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Research Article

From Caristi's Theorem to Ekeland's Variational Principle in 0_σ-Complete Metric-Like Spaces

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We discuss the extension of some fundamental results in nonlinear analysis to the setting of 0_σ-complete metric-like spaces. Then, we show that these extensions can be obtained via the corresponding results in standard metric spaces.

1. Introduction

It is well known that the Banach-Caccioppoli's theorem [1, 2] is the starting point for the development of metric fixed point theory. Over the years, this theory has evolved by receiving the support and interest of many mathematicians. In fact, the fixed point theorems and constructive techniques have been successfully applied in pure and applied analysis, topology, and others. Consequently various generalizations and extensions of the Banach-Caccioppoli's theorem have appeared in the literature; see for instance [3–10]. Precisely, these contributes have investigated the basic problems of the metric fixed point theory: existence and uniqueness under different contractive conditions, convergence of successive approximations, and well posedness of the fixed point problem.

In particular, Hitzler and Seda [11] presented the notion of dislocated metric space and proposed their generalization of the Banach-Caccioppoli's theorem. Hitzler and Seda's idea is to apply this theorem in order to obtain a unique supported model for acceptable logic programs. Also, many authors developed the fixed point theory in the setting of dislocated metric spaces; see for instance [12].

In 2012 Amini-Harandi rediscovered the notion of dislocated metric space in [13], as a generalization of a partial metric space [14]. These spaces were called metric-like spaces and used to introduce different notions of convergence and Cauchy sequence.

Inspired by the ideas in [11, 15], we characterize those metric-like spaces for which every Caristi's mapping [16] has a fixed point in the sense of the Romaguera's characterization of partial metric 0-completeness [17]. This will be done by means of the notion of a 0_σ-complete metric-like space which is introduced in the sequel. Then, we present fixed point theorems in this setting, by using known and new classes of lower semicontinuous functions. Finally, as an application of our technique, we deduce Ekeland's variational principle in a 0_σ-complete metric-like space. The aim of this work is to underline the strong relation between standard metric spaces and their generalizations to better target the research on this topic. Applying the approach followed in this paper, for instance, the reader can obtain the extensions to a metric-like space of many recent results in fixed point theory.

2. Metric-Like Spaces

In this section we collect first known notions and notations and then auxiliary concepts and tools to develop our theory. For a comprehensive discussion, we refer the reader to [13].

2.1. Preliminaries. We start by recalling some basic definitions and properties of the setting which we will use.

Definition 1 (see [13]). A metric-like on a nonempty set X is a function $\sigma : X\times X \to [0,+\infty)$ such that, for all $x, y, z \in X$,

(σ_1) $\sigma(x, y) = 0$ implies $x = y$;

(σ_2) $\sigma(x, y) = \sigma(y, x)$;

(σ_3) $\sigma(x, y) \le \sigma(x, z) + \sigma(z, y)$.

A metric-like space is a pair (X,σ) such that X is a nonempty set and σ is a metric-like on X.

Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-balls $\{B_\sigma(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where

$$B_\sigma(x,\varepsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \varepsilon\} \quad \forall x \in X, \quad \varepsilon > 0. \tag{1}$$

Then a sequence $\{x_n\}$ in the metric-like space (X,σ) converges to a point $x \in X$ if and only if $\lim_{n\to+\infty}\sigma(x_n, x) = \sigma(x, x)$.

A sequence $\{x_n\}$ of elements of X is called σ-Cauchy if the limit $\lim_{m,n\to+\infty}\sigma(x_m, x_n)$ exists and is finite. The metric-like space (X,σ) is called complete if, for each σ-Cauchy sequence $\{x_n\}$, there is some $x \in X$ such that

$$\sigma(x,x) = \lim_{n\to+\infty}\sigma(x_n,x) = \lim_{n,m\to+\infty}\sigma(x_n,x_m). \tag{2}$$

If $\lim_{n,m\to+\infty}\sigma(x_n, x_m) = 0$, then $\{x_n\}$ is called a 0_σ-Cauchy sequence. If every 0_σ-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_σ, to a point $x \in X$ such that $\sigma(x, x) = 0$, then (X,σ) is called 0_σ-complete; see the paper of Romaguera [17] for a comparative discussion with partial metric spaces. Here we point out that every partial metric space is a metric-like space; see [13]. Also we give some examples of a metric-like space.

