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Research Article

Norm Attaining Arens Extensions on ℓ_1

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We study norm attaining properties of the Arens extensions of multilinear forms defined on Banach spaces. Among other related results, we construct a multilinear form on ℓ_1 with the property that only some fixed Arens extensions determined a priori attain their norms. We also study when multilinear forms can be approximated by ones with the property that only some of their Arens extensions attain their norms.

1. Introduction

The Bishop-Phelps theorem [1] states that the set of norm attaining forms on a real or complex Banach space is norm dense in the set of linear and continuous forms. Bishop and Phelps raised the question of extending their results to operators between Banach spaces. This question was answered in the negative by Lindenstrauss in his seminal paper [2], where he gave an example of a Banach space X such that the identity mapping on X cannot be approximated by norm attaining operators. However, if one considers the adjoint $T^* : Y^* \to X^*$ of an operator $T : X \to Y$ between Banach spaces, given by $T^*(y^*)(x) = y^*(T(x))$, for all $x \in X$, $y^* \in Y^*$, Lindenstrauss proved the denseness of those operators whose second adjoints attain their norms.

The theory of norm attaining operators has spread to the nonlinear setting. The denseness of the set of norm attaining multilinear mappings has been deeply studied in the last decades. Assuming the Radon-Nikodým property, this density has been established for multilinear forms (see [3]). However, a general result for multilinear mappings cannot be expected. The first counterexample was given in [4] for bilinear forms. Based on Lindenstrauss result and making use of the Arens extensions to the second duals (see next section for the definitions), Acosta [5] proved a Lindenstrauss type result for bilinear forms whose third Arens transpose attains its norm. Afterwards, in [6] the denseness of bilinear forms whose Arens extensions to the biduals attain their norms at the same point was established. It is worth mentioning that in [6, Example 2] an example of a bilinear mapping is given such that only one of their Arens extensions attains its norm. This asymmetry between the two Arens extensions reveals the importance of the stronger condition of attaining their norms simultaneously. The generalization of Lindenstrauss result to n-linear vector-valued mappings was finally obtained in [7] in its strongest form; that is, the space formed by those n-linear mappings whose Arens extensions attain their norms simultaneously at the same point is dense in the space of all n-linear mappings.

The aim of this paper is to study the norm attaining properties of the Arens extensions of multilinear forms on ℓ_1. On one hand, inspired by [6, Example 2], several examples of multilinear forms whose extensions suffer different kinds of asymmetries from the point of view of norm attainment are provided. These examples are built using multilinear forms on ℓ_1, which is the classical example of a non-Arens regular Banach space. For instance, if we fix a priori some of the Arens extensions, we can construct a multilinear form on ℓ_1 with the property that only these extensions attain their norms. Moreover, by undertaking a detailed study of the procedure used to generate such examples, we also get examples with stronger properties that allow a better understanding of the norm attaining behavior of the Arens extensions. These examples are presented as general results on existence of multilinear forms that fulfill the required norm attaining properties. On the other hand, we also deal with general

Banach spaces and study when Arens extensions attain their norms in terms of convergence of sequences.

The paper is organized as follows. Next section is devoted to fix the notation and to recall some of the basics on Arens extensions. Section 3 is involved with the norm attaining behavior of the Arens extensions of multilinear forms on general Banach spaces. We prove that if an extension of a multilinear form attains its norm at a point then the norm is achieved just considering sequential limits. As a converse, we prove that if the norm of an extension is achieved with limits of subsequences of a normalized Schauder basis, then such extension attains its norm at a point whose coordinates are in the bidual. In Section 4 we deal with multilinear forms on ℓ_1. We strengthen the results from the former section by proving a characterization of norm attaining extensions of bilinear forms at points with coordinates in $B_{\ell_1^{**}} \setminus \ell_1$ in terms of sequential limits of the images of subsequences of the canonical sequence $\{e_n\}_{n=1}^{\infty}$. It is also proved that, fixing a number of Arens extensions, there exists an n-linear form on $(\ell_1)^n$ of norm one such that only these extensions fixed a priori are norm attaining. Finally, we show that such n-linear forms are dense in the set of all n-linear forms of norm one that fulfill a condition in terms of sequential limits.

2. Background and Notation

In this paper X, Y, X_1, …, X_n are real or complex Banach spaces. Let $\mathscr{L}(X_1,\dots,X_n)$ denote the space of continuous n-linear forms A from $X_1\times\cdots\times X_n$ into $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) with the usual norm

$$\|A\| = \sup\left\{\left|A\left(x_1,\dots,x_n\right)\right| : x_i \in X_i,\ \left\|x_i\right\| \le 1,\ i=1,\dots,n\right\}. \tag{1}$$

When $X_1=\cdots=X_n=X$, we just write $\mathscr{L}(^nX)$. We denote by B_X the closed unit ball of X, by S_X the unit sphere, by X^* the strong dual, and by X^{**} the bidual of X.

We say that A is *norm attaining* (or A attains its norm) if there exist $x_i \in X_i$, $\|x_i\|=1$, $i=1,\dots,n$, such that $\|A\| = |A(x_1,\dots,x_n)|$.

Arens [8] found a natural way to extend a continuous bilinear mapping $A : X_1\times X_2 \mapsto Y$ to a continuous bilinear mapping from $X_1^{**}\times X_2^{**}$ into Y^{**}. His method consists in applying three times the operation defined as

$$\begin{gathered} A^t : Y^* \times X_1 \longmapsto X_2^*, \\ \left(y^*, x_1\right) \rightsquigarrow A^t\left(y^*, x_1\right)\left(x_2\right) = y^*\left(A\left(x_1, x_2\right)\right), \end{gathered} \tag{2}$$

$x_1\in X_1$, $x_2\in X_2$, and $y^*\in Y^*$. The first extension is defined as $A^{ttt} : X^{**}\times Y^{**} \mapsto Z^{**}$ and the second one is A^{TtttT}, where $B^T(x_1,x_2) = B(x_2,x_1)$ for any bilinear mapping B. These extensions, which are in general different, are known as Arens products. This procedure was generalized by Aron and Berner [9] to arbitrary multilinear mappings.

For our purposes we will use an alternative approach due to Davie and Gamelin [10]. The key of such approach is Goldstine theorem as it is based on limits in the *weak-startopology*, denoted by $w(X^{**},X^*)$. Consider Σ_n the group of all permutations of the set $\{1,\dots,n\}$. Given $\sigma\in\Sigma_n$ they defined the extension A_σ associated with σ of an n-linear form A defined on $X_1\times\cdots\times X_n$, by

$$A_\sigma\left(x_1^{**},\dots,x_n^{**}\right) = \lim_{d_{\sigma(1)}}\cdots\lim_{d_{\sigma(n)}} A\left(x_{d_1},\dots,x_{d_n}\right), \tag{3}$$

where $\{x_{d_i}\}_{d_i}$ is a bounded net ($\|x_{d_i}\| \le \|x_i^{**}\|$, for all d_i) $w(X^*,X)$ convergent to $x_i^{**} \in X_i^{**}$, for $i=1,\dots,n$. The mapping A_σ is called an *Arens extension* of A and the n! Arens extensions may be different from each other. When convenient, we will write $A_{\sigma(1),\dots,\sigma(n)}$ instead of A_σ. In particular, for $n=2$, $A_{Id}=A_{1,2}=A^{ttt}$ and $A_{2,1}=A^{TtttT}$, where Id is the identity permutation of the set $\{1,2\}$.

Note that the use of the $w(X^*,X)$ topology prevents us in general from using sequences in the above limits. However we will show that in the study of norm attaining multilinear forms one can reduce such iterated limits to sequential ones.

In [7] Lindenstrauss theorem is extended to multilinear forms by using the Arens extensions.

Theorem 1 ([7, Theorem 2.1]). *Let X_i be Banach spaces ($1\le i\le n$). Then the set of n-linear forms on $X_1\times\cdots\times X_n$ such that all their Arens extensions to $X_1^{**}\times\cdots\times X_n^{**}$ attain their norms at the same n-tuple is dense in the space $\mathscr{L}(X_1,\dots,X_n)$.*

Let ℓ_1 denote the space of all absolutely summing sequences in $\mathbb{K}$ with its usual norm. In [6] the following example is provided. It illustrates that, although all Arens extensions have the same norm, the fact that one of them attains its norm does not imply that the other extensions should attain their norms too. More precisely, it shows a bounded bilinear form A whose first extension $A_{1,2}$ is not norm attaining, whereas the second one $A_{2,1}$ is norm attaining. This example brings into relief that the extensions of a bilinear form may have different behaviors from the point of view of attaining their norms and is the core of our study.

