y)^{\odot}Z^*A^{\odot}, \tag{2}$$

where $A \in \mathscr{B}(\mathscr{H})$, $G \in \mathscr{B}(\mathscr{K})$, and $Y, Z \in \mathscr{B}(\mathscr{K},\mathscr{H})$.

Duan questioned whether the GSMW formula can be extended in some weaker assumptions. This problem is worthy of being followed up.

2. Main Result

The following two lemmas are used to prove the main result.

Lemma 1. *If $A \in \mathscr{B}(\mathscr{H})$ and $P=P^2 \in \mathscr{B}(\mathscr{H})$, then $AP=A$ if and only if $\mathscr{N}(P) \subset \mathscr{N}(A)$.*

Lemma 2. *Let $A \in \mathscr{B}(\mathscr{H})$, $G \in \mathscr{B}(\mathscr{K})$, and $Y, Z \in \mathscr{B}(\mathscr{K},\mathscr{H})$. Let also $B=A+YGZ^*$, $T=G^{\odot}+Z^*A^{\odot}Y$, and*

$X = A^{\odot} - A^{\odot}Y(G^{\odot} + Z^* A^{\odot}Y)^{\odot} Z^* A^{\odot}$. *Then, the following three statements are equivalent:*

(i) *the GSMW formula holds;*

(ii) $A^{\odot}(YGZ^* - YT^{\odot}Z^*)X = A^{\odot}YT^{\odot}Z^* A^{\odot}YGZ^* X$;

(iii) $X(YGZ^* - YT^{\odot}Z^*)A^{\odot} = XYGZ^* A^{\odot}YT^{\odot}Z^* A^{\odot}$.

Proof. The GSMW formula holds if and only if $XBX = X$. But it is easy to see that $XAA^{\odot} = A^{\odot}AX = X$. Hence, we have

$$\begin{aligned} XBX &= \left(A^{\odot} - A^{\odot}YT^{\odot}Z^* A^{\odot}\right)\left(A + YGZ^*\right) X \\ &= \left(A^{\odot}A + A^{\odot}YGZ^* - A^{\odot}YT^{\odot}Z^* A^{\odot}A \right. \\ &\qquad \left. - A^{\odot}YT^{\odot}Z^* A^{\odot}YGZ^*\right) X \\ &= X + A^{\odot}YGZ^* X - A^{\odot}YT^{\odot}Z^* X \\ &\quad - A^{\odot}YT^{\odot}Z^* A^{\odot}YGZ^* X \\ &= X + A^{\odot}\left(YGZ^* - YT^{\odot}Z^*\right) X \\ &\quad - A^{\odot}YT^{\odot}Z^* A^{\odot}YGZ^* X \end{aligned} \tag{3}$$

and, meanwhile,

$$\begin{aligned} XBX &= X\left(A + YGZ^*\right)\left(A^{\odot} - A^{\odot}YT^{\odot}Z^* A^{\odot}\right) \\ &= X\left(AA^{\odot} + YGZ^* A^{\odot} - AA^{\odot}YT^{\odot}Z^* A^{\odot} \right. \\ &\qquad \left. - YGZ^* A^{\odot}YT^{\odot}Z^* A^{\odot}\right) \\ &= X + XYGZ^* A^{\odot} - XYT^{\odot}Z^* A^{\odot} \\ &\quad - XYGZ^* A^{\odot}YT^{\odot}Z^* A^{\odot} \\ &= X + X\left(YGZ^* - YT^{\odot}Z^*\right) A^{\odot} \\ &\quad - XYGZ^* A^{\odot}YT^{\odot}Z^* A^{\odot}. \end{aligned} \tag{4}$$

It is immediate that the three statements are equivalent. □

Now, the first main result of this paper is given as follows.

Theorem 3. *Let* $A \in \mathscr{B}(\mathscr{H})$, $G \in \mathscr{B}(\mathscr{K})$, *and* $Y, Z \in \mathscr{B}(\mathscr{K}, \mathscr{H})$. *Let also* $B = A + YGZ^*$ *and* $T = G^{\odot} + Z^* A^{\odot}Y$. *The GSMW formula holds if one of the two following statements holds:*

(i) $\mathscr{R}(Z^*) \subset \mathscr{R}(G^{\odot})$, $\mathscr{N}(T^{\odot}T) \subset \mathscr{N}(Y)$;

(ii) $\mathscr{N}(G^{\odot}) \subset \mathscr{N}(Y)$, $\mathscr{R}(Z^*) \subset \mathscr{R}(TT^{\odot})$.

Proof. Note that $Z^* A^{\odot}Y = T - G^{\odot}$.

(i) Assume that $\mathscr{R}(Z^*) \subset \mathscr{R}(G^{\odot})$, $\mathscr{N}(T^{\odot}T) \subset \mathscr{N}(Y)$. By Lemma 1, we have $Y(I - T^{\odot}T) = 0$ and $(I - G^{\odot}G)Z^* = 0$. Hence, $(A^{\odot}YGZ^* - A^{\odot}YT^{\odot}Z^*)X - A^{\odot}YT^{\odot}Z^* A^{\odot}YGZ^* X = A^{\odot}Y(I - T^{\odot}T)GZ^* X - A^{\odot}YT^{\odot}(I - G^{\odot}G)Z^* X = 0$.

(ii) Assume that $\mathscr{N}(G^{\odot}) \subset \mathscr{N}(Y)$, $\mathscr{R}(Z^*) \subset \mathscr{R}(TT^{\odot})$. By Lemma 1, we have $Y(I - GG^{\odot}) = 0$ and $(I - TT^{\odot})Z^* = 0$. Hence, $X(YGZ^* - YT^{\odot}Z^*)A^{\odot} - XYGZ^* A^{\odot}YT^{\odot}Z^* A^{\odot} = XYG(I - TT^{\odot})Z^* A^{-} - XY(I - GG^{\odot})T^{\odot}Z^* A^{\odot} = 0$.

By Lemma 2, The GSMW formula holds if one of (i) and (ii) holds. □

3. Concluding Remark

According to Theorem 3 in this paper, Theorem 5 and Corollary 6 in [9] still hold under weaker assumptions. It must be noted that there are no assumptions on $B^{\odot}$ in Theorem 3; hence, it also present more convenience than Theorem 3 and Corollary 4 in [9] in applications. The results are even robust for the finite dimensional case. Nevertheless, it remains undetermined whether these assumptions are the weakest. We would like to propose this unresolved issue as an open question for international research interest.

Conflict of Interests

The authors declare that there is no conflict of interests.

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