Example 2. Let $X = [0,+\infty)$ and $\sigma : X \times X \to [0,+\infty)$ be defined by

$$\sigma(x,y) = \begin{cases} 2 & \text{if } x = y = 0, \\ 1 & \text{otherwise.} \end{cases} \tag{3}$$

Then (X,σ) is a metric-like space, which is not a metric space or a partial metric space.

Example 3. Let $X = [0,+\infty)$ and $\sigma : X \times X \to [0,+\infty)$ be defined by

$$\sigma(x,y) = x + y, \tag{4}$$

for all $x, y \in [0,+\infty)$. Then (X,σ) is a complete metric-like space, which is not a metric space or a partial metric space.

Example 4. Let $X = [0,+\infty) \cap \mathbb{Q}$ and $\sigma : X \times X \to [0,+\infty)$ be defined by

$$\sigma(x,y) = x + y, \tag{5}$$

for all $x, y \in [0,+\infty) \cap \mathbb{Q}$. Then (X,σ) is a 0_σ-complete metric-like space, which is not a complete metric-like space.

2.2. Metric Induced by a Metric-Like. We introduce useful tools for developing our theory.

Let (X,σ) be a 0_σ-complete metric-like space and let $d : X \times X \to [0,+\infty)$ be defined by

$$d(x,y) = \begin{cases} \sigma(x,y) & \text{if } x \ne y, \\ 0 & \text{if } x = y. \end{cases} \tag{6}$$

Lemma 5. *Let (X,σ) be a metric-like space and $d : X \times X \to [0,+\infty)$ the function defined by* (6)*. Then (X,d) is a metric space. Moreover (X,d) is a complete metric space if and only if (X,σ) is a 0_σ-complete metric-like space.*

Proof. Clearly, $d(x, y) = d(y, x)$ and $d(x, y) = 0 \Leftrightarrow x = y$. Moreover, for all $x, y, z \in X$, we have

$$d(x,y) \le d(x,z) + d(z,y) \tag{7}$$

if $x = y$ or if $x \ne y$ and $z = x$ or $z = y$. Also, if x, y, z are distinct points, from

$$\sigma(x,y) \le \sigma(x,z) + \sigma(z,y), \tag{8}$$

we get

$$d(x,y) \le d(x,z) + d(z,y), \tag{9}$$

and so the triangle inequality holds. Thus d is a metric on X and hence (X, d) is a metric space.

Note that if $\{x_n\} \subset X$ is such that $x_n \ne x_m$ for all $n \ne m$, then $\lim_{m,n\to+\infty}\sigma(x_n, x_m) = 0$ if and only if $\lim_{m,n\to+\infty}d(x_n, x_m) = 0$.

Now, suppose that (X,σ) is a 0_σ-complete metric-like space and $\{x_n\}$ is a Cauchy sequence in (X, d). If $x_n = x_m$ for all $n \ge m$, then the sequence $\{x_n\}$ converges to $x = x_m \in X$. Then we can assume that $x_n \ne x_m$ for all $n \ne m$. In reason of the above observation, we get that $\{x_n\}$ is a 0_σ-Cauchy sequence in (X,σ). Using the fact that (X,σ) is a 0_σ-complete metric-like space, then there exists $z \in X$ such that $\lim_{n\to+\infty}\sigma(x_n, z) = \sigma(z, z) = 0$; that is, $\lim_{n\to+\infty}d(x_n, z) = 0$. Thus (X, d) is a complete metric space.