Example 2 ([6, Example 2]). The bilinear form $A\in\mathscr{L}(^2\ell_1)$, defined by

$$A\left(x_1,x_2\right) := \sum_{t_1=1}^{\infty} x_1\left(t_1\right)\left(\sum_{t_2=1}^{t_1}\frac{t_2}{t_2+1}x_2\left(t_2\right)\right), \tag{4}$$

is such that neither A nor $A_{Id}=A_{1,2}$ is norm attaining, but $A_{2,1}$ is norm attaining.

3. Norm Attaining Extensions of Multilinear Forms on General Banach Spaces

It is well known that, under the first axiom of separability, nets can be replaced with sequences, which turns out to be an advantage when dealing with limits. Our first result is just a lemma that will clarify how to pass from nets to sequences in the context of several indexes that will be helpful in the context of multilinear mappings. We give the proof for the sake of completeness.

Lemma 3. *Let $n \in \mathbb{N}$. For each $j = 1, \ldots, n$, let D_j be an infinite directed set. Consider a family $\{a_{\alpha_1,\ldots,\alpha_n}\}_{(\alpha_1,\ldots,\alpha_n)\in D_1\times\cdots\times D_n}$ of real or complex numbers. If the iterated limit $a := \lim_{\alpha_1\in D_1}\cdots\lim_{\alpha_n\in D_n} a_{\alpha_1,\ldots,\alpha_n}$ is finite then there exist strictly increasing sequences $\{\alpha_j(m)\}_{m=1}^{\infty}$ in D_j, $1 \le j \le n$, such that $\lim_{m_1\to\infty}\cdots\lim_{m_n\to\infty} a_{\alpha_1(m_1),\ldots,\alpha_n(m_n)} = a$.*

Proof. We proceed by induction on n. For $n = 1$, since $\lim_{\alpha_1\in D_1} a_{\alpha_1} = a$, for each $k \in \mathbb{N}$ there exists $\alpha_1(k) \in D_1$ such that $|a_{\alpha_1} - a| < 1/k$ for all $\alpha_1 \ge \alpha_1(k)$. Besides, by the condition on D_1, we can choose the sequence $\{\alpha_1(k)\}_{k\in\mathbb{N}}$ strictly increasing.

Assume that the result is true for $n-1$ and let us prove it for n. So, if we assume that $a = \lim_{\alpha_1\in D_1}\cdots\lim_{\alpha_n\in D_n} a_{\alpha_1,\ldots,\alpha_n}$ is finite, define

$$b_{\alpha_1,\ldots,\alpha_{n-1}} := \lim_{\alpha_n\in D_n} a_{\alpha_1,\ldots,\alpha_{n-1},\alpha_n}. \tag{5}$$

By the assumption applied to the family $\{b_{\alpha_1,\ldots,\alpha_{n-1}}\}_{(\alpha_1,\ldots,\alpha_{n-1})\in D_1\times\cdots\times D_{n-1}}$, for each $j = 1, \ldots, n-1$ there exists a strictly increasing sequence $\{\alpha_j(m_j)\}_{m_j\in\mathbb{N}}$ such that

$$a = \lim_{m_1\to\infty}\cdots\lim_{m_{n-1}\to\infty} b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})}. \tag{6}$$

Let us construct the sequence $\{\alpha_n(k)\}_{k\in\mathbb{N}}$ by induction on k.

Since $b_{\alpha_1(1),\ldots,\alpha_{n-1}(1)} = \lim_{\alpha_n\in D_n} a_{\alpha_1(1),\ldots,\alpha_{n-1}(1),\alpha_n}$, there exists $\alpha_n(1) \in D_n$ such that

$$\left|b_{\alpha_1(1),\ldots,\alpha_{n-1}(1)} - a_{\alpha_1(1),\ldots,\alpha_{n-1}(1),\alpha_n}\right| < 1 \tag{7}$$

for all $\alpha_n \ge \alpha_n(1)$. Assume that we have found $\alpha_n(1), \ldots, \alpha_n(k-1) \in D_n$ with $\alpha_n(1) < \cdots < \alpha_n(k-1)$ such that $|b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})} - a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n}| < (1/l)$ for all $\alpha_n \ge \alpha_n(l)$, all $1 \le m_1, \ldots, m_{n-1} \le l$, and all $l = 1, \ldots, k-1$.

Fix $1 \le m_1, \ldots, m_{n-1} \le k$. Since $b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})} = \lim_{\alpha_n\in D_n} a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n}$, there exists $\alpha_n(m_1, \ldots, m_{n-1}) \in D_n$, with $\alpha_n(m_1, \ldots, m_{n-1}) \ge \alpha_n(k-1)$, such that

$$\left|b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})} - a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n}\right| < \frac{1}{k} \tag{8}$$

for all $\alpha_n \ge \alpha_n(m_1, \ldots, m_{n-1})$. Take $\alpha_n(k) > \alpha_n(m_1, \ldots, m_{n-1})$ for all $1 \le m_1, \ldots, m_{n-1} \le k$. Then

$$\left|b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})} - a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n}\right| < \frac{1}{k} \tag{9}$$

whenever $\alpha_n \ge \alpha_n(k)$. Hence the limit $\lim_{k\to\infty} a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n(k)}$ exists and is equal to $b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})}$. Now,

$$\begin{aligned} a &= \lim_{m_1\to\infty}\cdots\lim_{m_{n-1}\to\infty} b_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1})} \\ &= \lim_{m_1\to\infty}\cdots\lim_{m_{n-1}\to\infty}\lim_{m_n\to\infty} a_{\alpha_1(m_1),\ldots,\alpha_{n-1}(m_{n-1}),\alpha_n(m_n)} \end{aligned} \tag{10}$$

and the proof is over. □

Theorem 4. *Let $X_1, \ldots, X_n$ be infinite dimensional Banach spaces, $C \in \mathscr{L}(X_1, \ldots, X_n)$, and $\sigma \in \Sigma_n$. If the extension C_σ attains its norm then there exist sequences $\{x_{m_1}^1\}_{m_1=1}^{\infty}, \ldots, \{x_{m_n}^n\}_{m_n=1}^{\infty}$ with each $x_{m_k}^k \in B_{X_k}$, $m_k \in \mathbb{N}$, and $k = 1, \ldots, n$, such that*

$$\lim_{m_{\sigma(1)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty}\left|C\left(x_{m_1}^1, \ldots, x_{m_n}^n\right)\right| = \|C\|. \tag{11}$$

Proof. For simplicity we assume that $\sigma = Id$. Let $(x_1^{**}, \ldots, x_n^{**})$ be a point in $B_{X_1^{**}}\times\cdots\times B_{X_n^{**}}$, where C_σ attains its norm. Let $K = \{k : x_k^{**} \in X^{**} \setminus X\}$. By density, each x_k^{**} is the weak-star limit of a net $\{x_{\alpha_k}^k\}_{\alpha_k\in D_k}$ in B_{X_k}, $k \in K$. For $k \notin K$, set $D_k = \mathbb{N}$ and $x_{\alpha_k}^k := x_k^{**} \in X$ for all $\alpha_k \in D_k$. Then

$$\begin{aligned} \|C\| &= \|C_\sigma\| \\ &= \left|C_\sigma\left(x_1^{**}, \ldots, x_n^{**}\right)\right| \\ &= \lim_{\alpha_1\in D_1}\cdots\lim_{\alpha_n\in D_n}\left|C\left(x_{\alpha_1}^1, \ldots, x_{\alpha_n}^n\right)\right|. \end{aligned} \tag{12}$$

By Lemma 3 applied to $a_{\alpha_1,\ldots,\alpha_n} := |C(x_{\alpha_1}^1, \ldots, x_{\alpha_n}^n)|$, we obtain the desired sequences $\{x_{m_k}^k\}_{m_k=1}^{\infty}$, for every $1 \le k \le n$. □

Proposition 5. *Let X be an infinite dimensional Banach space, and let $\{x_n\}_{n=1}^{\infty}$ be a basic sequence. Then, any nonzero weak-star cluster point of $\{x_n\}_{n=1}^{\infty}$ belongs to $X^{**} \setminus X$.*

Proof. Let Z be the closed linear span of $\{x_n\}_{n=1}^{\infty}$ and let $\{x_n^*\}_{n=1}^{\infty}$ be the orthogonal functionals in Z^* associated with $\{x_n\}_{n=1}^{\infty}$. By the Hahn-Banach extension theorem, we can consider each x_n^* in X^*.

