Research Article

Synchronization of Chaotic Delayed Neural Networks via Impulsive Control

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This paper is concerned with the impulsive synchronization problem of chaotic delayed neural networks. By employing Lyapunov stability theorem, impulsive control theory and linear matrix inequality (LMI) technique, several new sufficient conditions ensuring the asymptotically synchronization for coupled chaotic delayed neural networks are derived. Based on these new sufficient conditions, an impulsive controller is designed. Moreover, the stable impulsive interval of synchronized neural networks is objectively estimated by combining the MATLAB LMI toolbox and one of the two given equations. Two examples with numerical simulations are given to illustrate the effectiveness of the proposed method.

1. Introduction

During implementation of artificial neural networks, time delay is a very familiar phenomenon due to the finite switching speed of neurons and amplifiers. The existence of time delay possesses an important source in what cause instability and even chaotic behavior in some type of delayed neural networks (DNNs) if the parameters and time delays are appropriately chosen (see [1–6]). These kinds of chaotic neural networks have been successfully applied in chemical biology [7], combinatorial optimization [8], associative memory [9], biological simulation [10], and so on. Especially, the synchronization problem of chaotic delayed neural networks has been extensively studied over the past few decades due to its potential applications in many areas, such as secure communications [11–13], image encryption [14], image processing [15], and harmonic oscillation generation [16]. Hence, it is of great theoretical and practical significance to investigate synchronization problem of chaotic delayed neural networks.

A wide variety of schemes have been proposed for the synchronization of chaotic systems, for example, adaptive control [17], slide mode control [18], coupling control [19], feedback control [20], and impulsive control [21–25]. It is worth noting that impulsive control is characterized by the abrupt changes in the system dynamics at certain instants, which is an advantage in reducing the amount of information transmission and improving the security and robustness against disturbances especially in telecommunication network and power grid, orbital transfer of satellite. In addition, impulsive control allows the stabilization and synchronization of chaotic systems using only small control impulses. Thus, it has been widely used to synchronize chaotic neural networks (see [21, 23, 24]). In [21], Zhao and Zhang obtained some new criteria for the impulsive exponential antisynchronization of two chaotic delayed neural networks by establishing an integral delay inequality via the inequality method. Li et al. [23] investigated the synchronization scheme of coupled neural networks with time delays by utilizing the stability theory for impulsive functional differential equations. Zhang and Sun [24] studied the robust synchronization model of coupled delayed neural networks under general impulsive control. However, all above results do not efficiently utilize the so-called sector nonlinearity property of the activation functions of the neural networks, which leads to some conservatism of the results.

Motivated by the above discussions, the impulsive synchronization problem for chaotic delayed neural networks has not been fully investigated yet, which is still open and remains challenging. The aim of this paper is to study

the synchronization of chaotic neural networks with time-varying delay. Some novel sufficient conditions which guarantee the coupled chaotic delayed neural networks can be asymptotically synchronized are derived based on Lyapunov stability theorem, impulsive control theory, and linear matrix inequality (LMI) technique. Moreover, the stable impulsive interval of synchronized neural networks is objectively estimated by combining the MATLAB LMI toolbox and one of the two given equations.

The organization of this paper is as follows. In Section 2, the impulsive synchronization problem is described and some necessary definitions and lemmas are given. Some new synchronization criteria are obtained in Section 3. In Section 4, two illustrative examples are given to show the effectiveness of the proposed method. Finally, conclusions are given in Section 5.

Notations. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}^+$ denote the set of positive real numbers, and let $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote the n dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. $\mathbb{N}$ denotes the set of positive integers. $\|z\|$ is the Euclidean norm of the vector z. For any matrix $P \in \mathbb{R}^{n \times n}$, $P > 0$ denotes that P is a symmetric and positive definite matrix. If P_1, P_2 are symmetric matrices, then $P_1 \le P_2$ means that $P_1 - P_2$ is a negative semidefinite matrix. $\lambda_m(P), \lambda_M(P)$ denote the minimum and maximum eigenvalue of matrix P, respectively. P^T and P^{-1} mean the transpose of P and the inverse of a square matrix P. I denotes the identity matrix with appropriate dimensions. $S_\rho = \{x \in \mathbb{R}^n \mid \|x\| < \rho\}$, $\mathcal{K} = \{\psi \in C(\mathbb{R}^+, \mathbb{R}^+) \mid \psi(t) \text{ is strictly increasing and } \psi(0) = 0\}$, $\mathcal{K}^* = \{\psi \in \mathcal{K} \mid \psi(t) < t \text{ for } t > 0\}$, $\Sigma = \{\psi \in C(\mathbb{R}^+, \mathbb{R}^+) \mid \psi(0) = 0, \psi(t) > 0, \text{ for } t > 0\}$, $PC = \{\psi : [-\tau, 0] \rightarrow \mathbb{R}^n, \psi(t) \text{ is continuous everywhere except at the finite number of points } \bar{t}, \text{ where } \psi(\bar{t}^+), \psi(\bar{t}^-) \text{ exist and } \psi(\bar{t}^+) = \psi(\bar{t})\}$. The notation $\star$ always denotes the symmetric block in one symmetric matrix.