Now, suppose that (X, d) is a complete metric space and $\{x_n\}$ is a 0_σ-Cauchy sequence in (X,σ). Without loss of generality, assume that $x_n \ne x_m$ for all $n \ne m$. Then

$$\lim_{m,n\to+\infty}d(x_n,x_m) = \lim_{m,n\to+\infty}\sigma(x_n,x_m) = 0. \tag{10}$$

Hence, $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is complete, there exists $z \in X$ such that $\lim_{n\to+\infty}d(x_n, z) = 0$. Thus, $\lim_{n\to+\infty}\sigma(x_n, z) = 0 = \sigma(z, z)$ and so (X,σ) is 0_σ-complete. □

Let $X_0 := \{x \in X : \sigma(x, x) = 0\}$; we have the following proposition.

Proposition 6. *Let (X,σ) be a metric-like space and $d : X \times X \to [0,+\infty)$ the metric defined by* (6)*. Let x be a point of X and let $\{x_n\} \subseteq X$ be such that $\lim_{n\to+\infty}d(x_n, x) = 0$. If $x_n \ne x$ for infinity values of n, then $x \in X_0$. Moreover X_0 is a closed subset of (X, d).*

Proof. From $\sigma(x_n, x) = d(x_n, x)$ for every $n \in \mathbb{N}$ such that $x_n \neq x$ it follows that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to+\infty}\sigma(x_{n_k}, x) = 0$. This implies $\sigma(x, x) = 0$, since $\sigma(x, x) \leq 2\sigma(x_{n_k}, x) \to 0$, as $k \to +\infty$ and so $x \in X_0$. Then for each sequence $\{x_n\} \subseteq X_0$ we have $\sigma(x_n, x) = d(x_n, x)$ and so X_0 is a closed subset of (X, d). □

Definition 7. Let (X, σ) be a metric-like space and $T : X \to X$ a mapping. T is called 0_σ-continuous if, for all $x \in X_0$ and $\{x_n\} \subseteq X$ with $x_n \to x$ as $n \to +\infty$, we have $\sigma(Tx_n, Tx) \to 0$.

Remark 8. Let (X, σ) be a metric-like space and $T : X \to X$ a mapping. If T is 0_σ-continuous, then $T(X_0) \subseteq X_0$. In fact, if $x \in X_0$ and $\{x_n\} \subseteq X$ is a sequence such that $x_n \to x$ as $n \to +\infty$, then $\sigma(Tx_n, Tx) \to 0$ and so $\sigma(Tx, Tx) = 0$.

Proposition 9. *Let (X, σ) be a metric-like space, $d : X\times X \to [0, +\infty)$ the metric defined in (6), and $T : X \to X$ a mapping such that $T(X_0) \subseteq X_0$. Then T is continuous in (X, d) if and only if T is 0_σ-continuous in (X, σ).*

Proof. First, we assume that T is 0_σ-continuous in (X, σ) and let $\{x_n\} \subseteq X$ be a sequence convergent to a point $x \in X$ in (X, d). Clearly, $\lim_{n\to+\infty}d(Tx_n, Tx) = 0$ if $Tx_n = Tx$ for all $n \geq m \in \mathbb{N}$. Then, without loss of generality, we assume that $Tx_n \neq Tx$ for all $n \in \mathbb{N}$. This implies that $x_n \neq x$ for all $n \in \mathbb{N}$ and hence $\sigma(x_n, x) = d(x_n, x) \to 0$. By Proposition 6, we get that $x \in X_0$. Then $d(Tx_n, Tx) = \sigma(Tx_n, Tx) \to 0$ and T is continuous in (X, d).

Now, we assume that T is continuous in (X, d), x is a given point in X_0, and $\{x_n\} \subseteq X$ is a sequence convergent to x. Without loss of generality, we assume that $Tx_n \neq Tx$ for all $n \neq m$. From $\sigma(Tx_n, Tx) = d(Tx_n, Tx) \to 0$ as $n \to +\infty$, it follows that T is 0_σ-continuous in (X, σ). □

Definition 10. Let (X, σ) be a metric-like space. A mapping $T : X \to X$ is a contraction if there exists $k \in [0, 1)$ such that

$$\sigma(Tx, Ty) \leq k\sigma(x, y), \quad \forall x, y \in X. \tag{11}$$

Remark 11. Let (X, σ) be a metric-like space. Every contraction $T : X \to X$ is a 0_σ-continuous mapping. In fact, for all $x \in X_0$ and all sequences $\{x_n\} \subseteq X$ with $x_n \to x$ as $n \to +\infty$, we get $\sigma(Tx_n, Tx) \leq k\sigma(x_n, x) \to 0$.