Let $x^{**} \in X^{**}$ be a nonzero cluster point of $\{x_n\}_{n=1}^{\infty}$, and let $\{x_d\}_{d\in D}$ be a subnet of $\{x_n\}_{n=1}^{\infty}$ weak-star converging to x^{**}.

We first prove that x^{**} is none of the vectors x_n. Assume that this is not the case; that is, $x^{**} = x_{n_0}$ for some n_0. Since $\{x_d\}_{d\in D}$ weak-star converges to x^{**}, the net $\{\langle x_d, x_{n_0}^*\rangle\}_{d\in D}$ converges to $\langle x^{**}, x_{n_0}^*\rangle = 1$. Then, there is $\tilde{d} \in D$ such that

$$\left|\left\langle x_d, x_{n_0}^*\right\rangle\right| > \frac{1}{2} \tag{13}$$

for all $d \ge \tilde{d}$. Since D is cofinal, there is $d_1 \in D$ such that $d_1 \ge \tilde{d}$ and $d_1 \ge n_0 + 1 > n_0$. By the biorthogonality of $\{x_n^*\}_{n=1}^{\infty}$ it follows that $\langle x_{d_1}, x_{n_0}^*\rangle = 0$, which contradicts (13).

We prove now that $x^{**} \notin Z$. Let us assume that $x^{**} \in Z$. Then there is a unique sequence of scalars $\{a_n\}_{n=1}^{\infty}$ so that $x^{**} = \sum_{n=1}^{\infty} a_n x_n$. Let $\epsilon > 0$ and take $n_1 := 1$. Since $\{\langle x_d, x_1^*\rangle\}_{d\in D}$ converges to $\langle x^{**}, x_1^*\rangle = a_1$, there is $\tilde{\tilde{d}} \in D$ so that $|\langle x_d, x_1^*\rangle - a_1| < \epsilon$ for all $d \ge \tilde{\tilde{d}}$. Since D is cofinal, there is $\tilde{d}_1 \in D$ such that $\tilde{d}_1 \ge 2$. Let $\tilde{d}_2 \ge \tilde{d}_1, \tilde{\tilde{d}}$. Then $n_2 := \tilde{d}_2 \ge \tilde{d}_1 > 1 = n_1$. Therefore $\langle x_{d_2}, x_1^*\rangle = \langle x_{n_2}, x_{n_1}^*\rangle = 0$ and $|\langle x_{d_2}, x_1^*\rangle - a_1| < \epsilon$. Hence, $|a_1| < \epsilon$. This shows that $a_1 = 0$. Reiterating this process we can prove that $a_n = 0$ for all $n \in \mathbb{N}$, which contradicts the fact that $x^{**} \ne 0$.

To finish the proof, since x^{**} belongs to the $w(X^{**}, X^*)$ closure of Z, if we assume that $x^{**} \in X$, then x^{**} actually belongs to the $w(X, X^*)$ closure of Z. This closure coincides with the norm closure, that is, with Z. As we have already proved, this is impossible. Therefore, $x^{**} \notin X$. □

Theorem 6. *Let $n \in \mathbb{N}$. For each $1 \leq j \leq n$ let X_j be a Banach space with a normalized Schauder basis $\{x_n^j\}_{n=1}^{\infty}$. Let $C \in \mathscr{L}(X_1, \ldots, X_n)$ and $\sigma \in \Sigma_n$. If there exist strictly increasing sequences of natural numbers $\{k(j, m_j)\}_{m_j=1}^{\infty}$, $j = 1, \ldots, n$, such that*

$$\lim_{m_{\sigma(1)} \to \infty} \cdots \lim_{m_{\sigma(n)} \to \infty} \left| C\left(x_{k(1,m_1)}^1, \ldots, x_{k(n,m_n)}^n\right) \right| = \|C\| \quad (14)$$

then C_σ attains its norm at a point in $(B_{X_1^{**}} \setminus X_1) \times \cdots \times (B_{X_n^{**}} \setminus X_n)$.

Proof. Consider any $1 \leq j \leq n$. Let x_j^{**} be a cluster point of the subsequence $\{x_{k(j,m_j)}^j\}_{m_j=1}^{\infty}$ and hence of the sequence $\{x_n^j\}_{n=1}^{\infty}$. As the Schauder basis is normalized, $x_j^{**} \in B_{X_j^{**}}$ and by Proposition 5 $x_j^{**} \notin X_j$. Let $\{x_{k(j,d_j)}\}_{d_j \in D_j}$ be a subnet of $\{x_{k(j,m_j)}\}_{m_j=1}^{\infty}$ that weak-star converges to x_j^{**}. Then

$$\begin{aligned} \|C_\sigma\| &= \|C\| \\ &= \lim_{m_{\sigma(1)} \to \infty} \cdots \lim_{m_{\sigma(n)} \to \infty} \left| C\left(x_{k(1,m_1)}^1, \ldots, x_{k(n,m_n)}^n\right) \right| \\ &= \lim_{d_{\sigma(1)} \in D_{\sigma(1)}} \cdots \lim_{d_{\sigma(n)} \in D_{\sigma(n)}} \left| C\left(x_{k(1,d_1)}^1, \ldots, x_{k(n,d_n)}^n\right) \right| \\ &= \left| C_\sigma\left(x_1^{**}, \ldots, x_n^{**}\right) \right|. \end{aligned} \quad (15)$$

□

4. Norm Attaining Extensions of Multilinear Forms on ℓ_1

Our aim in this section is to show that, when working with the space ℓ_1, one can strengthen the results in Section 3. But before, let us recall some well known facts on ℓ_1 that we need to use later. First is that, since ℓ_1^{**} is the third dual of c_0, then ℓ_1 is a complemented subspace of ℓ_1^{**}. Actually, $\ell_1^{**} = c_0^* \oplus c_0^{\perp} = \ell_1 \oplus c_0^{\perp}$, where a linear form belongs to $c_0^{\perp}$ if it vanishes on c_0. Moreover, ℓ_1^{**} is 1-sum of ℓ_1 and $c_0^{\perp}$ [11, page 158]; that is, if we denote by $\pi : \ell_1^{**} \to \ell_1$ the projection of ℓ_1^{**} onto ℓ_1, we have that $\|x^{**}\| = \|\pi(x^{**})\| + \|x^{\perp}\|$ for every x^{**} in ℓ_1^{**}, where $x^{\perp} = x^{**} - \pi(x^{**})$. If A is in $\mathscr{L}(^n\ell_1^{**})$, then

$$\|A\| = \sup_{k_1, \ldots, k_n \in \mathbb{N}} \left| A\left(e_{k_1}, \ldots, e_{k_n}\right) \right|, \quad (16)$$

where $\{e_k\}_{k=1}^{\infty}$ is the canonical basis of ℓ_1.

We have seen that, even if the norm of an extension of a multilinear functional is attained in points of the bidual, we can deal with sequential limits of points in the unit ball of the space. We now prove that, when dealing with bilinear forms defined on $\ell_1 \times \ell_1$, sequences in the unit ball of ℓ_1 can be replaced with subsequences of the canonical basis of ℓ_1, and so a full characterization works.

Lemma 7. *Let $A \in \mathscr{L}(^n\ell_1^{**})$ with $\|A\| = 1$, $x_1^{**}, \ldots, x_n^{**} \in B_{\ell_1^{**}} \setminus \ell_1$, and $x_i^{\perp} = x_i^{**} - \pi(x_i^{**})$, $i = 1, \ldots, n$. If A attains its norm at $(x_1^{**}, \ldots, x_n^{**})$ then A attains its norm at $(x_1^{\perp}/\|x_1^{\perp}\|, \ldots, x_n^{\perp}/\|x_n^{\perp}\|)$ too.*

Proof. Let us prove it first for $n = 1$, that is, for A being linear. If we assume that $|A(x_1^{\perp})| < \|x_1^{\perp}\|$ then for some $\varepsilon \in \mathbb{K}$ with $|\varepsilon| = 1$

$$\begin{aligned} 1 &= A\left(\varepsilon x_1^{**}\right) \\ &= A\left(\varepsilon x_1^{\perp}\right) + A\left(\varepsilon \pi\left(x_1^{**}\right)\right) \\ &< \left\|x_1^{\perp}\right\| + \left\|\pi\left(x_1^{**}\right)\right\| = \left\|x_1^{**}\right\| = 1 \end{aligned} \quad (17)$$

which is a contradiction.

Assume now that A is bilinear. The associated linear mapping $A_1(y) := A(y, x_2^{**})$, $y \in \ell_1^{**}$, attains its norm at $x_1^{**} \in B_{\ell_1^{**}} \setminus \ell_1$ and so, by the linear case, A_1 attains its norm at $x_1^{\perp}/\|x_1^{\perp}\|$. Now, if we consider the other associated linear mapping $A_2(y) := A(x_1^{\perp}/\|x_1^{\perp}\|, y)$, $y \in \ell_1^{**}$, it attains its norm at x_2^{**}. Then, A_2 also attains its norm at $x_2^{\perp}/\|x_2^{\perp}\|$. That is, $|A(x_1^{\perp}/\|x_1^{\perp}\|, x_2^{\perp}/\|x_2^{\perp}\|)| = 1$.

An easy induction yields the general case. □

Lemma 8. *Let M and N be subsets of $\mathbb{N}$, $0 < \beta < 1$, and for each $n \in \mathbb{N}$ let $a_n \geq 0$ be such that $\sum_{n=1}^{\infty} a_n = 1$. If $\sum_{t \in M} a_t + \sum_{t \in N} a_t > 2 - \beta$ then $\sum_{t \in M \cap N} a_t > 1 - \beta$.*

Proof. Since

$$1 = \sum_{n=1}^{\infty} a_n \geq \sum_{t \in M \setminus N} a_t + \sum_{t \in N \setminus M} a_t + \sum_{t \in M \cap N} a_t \quad (18)$$

it follows that

$$\sum_{t \in M \setminus N} a_t + \sum_{t \in N \setminus M} a_t \leq 1 - \sum_{t \in M \cap N} a_t. \quad (19)$$

Combining this with the hypothesis we finally get that

$$\begin{aligned} 2 - \beta &< \sum_{t \in M \setminus N} a_t + \sum_{t \in M \cap N} a_t + \sum_{t \in N \setminus M} a_t + \sum_{t \in M \cap N} a_t \\ &\leq 2 \sum_{t \in M \cap N} a_t - \sum_{t \in M \cap N} a_t + 1 = \sum_{t \in M \cap N} a_t + 1. \end{aligned} \quad (20)$$

□

Theorem 9. *Given a bilinear form $A \in \mathscr{L}(^2\ell_1)$ of norm one, the following are equivalent:*

(a) $\lim_i \lim_j |A(e_{m_i}, e_{n_j})| = 1$ *for some strictly increasing sequences of natural numbers $(m_i)_{i=1}^{\infty}$ and $(n_j)_{j=1}^{\infty}$;*

(b) *there exist $x_1^{**}, x_2^{**} \in \ell_1^{**} \setminus \ell_1$ of norm one such that $|A_{Id}(x_1^{**}, x_2^{**})| = 1$.*

Proof. $(a) \Rightarrow (b)$ is a consequence of Theorem 6.

$(b) \Rightarrow (a)$: notice that ℓ_1 is an L-summand space in its bidual so $\|x_s^{**}\| = \|\pi(x_s^{**})\| + \|x_s^{**} - \pi(x_s^{**})\|$, for $s = 1, 2$, where π is the projection from ℓ_1^{**} onto ℓ_1. For each $n \in \mathbb{N}$ let π_n denote the projection from ℓ_1^{**} onto ℓ_1^n. Note that π_n is weak-star continuous.

By Lemma 7 we can assume that $\pi(x_1^{**}) = \pi(x_2^{**}) = 0$. Consider the linear form $A_{Id}(\cdot, x_2^{**})$ of ℓ_1^* with norm one defined by $A_{Id}(x, x_2^{**}) = \lim_{d_2} A(x, x_{d_2})$ for all x in ℓ_1, whenever $\{x_{d_2}\}_{d_2\in D_2}$ is a net in the unit ball of ℓ_1 weak-star convergent to x_2^{**}.

Let us see that there exists a strictly increasing sequence of natural numbers $\{m_i\}_{i=1}^{\infty}$ with $\lim_i|A_{Id}(e_{m_i}, x_2^{**})| = 1$. If this is not the case, then there exists $\epsilon > 0$ and there exists a natural number r with $|A_{Id}(e_k, x_2^{**})| \leq 1-\epsilon$ for all $k > r$. Let $\{x_{d_1}\}_{d_1\in D_1}$ be a net in the unit ball of ℓ_1 weak-star convergent to x_1^{**}. Since $\pi(x_1^{**}) = 0$ and $\pi_r(x_{d_1})$ converges to $\pi_r(x_1^{**}) = 0$ then $\{x_{d_1} - \pi_r(x_{d_1})\}_{d_1\in D_1}$ weak-star converges to x_1^{**}. Moreover, $\|x_{d_1} - \pi_r(x_{d_1})\| \leq \|x_{d_1}\| \leq 1$ and so by replacing x_{d_1} with $x_{d_1} - \pi_r(x_{d_1})$ we can assume that $\pi_r(x_{d_1}) = 0$; that is, $x_{d_1}(t) = 0$ for all $t = 1,\ldots,r$.

Therefore for all $d_1 \in D_1$

$$\begin{aligned}\left|A_{Id}\left(x_{d_1}, x_2^{**}\right)\right| &= \left|\sum_{t=1}^{\infty} x_{d_1}(t)\, A_{Id}\left(e_t, x_2^{**}\right)\right| \\ &\leq \sum_{t=r+1}^{\infty}\left|x_{d_1}(t)\right|\left|A_{Id}\left(e_t, x_2^{**}\right)\right| \\ &\leq 1-\epsilon,\end{aligned} \tag{21}$$

contradicting the fact that $|\lim_{d_1} A_{Id}(x_{d_1}, x_2^{**})| = |A_{Id}(x_1^{**}, x_2^{**})| = 1$.

Without loss of generality assume that for all $i \in \mathbb{N}$

$$1-\left|A_{Id}\left(e_{m_i}, x_2^{**}\right)\right| \leq 2^{-(2i+2)}. \tag{22}$$

By using induction, let us find a strictly increasing sequence of natural numbers $\{n_j\}_{j=1}^{\infty}$ such that $|A(e_{m_i}, e_{n_j})| \geq 1-2^{-i}$ for all $1 \leq i \leq j$.

Let $\{x_{d_2}\}_{d_2\in D_2}$ be a net in the unit ball of ℓ_1 weak-star convergent to x_2^{**}. Since $|\lim_{d_2} A_{Id}(e_{m_1}, x_{d_2})| = |A_{Id}(e_{m_1}, x_2^{**})| > 1-2^{-4}$, there exists d_0 in D_2 with $|A(e_{m_1}, x_{d_0})| > 2^{-1}$. Then

$$\begin{aligned}2^{-1} &< \left|A\left(e_{m_1}, x_{d_0}\right)\right| \leq \sum_{t\in\mathbb{N}}\left|x_{d_0}(t)\right|\left|A\left(e_{m_1}, e_t\right)\right| \\ &\leq \sup_{t\in\mathbb{N}}\left\{\left|A\left(e_{m_1}, e_t\right)\right|\right\}\sum_{t\in\mathbb{N}}\left|x_{d_0}(t)\right| \\ &\leq \sup_{t\in\mathbb{N}}\left\{\left|A\left(e_{m_1}, e_t\right)\right|\right\}.\end{aligned} \tag{23}$$

Let n_1 be a natural number with $|A_{Id}(e_{m_1}, e_{n_1})| > 2^{-1}$. Now, assume that we have found $n_1 < \cdots < n_r$ with $|A(e_{m_i}, e_{n_j})| > 1-2^{-i}$ for $1 \leq i \leq j \leq r$ and let us find n_{r+1}. Considering that $\pi(x_2^{**}) = 0$, by replacing x_{d_2} with $x_{d_2} - \pi_{n_r}(x_{d_2})$, we can assume that $\pi_{n_r}(x_{d_2}) = 0$; that is, $x_{d_2}(t) = 0$ for all $t = 1,\ldots,n_r$ and all $d_2 \in D_2$.