Finally, we introduce the notion of 0_σ-lower semicontinuous function.

Definition 12. Let (X, σ) be a metric-like space and $d : X \times X \to [0, +\infty)$ the metric defined by (6). Assume that $X_0 \neq \emptyset$. A function $\phi : X \to [0, +\infty)$ is called 0_σ-lower semicontinuous if, for all $x \in X_0$ and every sequence $\{x_n\} \subseteq X$ with $\lim_{n\to+\infty}\sigma(x_n, x) = 0$, we have

$$\phi(x) \leq \liminf_{n\to+\infty} \phi(x_n). \tag{12}$$

Lemma 13. *Let (X, σ) be a metric-like space with $X_0 \neq \emptyset$ and $d : X \times X \to [0, +\infty)$ the metric defined by (6). Then a function $\phi : X \to [0, +\infty)$ is lower semicontinuous in (X, d) if and only if ϕ is 0_σ-lower semicontinuous in (X, σ).*

Proof. First, we assume that the function ϕ is 0_σ-lower semicontinuous in (X, σ) and let $\{x_n\} \subseteq X$ be a sequence convergent to $x \in X$ in (X, d). If $x_n = x_m$ for all $n \in \mathbb{N}$ with $n \geq m$, then (12) holds since $x = x_m$. Then we can assume that $x_n \neq x_m$ for all $n \neq m$ and also that $x \neq x_n$ for all $n \in \mathbb{N}$. This implies $\sigma(x_n, x) = d(x_n, x) \to 0$ as $n \to +\infty$ and hence (12) holds, since ϕ is 0_σ-lower semicontinuous in (X, σ) and hence ϕ is lower semicontinuous in (X, d).

Now, we assume that the function ϕ is lower semicontinuous in (X, d) and let $\{x_n\} \subseteq X$ be a sequence convergent to $x \in X_0$ in (X, σ). This implies $d(x_n, x) = \sigma(x_n, x) \to 0$ as $n \to +\infty$ and hence (12) holds, since ϕ is lower semicontinuous in (X, d) and so ϕ is 0_σ-lower semicontinuous in (X, σ). □

3. Fixed Point Theorems

The significance of the results given in the previous section will become clear as we proceed with the following applications of fixed points.

3.1. Caristi Type Fixed Point Theorems. The following theorem is an extension of the result of Caristi [16, Theorem (2.1)′] in the setting of metric-like spaces. First, we say that a mapping $f : X \to X$ satisfying the condition

$$\sigma(x, fx) \leq \phi(x) - \phi(fx), \quad \text{for each } x \in X, \tag{13}$$

where $\phi : X \to [0, +\infty)$ is a 0_σ-lower semicontinuous function, is a Caristi's mapping on (X, σ). Also, a point $z \in X$ such that $z = fz$ is called a fixed point of f.

Theorem 14. *Let (X, σ) be a 0_σ-complete metric-like space. Then any Caristi's mapping on (X, σ) has a fixed point z in X with $\sigma(z, z) = 0$.*

Proof. Let $d : X\times X \to [0, +\infty)$ be the metric defined by (6). Then, by Lemma 5, (X, d) is a complete metric space. From $d(x, fx) \leq \sigma(x, fx)$ for all $x \in X$ and (13), we get

$$d(x, fx) \leq \phi(x) - \phi(fx), \quad \text{for each } x \in X. \tag{14}$$

Since ϕ is lower semicontinuous in (X, d) by Lemma 13, then by Caristi's theorem f has a fixed point z. Finally, by (13), we get $\sigma(z, z) = 0$. □

Example 15. Let $X = \{-2, -1\} \cup [0, +\infty)$ and $\sigma : X \times X \to [0, +\infty)$ be defined by $\sigma(x, y) = |x - y|$ if $x \neq y$ and

$$\sigma(x, x) = \begin{cases} 1 & \text{if } x \in \{-2, -1\}, \\ 0 & \text{if } x \in [0, +\infty). \end{cases} \tag{15}$$

Clearly, (X, σ) is a 0_σ-complete metric-like space. Also, notice that $X_0 := \{x \in X : \sigma(x, x) = 0\} = [0, +\infty)$. Consider the mapping $f : X \to X$ defined by

$$fx = \begin{cases} 0 & \text{if } x \notin [1, 2], \\ 1 & \text{if } x \in [1, 2]. \end{cases} \tag{16}$$

Then, we get

$$\sigma(x, fx) = \begin{cases} |x| & \text{if } x \notin [1, 2], \\ x - 1 & \text{if } x \in [1, 2]. \end{cases} \tag{17}$$

It is easy to show that the function $\phi : X \to [0, +\infty)$, defined by $\phi(x) = |x|$ for all $x \in X$, is a 0_σ-lower semicontinuous function. Also we get $\sigma(x, fx) = \phi(x) - \phi(fx)$ and so f is a Caristi's mapping. Thus Theorem 14 ensures that f has a fixed point; here 0 and 1 are fixed points of f.

The following results are some consequences of Theorem 14. In particular, the next theorem is the metric-like counterpart of Theorem 2.1 in [16].

Theorem 16. *Let (X, σ) be a 0_σ-complete metric-like space and $T : X \to X$ a 0_σ-continuous mapping with $\mathrm{Fix}(T) \subseteq X_0$. Suppose that $f : X \to X$ is a mapping and there exists a negative real number r such that*

$$\sigma(fx, Tfx) \leq \sigma(x, Tx) + r\sigma(x, fx), \quad \text{for each } x \in X. \tag{18}$$

Then f has a fixed point z in X with $\sigma(z, z) = 0$.

Proof. Let $d : X \times X \to [0, +\infty)$ be the metric defined by (6). Then, by Proposition 9, the mapping T is continuous in (X, d). This implies that the function $\phi : X \to [0, +\infty)$ defined by

$$\phi(y) = \frac{-1}{r} d(y, Ty) = \frac{-1}{r} \sigma(y, Ty) \tag{19}$$

is lower semicontinuous in (X, d) and hence is a 0_σ-lower semicontinuous function in (X, σ). From (18), we get

$$\sigma(x, fx) \leq \phi(x) - \phi(fx), \quad \text{for each } x \in X. \tag{20}$$

The existence of a fixed point follows by an application of Theorem 14. □

Theorem 17. *Let (X, σ) be a 0_σ-complete metric-like space and $f : X \to X$ a mapping. Assume that there exists $k \in [0, 1)$ such that*

$$\sigma(fx, f^2x) \leq k\sigma(x, fx), \quad \forall x \in X. \tag{21}$$

If one of the following conditions holds, then f has a fixed point z in X with $\sigma(z, z) = 0$:

(i) *the function $h : X \to [0, +\infty)$ defined by $h(x) = \sigma(x, fx)$ is 0_σ-lower semicontinuous;*

(ii) *the mapping f is 0_σ-continuous.*

Proof. Note that (ii) implies (i). In fact, let $x \in X_0$ and $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to +\infty$ and assume that f is 0_σ-continuous. From

$$\begin{aligned} h(x) = \sigma(x, fx) &\leq \sigma(x, x_n) \\ &\quad + \sigma(x_n, fx_n) + \sigma(fx_n, fx) \\ &= \sigma(x, x_n) + h(x_n) + \sigma(fx_n, fx), \end{aligned} \tag{22}$$

we get

$$h(x) \leq \liminf_{n \to +\infty} h(x_n). \tag{23}$$

This ensures that the function h is 0_σ-lower semicontinuous.