By (22), consider x_0 an element of the net $\{x_{d_2}\}_{d_2\in D_2}$ such that

$$\left|A\left(e_{m_i}, x_0\right)\right| \geq 1-2^{-2i} \quad \text{for } i = 1,\ldots,r+1. \tag{24}$$

For each $i = 1,\ldots,r+1$ define the sets

$$T_i := \left\{t \in \mathbb{N} : t > n_r, \left|A_{Id}\left(e_{m_i}, e_t\right)\right| \geq 1-2^{-i}\right\}. \tag{25}$$

Therefore, for every $i = 1,\ldots,r+1$,

$$\begin{aligned}1-2^{-2i} &\leq \left|A\left(e_{m_i}, x_0\right)\right| \\ &\leq \sum_{t\in T_i}\left|x_0(t)\right|\left|A\left(e_{m_i}, e_t\right)\right| + \sum_{t\notin T_i}\left|x_0(t)\right|\left|A\left(e_{m_i}, e_t\right)\right| \\ &\leq \sum_{t\in T_i}\left|x_0(t)\right| + \left(1-2^{-i}\right)\sum_{t\notin T_i}\left|x_0(t)\right| \\ &\leq \sum_{t\in T_i}\left|x_0(t)\right| + \left(1-2^{-i}\right)\left(1-\sum_{t\in T_i}\left|x_0(t)\right|\right) \\ &= \left(1-2^{-i}\right) + 2^{-i}\sum_{t\in T_i}\left|x_0(t)\right|,\end{aligned} \tag{26}$$

where in the first inequality we have used (24). Thus $2^{-i}\sum_{t\in T_i}|x_0(t)| \geq 2^{-i} - 2^{-2i}$ and so

$$\sum_{t\in T_i}\left|x_0(t)\right| \geq 1-2^{-i}. \tag{27}$$

We use now finite induction and Lemma 8 to see that $\bigcap_{i=1}^{r+1} T_i \neq \emptyset$. Indeed, by (27)

$$\sum_{t\in T_1}\left|x_0(t)\right| + \sum_{t\in T_2}\left|x_0(t)\right| > 2-\left(\frac{1}{2}+\frac{1}{2^2}\right). \tag{28}$$

Lemma 8 yields that $\sum_{t\in T_1\cap T_2}|x_0(t)| > 1-((1/2)+(1/2^2))$. If for some $1 \leq l < r+1$ we assume that

$$\sum_{t\in\cap_{j=1}^{l} T_j}\left|x_0(t)\right| > 1-\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^l}\right), \tag{29}$$

then

$$\begin{aligned}&\sum_{t\in\cap_{j=1}^{l} T_j}\left|x_0(t)\right| + \sum_{t\in T_{l+1}}\left|x_0(t)\right| \\ &\qquad > 2-\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^l}+\frac{1}{2^{l+1}}\right).\end{aligned} \tag{30}$$

Once more, Lemma 8 yields that

$$\sum_{t\in\cap_{j=1}^{l+1} T_j}\left|x_0(t)\right| > 1-\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{l+1}}\right). \tag{31}$$

Therefore, we can conclude that

$$\sum_{t\in\cap_{j=1}^{r+1} T_j}\left|x_0(t)\right| > 1-\left(\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{r+1}}\right), \tag{32}$$

and so $\cap_{j=1}^{r+1} T_j \neq \emptyset$. We define $n_{r+1} := \min(\cap_{j=1}^{r+1} T_j)$. Note that $n_{r+1} > n_r$.

From (25) it follows that

$$\left|A\left(e_{m_i}, e_{n_{r+1}}\right)\right| \geq 1-2^{-i} \tag{33}$$

for all $i = 1,\ldots,r+1$.

By (33)

$$1 \geq \liminf_j \left|A\left(e_{m_i}, e_{n_j}\right)\right| \geq 1 - 2^{-i}. \tag{34}$$

Then

$$\lim_i \liminf_j \left|A\left(e_{m_i}, e_{n_j}\right)\right| = 1. \tag{35}$$

To finish the proof, we show that the lim inf can be replaced with lim by just choosing a suitable subsequence of $\{e_{n_j}\}_{j=1}^{\infty}$.

Let us proceed once more by induction. By (33), $|A(e_{m_1}, e_{n_j})| \geq 1 - (1/2)$ for all $j \geq 1$. Then, there exists a subsequence $\{e_{n_{j_k}}\}_{k=1}^{\infty}$ of $\{e_{n_j}\}_{j=1}^{\infty}$ such that $\lim_k |A(e_{m_1}, e_{n_{j_k}})|$ exists and it is greater than or equal to $1 - (1/2)$. To make the notation clear, we write $n(1,k) := n_{j_k}$ and so

$$\lim_k \left|A\left(e_{m_1}, e_{n(1,k)}\right)\right| \geq 1 - \frac{1}{2}. \tag{36}$$

Assume that we have a chain of sequences $\{e_{n(1,j)}\}_{j=1}^{\infty}, \ldots, \{e_{n(p,j)}\}_{j=1}^{\infty}$ with each of them being a subsequence of the previous one, such that $\lim_j |A(e_{m_i}, e_{n(i,j)})| \geq 1 - (1/2^i)$, for all $i = 1, \ldots, p$. Let us construct a subsequence $\{e_{n(p+1,j)}\}_{j=1}^{\infty}$ of $\{e_{n(p,j)}\}_{j=1}^{\infty}$ such that $|A(e_{m_{p+1}}, e_{n(p+1,j)})| \geq 1 - (1/2^{p+1})$ for all $j \in \mathbb{N}$. Indeed, since $|A(e_{m_{p+1}}, e_{n(p,j)})| \geq 1 - (1/2^{p+1})$ for all $j \geq p+1$, there exists a subsequence $\{e_{n(p,j)_l}\}_{l=1}^{\infty}$ of $\{e_{n(p,j)}\}_{j=1}^{\infty}$ such that $\lim_l |A(e_{m_{p+1}}, e_{n(p,j)_l})|$ exists and it is greater than or equal to $1 - (1/2^{p+1})$. We write $n(p+1, l) := n(p,j)_l$ and so

$$\lim_l \left|A\left(e_{m_{p+1}}, e_{n(p+1,l)}\right)\right| \geq 1 - \frac{1}{2^{p+1}}. \tag{37}$$

So we have countably many sequences $\{e_{n(1,j)}\}_{j=1}^{\infty}$, $\{e_{n(2,j)}\}_{j=1}^{\infty}, \ldots$, with each of them being a subsequence of the previous one, such that $\lim_j |A(e_{m_i}, e_{n(i,j)})| \geq 1 - (1/2^i)$, for all $i = 1, 2, \ldots$. The diagonal sequence $\{e_{n(j,j)}\}_{j=1}^{\infty}$ is the one we were looking for. Note that $\{e_{n(j,j)}\}_{j=i}^{\infty}$ is a subsequence of $\{e_{n(i,j)}\}_{j=1}^{\infty}$ and then there exists

$$\lim_j \left|A\left(e_{m_i}, e_{n(j,j)}\right)\right| \geq 1 - \frac{1}{2^i}, \tag{38}$$

for all $i \in \mathbb{N}$.

Therefore, we have found sequences $\{e_{m_i}\}_{i=1}^{\infty}$ and $\{e_{n(j,j)}\}_{j=1}^{\infty}$, with $\{m_i\}_{i=1}^{\infty}$ and $\{n(j,j)\}_{j=1}^{\infty}$ strictly increasing, for which there exists

$$\lim_i \lim_j \left|A\left(e_{m_i}, e_{n(j,j)}\right)\right| = 1. \tag{39}$$

This concludes the case with $\pi(x_1^{**}) = \pi(x_2^{**}) = 0$.

If $\pi(x_1^{**}) \neq 0$ or $\pi(x_2^{**}) \neq 0$, then $y_1^{**} = x_1^{**} - \pi(x_1^{**})$ and $y_2^{**} = x_2^{**} - \pi(x_2^{**})$ are nonzero points of $\ell_1^{**} \setminus \ell_1$ with $|A(y_1^{**}/\|y_1^{**}\|, y_2^{**}/\|y_2^{**}\|)| = 1$, and the former case gives us the desired result. □

Corollary 10. *Given a bilinear form $A \in \mathscr{L}(^2\ell_1)$ of norm one and $\sigma \in \Sigma_2$, the following are equivalent:*

(a) $\lim_{(i,\sigma(1),\sigma)} \lim_{(i,\sigma(2),\sigma)} |A(e_{m(i,1,\sigma)}, e_{m(i,2,\sigma)})| = 1$ *for some strictly increasing sequences of natural numbers $(m(i,1,\sigma))_{i=1}^{\infty}$ and $(m(i,2,\sigma))_{i=1}^{\infty}$;*

(b) *there exist $x_{1,\sigma}^{**}, x_{2,\sigma}^{**} \in \ell_1^{**} \setminus \ell_1$ of norm one such that $|A_\sigma(x_{1,\sigma}^{**}, x_{2,\sigma}^{**})| = 1$.*

Remark 11. We do not know if Theorem 9 is valid for n-linear mappings with $n > 2$. Our conjecture is the following. Let $n \in \mathbb{N}$, $A \in \mathscr{L}(^n\ell_1)$, and $\sigma \in \Sigma_n$. If A_σ attains its norm on ℓ_1^{**} but only in n-tuples that belong to $(B_{\ell_1^{**}} \setminus \ell_1)^n$, then there exist increasing sequences of natural numbers $\{k(j, m_j)\}_{m_j=1}^{\infty}$, $j = 1, \ldots, n$, such that

$$\lim_{m_{\sigma(1)} \to \infty} \cdots \lim_{m_{\sigma(n)} \to \infty} \left|A\left(e_{k(1,m_1)}, \ldots, e_{k(n,m_n)}\right)\right| = \|A\|. \tag{40}$$

Next we give the following lemma.