Now, we prove that f has a fixed point in X if (i) holds. By (21), we have

$$\sigma(x, fx) - k\sigma(x, fx) \leq \sigma(x, fx) - \sigma(fx, f^2x), \quad \forall x \in X. \tag{24}$$

This implies

$$\sigma(x, fx) \leq \phi(x) - \phi(fx), \quad \forall x \in X, \tag{25}$$

where $\phi : X \to [0, +\infty)$ is defined by $\phi(t) = (1 - k)^{-1}\sigma(t, ft)$, for all $t \in X$.

Now, by (i), the function ϕ is 0_σ-lower semicontinuous. Thus, the existence of a fixed point follows by an application of Theorem 14. □

3.2. Banach-Caccioppoli, Ćirić, and Khamsi Type Results. First, we deduce the Banach-Caccioppoli's theorem in the setting of a metric-like space by Theorem 17.

Theorem 18. *Let (X, σ) be a 0_σ-complete metric-like space and let $f : X \to X$ be a contraction. Then f has a unique fixed point z in X with $\sigma(z, z) = 0$.*

Proof. Let $k \in [0, 1)$ such that (11) holds true. Then $\sigma(fx, f^2x) \leq k\sigma(x, fx)$ for all $x \in X$; that is (21) holds true. Since, by Remark 11, the mapping f is 0_σ-continuous, then the existence of a fixed point follows by an application of Theorem 17. In view of the fact that f is a contraction, the uniqueness of the fixed point z, with $\sigma(z, z) = 0$, is an easy consequence of (11). □

The proof of the following Ćirić type theorem (see [4]) proceeds on the same lines of the proof of Theorem 18 and so we omit it.

Theorem 19. *Let (X, σ) be a 0_σ-complete metric-like space and let $f : X \to X$ be a mapping. Assume that there exists $k \in [0, 1)$ such that*

$$\begin{aligned} \sigma(fx, fy) \leq k \max \Big\{ &\sigma(x, y), \sigma(x, fx), \sigma(y, fy), \\ &\frac{1}{2}\sigma(x, fy), \frac{1}{2}\sigma(y, fx) \Big\}, \end{aligned} \tag{26}$$

for all $x, y \in X$. Then f has a unique fixed point in X if one of the following conditions holds:

(i) *the function $h : X \to [0, +\infty)$ defined by $h(x) = \sigma(x, fx)$ is 0_σ-lower semicontinuous;*

(ii) *the mapping f is 0_σ-continuous.*

If the mapping f satisfies condition (26) and $x \in \mathrm{Fix}(f)$, then $\sigma(x, x) = 0$.

In what follows we denote by Θ the family of all functions $\theta : [0, +\infty) \to [0, +\infty)$ nondecreasing, continuous at $t = 0$ with $\theta(0) = 0$, such that there exist $c > 0$ and $\delta > 0$ satisfying the condition $\theta(t) \geq ct$, for each $t \in [0, \delta]$. Since

θ is continuous at $t = 0$, then there exists $\varepsilon_0 > 0$ such that $\theta^{-1}([0, \varepsilon_0]) \subseteq [0, \delta]$.

We recall the following result due to Khamsi; see [18].

Theorem 20 (see [18, Theorem 2]). *Let (X, d) be a complete metric space. Define the relation $\preceq$ by*

$$x \preceq y \quad iff \;\; \theta(d(x, y)) \leq \phi(y) - \phi(x), \tag{27}$$

where $\theta \in \Theta$ and $\phi : X \to [0, +\infty)$ is a lower semicontinuous function. Then $(X, \preceq)$ has a minimal element x^*; that is, if $x \preceq x^*$, then we must have $x = x^*$.

From this theorem, Khamsi deduced some generalizations of Caristi's fixed point theorem.

Theorem 21 (see [18, Theorem 3]). *Let (X, d) be a complete metric space. Let $f : X \to X$ be a mapping such that for all $x \in X$*

$$\theta(d(x, fx)) \leq \phi(x) - \phi(fx), \tag{28}$$

where the function $\theta \in \Theta$ and $\phi : X \to [0, +\infty)$ is a lower semicontinuous function. Then f has a fixed point.

Theorem 22 (see [18, Theorem 4]). *Let (X, d) be a complete metric space. Let $F : X \to 2^X$ be a multivalued mapping such that Fx is nonempty. Assume that for all $x \in X$ there exists $y \in Fx$ such that*

$$\theta(d(x, y)) \leq \phi(x) - \phi(y), \tag{29}$$

where the function $\theta \in \Theta$ and $\phi : X \to [0, +\infty)$ is a lower semicontinuous function. Then F has a fixed point; that is, there exists $z \in X$ such that $z \in Fz$.

Now, in the setting of a 0_σ-complete metric-like space, we deduce the following results.

Theorem 23. *Let (X, σ) be a 0_σ-complete metric-like space. Let $f : X \to X$ be a mapping such that for all $x \in X$*

$$\theta(\sigma(x, fx)) \leq \phi(x) - \phi(fx), \tag{30}$$

where the function $\theta \in \Theta$ and $\phi : X \to [0, +\infty)$ is a 0_σ-lower semicontinuous function. Then f has a fixed point.

Proof. Let $d : X \times X \to [0, +\infty)$ be the metric on X defined by (6). By Lemma 13, (X, d) is a complete metric space and, by Lemma 5, ϕ is a lower semicontinuous function in (X, d). Next, from $d(x, y) \leq \sigma(x, y)$ for all $x, y \in X$, we get

$$\theta(d(x, fx)) \leq \phi(x) - \phi(fx), \tag{31}$$

for all $x, y \in X$. By an application of Theorem 21, we obtain that f has a fixed point. □

Theorem 24. *Let (X, σ) be a 0_σ-complete metric-like space. Let $F : X \to 2^X$ be a multivalued mapping such that Fx is nonempty. Assume that for all $x \in X$ there exists $y \in Fx$ such that*

$$\theta(\sigma(x, y)) \leq \phi(x) - \phi(y), \tag{32}$$

where the function $\theta \in \Theta$ and $\phi : X \to [0, +\infty)$ is a 0_σ-lower semicontinuous function. Then F has a fixed point.

Proof. The multivalued mapping F has a selection f that satisfies the condition (30). Then by Theorem 23, the multivalued mapping F has a fixed point. □

Example 25. Let again $X = \{-2, -1\} \cup [0, +\infty)$ and $\sigma : X \times X \to [0, +\infty)$ be defined by $\sigma(x, y) = |x - y|$ if $x \neq y$ and

$$\sigma(x, x) = \begin{cases} 1 & \text{if } x \in \{-2, -1\}, \\ 0 & \text{if } x \in [0, +\infty), \end{cases} \tag{33}$$

so that (X, σ) is a 0_σ-complete metric-like space and $X_0 := \{x \in X : \sigma(x, x) = 0\} = [0, +\infty)$. Then, consider the multivalued mapping $F : X \to 2^X$ defined by

$$Fx = \begin{cases} \{0\} & \text{if } x \in \{-2, -1\}, \\ \left[\dfrac{x}{4}, \dfrac{x}{2}\right] & \text{if } x \notin \{-2, -1\} \cup [1, 2], \\ \left[\dfrac{x}{2}, 1\right] & \text{if } x \in [1, 2]. \end{cases} \tag{34}$$

Clearly, for all $x \in X$, $Fx \neq \emptyset$ and there exists $y \in Fx$ given by

$$y = \begin{cases} 0 & \text{if } x \in \{-2, -1\}, \\ \dfrac{x}{4} & \text{if } x \notin \{-2, -1\} \cup [1, 2], \\ 1 & \text{if } x \in [1, 2], \end{cases} \tag{35}$$

such that

$$\sigma(x, fx) = \begin{cases} |x| & \text{if } x \in \{-2, -1\}, \\ \dfrac{3}{4}x & \text{if } x \notin \{-2, -1\} \cup [1, 2], \\ x - 1 & \text{if } x \in [1, 2]. \end{cases} \tag{36}$$

Thus Theorem 24 is applicable in this case with $\phi(x) = |x|$ for all $x \in X$ and $\theta(t) = t$ for all $t \in [0, +\infty)$.