Lemma 12. *Let $\{k_1(h), \ldots, k_n(h)\}_{h=1}^{\infty}$ be a sequence in $\mathbb{N}^n$ such that each $\{k_j(h)\}_{h=1}^{\infty}$ is strictly increasing, $j = 1, \ldots, n$, and let the n-linear mapping $A : \ell_1^n \to \mathbb{R}$ be defined by*

$$A\left(e_{k_1}, \ldots, e_{k_n}\right) = \begin{cases} \left(\dfrac{k_1(h)}{k_1(h)+1}\right)^n & \text{if } (k_1, \ldots, k_n) = (k_1(h), \ldots, k_n(h)) \\ & \quad \text{for some } h \in \mathbb{N}. \\ 0 & \text{otherwise}. \end{cases} \tag{41}$$

One has that $\|A\| = 1$ and there is no permutation σ such that A_σ attains its norm (at any n-tuple of $B_{\ell_1^{**}} \times \cdots \times B_{\ell_1^{**}}$).

Proof. Note first that, for arbitrary $x_i := \sum_{k=1}^{\infty} a_{k,i} e_k \in B_{\ell_1}$, $i = 1, \ldots, n$, if we fix $1 \leq j \leq n$ then

$$\begin{aligned} |A(x_1, \ldots, x_n)| &\leq \|x_1\| \cdots \|x_{j-1}\| \cdot \|x_{j+1}\| \cdots \|x_n\| \\ &\quad \times \sum_{h=1}^{\infty} \frac{k_1(h)}{k_1(h)+1} \left|a_{k_j(h),j}\right| \\ &\leq \sum_{h=1}^{\infty} \frac{k_1(h)}{k_1(h)+1} \left|a_{k_j(h),j}\right|. \end{aligned} \tag{42}$$

Thus, for any $\sigma \in \Sigma_n$, A_σ, any $x_1^{**}, \ldots, x_{k-j}^{**}, x_{j+1}^{**}, \ldots, x_n^{**} \in B_{\ell_1^{**}}$, and any $x_j \in B_{\ell_1}$, by taking nets in B_{ℓ_1} weak-star convergent if necessary, we get

$$\begin{aligned} &\left|A_\sigma\left(x_1^{**}, \ldots, x_{j-1}^{**}, x_j, x_{j+1}^{**}, \ldots, x_n^{**}\right)\right| \\ &\quad \leq \sum_{h=1}^{\infty} \frac{k_1(h)}{k_1(h)+1} \left|a_{k_j(h),j}\right| < 1. \end{aligned} \tag{43}$$

Hence if there exist a permutation $\sigma \in \Sigma_n$ and $x_1^{**}, \ldots, x_n^{**}$ in $B_{\ell_1^{**}}$ such that $\|x_j^{**}\| = 1$ for every j and

$$\left|A_\sigma\left(x_1^{**}, \ldots, x_n^{**}\right)\right| = 1, \tag{44}$$

we have that $x_1^{**}, \ldots, x_n^{**} \in B_{\ell_1^{**}} \setminus B_{\ell_1}$. Moreover, by Lemma 7 it can also be assumed that x_j^{**} belongs to $c_0^{\perp}$ for $j = 1, \ldots, n$. Finally, by making a rearrangement of coordinates, if necessary, we can assume that σ is the identity permutation.

We define $B : \ell_1 \times \ell_1 \to \mathbb{R}$ by $B(x, y) = A_{Id}(x, y, x_3^{**}, \ldots, x_n^{**})$. Clearly

$$\left|B_{Id}\left(x_1^{**}, x_2^{**}\right)\right| = \left|A_{Id}\left(x_1^{**}, \ldots, x_n^{**}\right)\right| = 1. \tag{45}$$

By Theorem 9, there exist two sequences (e_{n_j}) and (e_{m_l}) such that

$$\lim_{j \to \infty} \lim_{l \to \infty} \left|B\left(e_{n_j}, e_{m_l}\right)\right| = 1. \tag{46}$$

Thus there exist j, l such that

$$\left|B\left(e_{n_j}, e_{m_l}\right)\right| > \frac{1}{2}. \tag{47}$$

But there exists h_0 such that $n_j < k_1(h)$ and $m_l < k_2(h)$ for every $h \geq h_0$ and we get that

$$\left(n_j, m_l, k_3, \ldots, k_n\right) \notin \left\{\left(k_1(h), \ldots, k_n(h)\right) : h \geq h_0\right\}, \tag{48}$$

for every $k_3, \ldots, k_n \in \mathbb{N}$ with $k_3 > k_3(h_0), \ldots, k_n > k_n(h_0)$. Now consider a net $\{x_{d_j}\}_{d_j \in D_j}$ in B_{ℓ_1} weak-star convergent to x_j^{**} for $j = 3, \ldots, n$. Since x_j^{**} belongs to $c_0^{\perp}$, as in the proof of Theorem 9, we can assume additionally that, for every $d_j \in D_j$, the first $k_j(h_0)$ components of x_{d_j} are 0. Hence

$$A\left(e_{n_j}, e_{m_l}, e_{k_3}, \ldots, e_{k_{n-1}}, x_{d_n}\right) = 0, \tag{49}$$

for every $d_n \in D_n$. Hence

$$A_{Id}\left(e_{n_j}, e_{m_l}, e_{k_3}, \ldots, e_{k_{n-1}}, x_n^{**}\right) = 0, \tag{50}$$

for every $k_3 > k_3(h_0), \ldots, k_n > k_n(h_0)$. By induction we obtain the contradiction

$$B\left(e_{n_j}, e_{m_l}\right) = A_{Id}\left(e_{n_j}, e_{m_l}, x_3^{**}, \ldots, x_n^{**}\right) = 0. \tag{51}$$

□

Theorem 13. *Given a subset $P \subseteq \Sigma_n$, there exists an n-linear form $A(P) \in \mathscr{L}(^n\ell_1)$ with $\|A(P)\| = 1$ such that $A(P)_\sigma$ is norm attaining if and only if $\sigma \in P$.*

Proof. The proof will be divided into two cases.

If P is the empty set, consider $A(P) \in \mathscr{L}(^n\ell_1)$

$$A(P)\left(e_{k_1}, e_{k_2}, \ldots, e_{k_n}\right) = \begin{cases} \left(\dfrac{k_1}{k_1+1}\right)^n & \text{if } k_1 = k_2 = \cdots = k_n, \\ 0 & \text{otherwise.} \end{cases} \tag{52}$$

By Lemma 12, $A(P)$ does not attain its norm at any point of the unit ball of ℓ_1^{**}.

If P is not empty, consider

$$A(P)\left(e_{k_1}, e_{k_2}, \ldots, e_{k_n}\right) = \begin{cases} \displaystyle\prod_{i=1}^{n} \frac{k_i}{k_i+1} & \text{if } \exists \sigma \in P, k_{\sigma(1)} \leq k_{\sigma(2)} \leq \cdots \leq k_{\sigma(n)}, \\ 0 & \text{otherwise.} \end{cases} \tag{53}$$

Clearly, $\|A(P)\| = 1$. A similar argument to the one given in (43) shows that, for any $\sigma \in \Sigma_n$, A_σ does not attain its norm at any n-tuple in $B_{\ell_1^{**}} \times \cdots \times B_{\ell_1^{**}}$ with at least a coordinate j belonging to B_{ℓ_1}. If $\sigma \in P$ then

$$\lim_{k_{\sigma(1)} \to \infty} \cdots \lim_{k_{\sigma(n)} \to \infty} A(P)\left(e_{k_1}, \ldots, e_{k_n}\right) = 1. \tag{54}$$

Hence, considering x^{**} a weak-star cluster point of the sequence $\{e_k\}_{k=1}^{\infty}$ we obtain

$$A(P)_\sigma\left(x^{**}, \ldots, x^{**}\right) = 1. \tag{55}$$

Thus, $A(P)_\sigma$ is norm attaining.

Now we see that $A(P)_\sigma$ does not attain its norm whenever σ is not in P. For simplicity we will assume that σ is the identity permutation. Let us assume that $A(P)_{Id}$ does attain its norm at $(x_1^{**}, \ldots, x_n^{**}) \in B_{\ell_1^{**}} \times \cdots \times B_{\ell_1^{**}}$. By the above observation, x_i^{**} is a point in $B_{\ell_1^{**}} \setminus \ell_1$ for $i = 1, \ldots, n$. By Lemma 7 we can assume that $\pi(x_i^{**}) = 0$ for $i = 1, \ldots, n$. Let $\{x_{d_i}\}_{d_i \in D_i}$ be nets in the unit sphere of ℓ_1 weak-star convergent to x_i^{**}, for $i = 1, \ldots, n$.

Let $l_0 = 0$. Since $|A(P)_{Id}(x_1^{**}, \ldots, x_n^{**})| = 1$ there exists $d_1^0 \in D_1$ with

$$\left|A(P)_{Id}\left(x_{d_1^0}, x_2^{**}, \ldots, x_n^{**}\right)\right| > 1 - 2^{-n}. \tag{56}$$

Let l_1 be such that $\|\pi_{l_1}(x_{d_1^0})\| > 1/2$. Now, using (56) and since $\pi(x_2^{**}) = 0$ we can find $d_2 \in D_2$ and a natural number l_2 with $|A(P)_{Id}(x_{d_1^0}, x_{d_2^0}, x_3^{**}, \ldots, x_n^{**})| > 1 - 2^{-n}$ and $\|\pi_{l_2}(x_{d_2^0})\| - \|\pi_{l_1}(x_{d_2^0})\| > 1/2$. In general, by using finite induction over i, we can find $d_i^0 \in D_i$ and a natural number l_i such that $|A(P)_{Id}(x_{d_1^0}, \ldots, x_{d_i^0}, x_{i+1}^{**}, \ldots, x_n^{**})| > 1 - 2^{-n}$ and $\|\pi_{l_i}(x_{d_i^0})\| - \|\pi_{l_{i-1}}(x_{d_i^0})\| > 1/2$, for $i = 2, \ldots, n$.

But then, if we denote by

$$C := \left\{\left(t_1, \ldots, t_n\right) \in \mathbb{N}^n : l_{i-1} < t_i \leq l_i \text{ for } i = 1, \ldots n\right\} \tag{57}$$

since Id is not in P, we have $A(P)(e_{t_1},\ldots,e_{t_n}) = 0$ for all $(t_1,\ldots,t_n)\in C$. Therefore,

$$\begin{aligned}
1-2^{-n} &< \left|A(P)\left(x_{d_1^0},\ldots,x_{d_n^0}\right)\right| \\
&\leq \sum_{(t_1,\ldots,t_n)} \prod_{i=1}^{n} \left|x_{d_i^0}(t_i)\right| A(P)\left(e_{t_1},\ldots,e_{t_n}\right) \\
&= \sum_{(t_1,\ldots,t_n)\notin C} \prod_{i=1}^{n} \left|x_{d_i^0}(t_i)\right| A(P)\left(e_{t_1},\ldots,e_{t_n}\right) \\
&< \sum_{(t_1,\ldots,t_n)\notin C} \prod_{i=1}^{n} \left|x_{d_i^0}(t_i)\right| \\
&\leq 1-\sum_{(t_1,\ldots,t_n)\in C} \prod_{i=1}^{n} \left|x_{d_i^0}(t_i)\right| \\
&= 1-\prod_{i=1}^{n}\left(\left\|\pi_{k_i}\left(x_{d_i^0}\right)\right\|-\left\|\pi_{k_{i-1}}\left(x_{d_i^0}\right)\right\|\right) \\
&< 1-2^{-n}
\end{aligned} \tag{58}$$

which is a contradiction. Hence $A(P)_{Id}$ does not attain its norm. □

Theorem 14. *Let $A\in\mathscr{L}(^n\ell_1)$ of norm one such that, for every $\epsilon>0$ and every $\sigma\in\Sigma_n$, there exist subsequences $\{e_{k(i,m_i,\sigma)}\}_{m_i=1}^{\infty}$(that depend on σ and ϵ) of the sequence $\{e_k\}_{k=1}^{\infty}$ so that*

$$\lim_{m_{\sigma(1)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty}\left|A\left(e_{k(1,m_1,\sigma)},\ldots,e_{k(n,m_n,\sigma)}\right)\right|>1-\epsilon. \tag{59}$$

Then for every $\epsilon>0$ and each subset $P\subseteq\Sigma_n$, there exists $A(P,\epsilon)\in\mathscr{L}(^n\ell_1)$ with $\|A(P,\epsilon)\|=1$ such that $\|A(P,\epsilon)-A\|\leq\epsilon$, and $A(P,\epsilon)_\sigma$ is norm attaining if and only if $\sigma\in P$.

Proof. Consider the n-linear form

$$B\left(x_1,\ldots,x_n\right)=\sum_{k_1,\ldots,k_n\in\mathbb{N}} B\left(e_{k_1},\ldots,e_{k_n}\right)\prod_{i=1}^{n} x_i\left(k_i\right) \tag{60}$$

for $x_1,\ldots,x_n\in\ell_1$, where

$$B\left(e_{k_1},\ldots,e_{k_n}\right) = \begin{cases} A\left(e_{k_1},\ldots,e_{k_n}\right) & \text{if } 1-\dfrac{\epsilon}{2}\geq\left|A\left(e_{k_1},\ldots,e_{k_n}\right)\right| \\ \left(1-\dfrac{\epsilon}{2}\right)\operatorname{sign}\left(A\left(e_{k_1},\ldots,e_{k_n}\right)\right), & \text{if } \left|A\left(e_{k_1},\ldots,e_{k_n}\right)\right|>1-\dfrac{\epsilon}{2}. \end{cases} \tag{61}$$

We have $\|B\|\leq 1-(\epsilon/2)$.

Fix a nonempty subset of permutations P, consider the n-linear form $A(P)$ from Theorem 13, and define the n-linear form $A(P,\epsilon)$ as follows:

$$\begin{aligned}
A(P,\epsilon)\left(e_{k_1},\ldots,e_{k_n}\right) := B\left(e_{k_1},\ldots,e_{k_n}\right) \\
+\operatorname{sign}\left(B\left(e_{k_1},\ldots,e_{k_n}\right)\right) \\
\times\frac{\epsilon}{2}A(P)\left(e_{k_1},\ldots,e_{k_n}\right)
\end{aligned} \tag{62}$$

if $k_1=k(1,m_1,\sigma),\ldots,k_n=k(n,m_n,\sigma)$, for some $\sigma\in P$, with $m_{\sigma(1)}\leq\cdots\leq m_{\sigma(n)}$ and $B(e_{k_1},\ldots,e_{k_n})$ otherwise.

Clearly,

$$\|A(P,\epsilon)\|\leq\|B\|+\frac{\epsilon}{2}\|A(P)\|\leq 1-\frac{\epsilon}{2}+\frac{\epsilon}{2}=1, \tag{63}$$

and hence $\|A(P,\epsilon)\|\leq 1$.