4. Ekeland's Variational Principle

As an application of our technique, we prove Ekeland's variational principle in the setting of metric-like spaces. For a comparative study, see also [19].

Theorem 26 (Ekeland's variational principle). *Let (X, σ) be a 0_σ-complete metric-like space with $X_0 \neq \emptyset$ and consider a function $\phi : X \to (-\infty, +\infty]$ that is 0_σ-lower semicontinuous, bounded from below, and not identical to $+\infty$. Let $\varepsilon > 0$ be given and let $x \in X$ be such that $\phi(x) \leq \inf_{t \in X} \phi(t) + \varepsilon$. Then there exists $y \in X$ such that*

(i) $\phi(y) \leq \phi(x)$;

(ii) $\sigma(x, y) \leq \max\{1, \sigma(x, x)\}$;

(iii) *for all $w \neq y$ in X, $\phi(w) > \phi(y) - \varepsilon\sigma(y, w)$.*

Proof. Let $d : X \times X \to [0, +\infty)$ be the metric defined by (6); then, by Lemma 5, we deduce that (X, d) is a complete metric space. Further, by Lemma 13 we deduce that the function ϕ is lower semicontinuous in (X, d). By the Ekeland's variational principle in metric space there exists $y \in X$ such that

(j) $\phi(y) \leq \phi(x)$;

(jj) $d(x, y) \leq 1$;

(jjj) for all $w \neq y$ in X, $\phi(w) > \phi(y) - \varepsilon d(y, w)$.

This implies that (i)–(iii) hold. In fact (i) reduces to (j). Next, if $y \neq x$, then $\sigma(x, y) = d(x, y) \leq 1$ and so (ii) holds. Finally, (iii) holds since $w \neq y$ implies $\sigma(y, w) = d(y, w)$. □

Building on Theorem 26, we present the second theorem of this section.

Theorem 27. *Let (X, σ) be a 0_σ-complete metric-like space and $\phi : X \to [0, +\infty)$ a 0_σ-lower semicontinuous function. Given $\varepsilon > 0$, then there exists $y \in X$ such that*

$$\begin{gathered} \phi(y) \leq \inf_{t \in X} \phi(t) + \varepsilon, \\ \text{for each } w \in X, \quad \phi(w) \geq \phi(y) - \varepsilon\sigma(y, w). \end{gathered} \tag{37}$$

Proof. To conclude, we recall that there exists at least a point x such that $\phi(x) \leq \inf_{t \in X} \phi(t) + \varepsilon$. This implies that (37) follows from (i) and (iii) of Theorem 26, respectively. □

Remark 28. By comparing Theorems 26 and 27, it is clear that the first is stronger than the second. In fact, the condition (ii) of Theorem 26, which gives the whereabouts of point x in X, does not have a counterpart in Theorem 27.

In view of Theorem 27, we can provide the following alternative proof of Theorem 14 described in this paper.

Proof. By an application of Theorem 27 with $\varepsilon = 1/2$, we get that there exists some point $y \in X$ such that

$$\text{for each } t \in X, \quad \phi(t) \geq \phi(y) - \frac{1}{2}\sigma(y, t), \tag{38}$$

where we assume that the function ϕ satisfies (13). The above inequality also holds for $t = fy$; therefore

$$\phi(y) - \phi(fy) \leq \frac{1}{2}\sigma(y, fy). \tag{39}$$

Next, putting $x = y$ in (13), we obtain

$$\sigma(y, fy) \leq \phi(y) - \phi(fy). \tag{40}$$

Combining together the above inequalities, we get

$$\sigma(y, fy) \leq \frac{1}{2}\sigma(y, fy). \tag{41}$$

This holds true unless $\sigma(y, fy) = 0$ and therefore we deduce that $fy = y$. Then, the existence of a fixed point is proved. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

All authors contributed equally and significantly to writing this paper. All authors read and approved the final paper.

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