By hypothesis, for each $\sigma\in\Sigma_n$, there exist sequences $\{e_{k(i,m_i,\sigma)}\}_{m_i=1}^{\infty}$, with the property that $\{k(i,m_i,\sigma)\}_{m_i=1}^{\infty}$ is strictly increasing, such that

$$\lim_{m_{\sigma(1)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty}\left|A\left(e_{k(1,m_1,\sigma)},\ldots,e_{k(n,m_n,\sigma)}\right)\right|>1-\frac{\epsilon}{2}. \tag{64}$$

From (64) there exists $m_{\sigma(1)}^0$ such that for every $m_{\sigma(1)}\geq m_{\sigma(1)}^0$

$$\lim_{m_{\sigma(2)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty}\left|A\left(e_{k(1,m_1,\sigma)},\ldots,e_{k(n,m_n,\sigma)}\right)\right|>1-\frac{\epsilon}{2}. \tag{65}$$

Taking $m_{\sigma(1)}\geq m_{\sigma(1)}^0$, there is $m_{\sigma(2)}^0$ that depends on $m_{\sigma(1)}$, such that for every $m_{\sigma(2)}\geq m_{\sigma(2)}^0$

$$\lim_{m_{\sigma(3)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty}\left|A\left(e_{k(1,m_1,\sigma)},\ldots,e_{k(n,m_n,\sigma)}\right)\right|>1-\frac{\epsilon}{2}. \tag{66}$$

Reiterating this process and assuming that we have fixed natural numbers $m_{\sigma(1)}$, $m_{\sigma(1)}$, …, $m_{\sigma(n-1)}$ with $m_{\sigma(1)}\geq m_{\sigma(1)}^0$, $m_{\sigma(2)}\geq m_{\sigma(2)}^0,\ldots,m_{\sigma(n-1)}\geq m_{\sigma(n-1)}^0$, where $m_{\sigma(n-1)}^0$ depends on $m_{\sigma(1)},m_{\sigma(2)},\ldots,m_{\sigma(n-2)}$, so that $\lim_{m_{\sigma(n)}}|A(e_{k(1,m_1,\sigma)},\ldots,e_{k(n,m_n,\sigma)})|>1-(\epsilon/2)$, we can find $m_{\sigma(n)}^0$ that depends on $m_{\sigma(1)},m_{\sigma(2)},\ldots,m_{\sigma(n-1)}$, such that for every $m_{\sigma(n)}\geq m_{\sigma(n)}^0$

$$\left|A\left(e_{k(1,m_1,\sigma)},\ldots,e_{k(n,m_n,\sigma)}\right)\right|>1-\frac{\epsilon}{2}. \tag{67}$$

Then,

$$\left|B\left(e_{k(1,m_1,\sigma)},\ldots,e_{k(n,m_n,\sigma)}\right)\right|=1-\frac{\epsilon}{2}, \tag{68}$$

and then $\|B\| = 1-(\epsilon/2)$. Moreover, given $\sigma\in P$ and $\delta>0$ we can take $m_{\sigma(1)}\leq\cdots\leq m_{\sigma(n)}$ big enough so that $A(P)(e_{k(1,m_1,\sigma)},\ldots,e_{k(n,m_n,\sigma)})\geq 1-\delta$. Hence,

$$\left|A(P,\epsilon)\left(e_{k(1,m_1,\sigma)},\ldots,e_{k(n,m_n,\sigma)}\right)\right|\geq 1-\frac{\epsilon}{2}+\frac{\epsilon}{2}(1-\delta) \tag{69}$$

and so $\|A(P,\epsilon)\|\geq 1-(\epsilon/2)+(\epsilon/2)(1-\delta)$. Thus, $\|A(P,\epsilon)\|=1$.

Notice that $|A(P,\epsilon)(x_1,\ldots,x_n) - B(x_1,\ldots,x_n)| \leq (\epsilon/2)$, for all $x_1,\ldots,x_n \in \ell_1$; hence

$$\|A(P,\epsilon) - A\| \leq \|A(P,\epsilon) - B\| + \|B - A\| \leq \epsilon. \tag{70}$$

Now we show that $A(P,\epsilon)_\sigma$ is norm attaining if and only if $\sigma \in P$.

If $\sigma \notin P$ and we assume that there is $(x_1^{**},\ldots,x_n^{**}) \in B_{\ell_1^{**}} \times \cdots \times B_{\ell_1^{**}}$ such that $A(P,\epsilon)_\sigma(x_1^{**},\ldots,x_n^{**}) = 1$, since $A(P)_\sigma(x_1^{**},\ldots,x_n^{**}) < 1$, then $1 < |B_\sigma(x_1^{**},\ldots,x_n^{**})| + (\epsilon/2)$. Hence $|B_\sigma(x_1^{**},\ldots,x_n^{**})| > 1 - (\epsilon/2)$, which is impossible.

Take now $\sigma \in P$. From (68) we have for $m_{\sigma(1)} \leq \cdots \leq m_{\sigma(n)}$

$$\left|A(P,\epsilon)\left(e_{k(1,m_1,\sigma)},\ldots,e_{k(n,m_n,\sigma)}\right)\right| = 1 - \frac{\varepsilon}{2} + \frac{\epsilon}{2}\prod_{i=1}^{n}\frac{m_i}{m_i+1}. \tag{71}$$

Hence

$$\begin{aligned}\lim_{m_{\sigma(1)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty}&\left|A(P,\epsilon)\left(e_{k(1,m_1,\sigma)},\ldots,e_{k(n,m_n,\sigma)}\right)\right| \\ &= \lim_{m_{\sigma(1)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty} 1 - \frac{\epsilon}{2} + \frac{\epsilon}{2}\prod_{i=1}^{n}\frac{m_i}{m_i+1} = 1\end{aligned} \tag{72}$$

so $A(P,\epsilon)_\sigma$ is norm attaining at a point $(x_1^{**},\ldots,x_n^{**})$, where each x_j^{**} is a weak-star cluster point of the sequence $\{e_{k(j,m_j,\sigma)}\}_{m_j=1}^{\infty}$, $j = 1,\ldots,n$.

If $P = \emptyset$, for every h, by taking $\varepsilon = (1/h)$, the process above gives the existence of a sequence of n-tuples $\{(k_1(h),\ldots,k_n(h))\}_{h=1}^{\infty}$ in $\mathbb{N}^n$ such that each $\{k_j(h)\}_{h=1}^{\infty}$ is strictly increasing, $j = 1,\ldots,n$, and

$$\left|A\left(e_{k_1(h)},\ldots,e_{k_n(h))}\right)\right| > 1 - \frac{1}{h}. \tag{73}$$

We define

$$C\left(e_{k_1},\ldots,e_{k_n}\right) = \begin{cases}\left(\dfrac{k_1(h)}{k_1(h)+1}\right)^n & \text{if } (k_1,\ldots,k_n) = (k_1(h),\ldots,k_n(h)) \\ & \qquad\qquad \text{for some } h \in \mathbb{N}, \\ 0 & \text{otherwise}\end{cases} \tag{74}$$

and $A(P,\epsilon)$ at $(e_{k_1},\ldots,e_{k_n})$ as

$$\begin{aligned}A(P,\epsilon)\left(e_{k_1},\ldots,e_{k_n}\right) := B\left(e_{k_1},\ldots,e_{k_n}\right) \\ + \operatorname{sign}\left(B\left(e_{k_1},\ldots,e_{k_n}\right)\right) \\ \times \frac{\epsilon}{2}C\left(e_{k_1},\ldots,e_{k_n}\right).\end{aligned} \tag{75}$$

Notice that as before $\|A(P,\epsilon)\| \leq \|B\| + (\epsilon/2)\|C\| = 1$, and, since

$$\begin{aligned}&\left|A(P,\epsilon)\left(e_{k_1(h)},\ldots,e_{k_n(h)}\right)\right| \\ &\quad = \left|B\left(e_{k_1(h)},\ldots,e_{k_n(h)}\right)\right| + \left(\frac{k_1(h)}{k_1(h)+1}\right)^n \\ &\quad = 1 - \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\left(\frac{k_1(h)}{k_1(h)+1}\right)^n,\end{aligned} \tag{76}$$

for every h such that $(1/h) < (\epsilon/2)$, we obtain that $\|A(P,\epsilon)\| = 1$. Since, by Lemma 12, C does not attain the norm at any point of $B_{\ell_1^{**}}$ neither can $A(P,\epsilon)$. To conclude the proof, notice $\|A(P,\epsilon) - A\| \leq \|A(P,\epsilon) - B\| + \|B - A\| \leq \epsilon/2 + \epsilon/2 = \epsilon$. □

Remark 15. If the conjecture in Remark 11 was true, we could get the following result. Let $A \in \mathcal{L}(^n\ell_1)$ of norm one. The following are equivalent.

(1) For every $\epsilon > 0$ and every $\sigma \in \Sigma_n$, there exist subsequences $\{e_{k(i,m_i)}\}_{m_i=1}^{\infty}$ (that depend on σ and ϵ) of the sequence $\{e_k\}_{k=1}^{\infty}$ so that

$$\lim_{m_{\sigma(1)}\to\infty}\cdots\lim_{m_{\sigma(n)}\to\infty}\left|A\left(e_{k(1,m_1)},\ldots,e_{k(n,m_n)}\right)\right| > 1 - \epsilon. \tag{77}$$

(2) For every $\epsilon > 0$ and each subset $P \subseteq \Sigma_n$, there exists $A(P,\epsilon) \in \mathcal{L}(^n\ell_1)$ with $\|A(P,\epsilon)\| = 1$ such that $\|A(P,\epsilon) - A\| \leq \epsilon$, and $A(P,\epsilon)_\sigma$ is norm attaining if and only if $\sigma \in P$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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