2. Problem Description and Preliminaries

The chaotic neural networks with variable delay can be described by

$$\dot{x}(t) = -Ax(t) + Bf(x(t)) + Cf(x(t - \tau(t))) + J, \quad t > 0,$$
$$x(s) = \phi(s), \quad s \in [-\tau, 0], \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is the neuron state vector; A is a positive diagonal matrix; $B = (b_{ij}) \in \mathbb{R}^{n \times n}$ and $C = (c_{ij}) \in \mathbb{R}^{n \times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively; J is the constant input vector; $f(x(\cdot)) = (f_1(x_1(\cdot)), f_2(x_2(\cdot)), \ldots, f_n(x_n(\cdot)))^T$ is the nonlinear neuron activation function which describes the behavior in which the neurons respond to each other; the time delay $\tau(t)$ is bounded as $0 \le \tau(t) \le \tau$, and the initial value condition $\phi(\cdot) \in PC([-\tau, 0], \mathbb{R}^n)$ is a piecewise right continuous function.

Throughout this paper we assume that $f(\cdot)$ satisfies assumption (H1).

(H1) (see [26]) Each function f_i is continuous, and there exist scalars l_i^- and l_i^+ such that

$$l_i^- \le \frac{f_i(a) - f_i(b)}{a - b} \le l_i^+ \tag{2}$$

for any $a, b \in \mathbb{R}$, $a \ne b$, where l_i^+ and l_i^- can be positive, negative, or zero.

Based on assumption (H1), we set

$$L_1 = \operatorname{diag}(l_1^+ + l_1^-, l_2^+ + l_2^-, \ldots, l_n^+ + l_n^-),$$
$$L_2 = \operatorname{diag}(l_1^+ l_1^-, l_2^+ l_2^-, \ldots, l_n^+ l_n^-). \tag{3}$$

Remark 1. In usual Lipschitz condition, it is assumed that $l_i^- = -l_i^+$. Clearly, the condition (H1) is quite general and includes the usual Lipschitz conditions as a special case.

To investigate the impulsive synchronization of chaotic neural networks and consider system (1) as the drive system, the corresponding response system is given by

$$\dot{y}(t) = -Ay(t) + Bf(y(t)) + Cf(y(t - \tau(t))) + J, \quad t > 0,$$
$$y(s) = \varphi(s), \quad s \in [-\tau, 0], \tag{4}$$

where $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T$ is the neuron state vector of the response system, the initial value condition $\varphi(\cdot) \in PC([-\tau, 0], \mathbb{R}^n)$ is a piecewise right continuous function.

At discrete time t_k, the state variables of the drive system are transmitted to the response system as the control input such that the state vectors of the response system are suddenly changed at these instants. Therefore, the impulsive controlled response system can be written as

$$\dot{y}(t) = -Ay(t) + Bf(y(t)) + Cf(y(t - \tau(t))) + J,$$
$$t \ge t_0 = 0, \quad t \ne t_k,$$
$$\Delta y(t_k) = y(t_k^+) - y(t_k^-) = W_k(y(t_k^-) - x(t_k^-)), \tag{5}$$
$$t = t_k, \quad k \in \mathbb{N},$$
$$y(s) = \varphi(s), \quad s \in [-\tau, 0],$$

where $\Delta y(t_k)$ denotes the state jumping at impulsive time instant $t = t_k$, $y(t_k^+), y(t_k^-)$ and $x(t_k^+), x(t_k^-)$ are the right-hand and left-hand limits of the functions $y(t)$ and $x(t)$ at t_k, respectively. Moreover, $y(t_k) = y(t_k^+)$, $x(t_k) = x(t_k^+)$. Suppose that the discrete time sequence $\{t_k\}$ satisfies $0 < t_1 < t_2 < \cdots$ and $\lim_{k \to \infty} t_k = \infty$. W_k is the impulsive matrix.

Let $e(t) = y(t) - x(t)$ be the synchronization error, and then we can obtain the error system between (1) and (5):

$$\dot{e}(t) = -Ae(t) + Bh(e(t)) + Ch(e(t - \tau(t))),$$
$$t \ge t_0 = 0, \quad t \ne t_k,$$
$$\Delta e(t_k) = e(t_k^+) - e(t_k^-) = W_k e(t_k^-), \quad t = t_k, \ k \in \mathbb{N}, \tag{6}$$
$$e(s) = \varphi(s) - \phi(s), \quad s \in [-\tau, 0],$$

where $h(e(\cdot)) = f(e(\cdot) + x(\cdot)) - f(x(\cdot))$.

The following definitions and lemmas which are useful in deriving synchronization criteria are used in the paper.

In general, the impulsive functional differential equation can be described by

$$\dot{x}(t) = f(t, x_t), \quad t \geq 0, \ t \neq t_k,$$
$$x(t_k^+) = J_k(x(t_k^-)), \quad t = t_k, \ k \in \mathbb{N}, \tag{7}$$
$$x(t) = x_0, \quad t \in [-\tau, 0],$$

where $f : [0, \infty) \times PC \to \mathbb{R}^n$ ensures that (7) has a zero solution. $J_k(x) : S_\rho \to \mathbb{R}^n$ for each $k \in \mathbb{N}^+$. For any $t \geq 0$, $x_t \in PC$ is defined by $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$. Assume that there exists a $\rho_1 > 0$ ($\rho_1 \leq \rho$) such that $x \in S(\rho_1)$ implies $J_k(x) \in S_\rho$ for all $k \in \mathbb{N}$.

Definition 2. The function $V(t, x) : [t_0, \infty) \times S_\rho \to \mathbb{R}^+$ is said to belong to class $\mathscr{V}_0$ if

(i) V is continuous in each of sets $[t_{k-1}, t_k) \times S_\rho$, $k \in \mathbb{N}$, and for each $x \in S_\rho$, $\lim_{(t,y) \to (t_k^-, x)} V(t, y) = V(t_k^-, x)$ exists;

(ii) V is locally Lipschitzian in $x \in S_\rho$ and for all $t \geq t_0, V(t, 0) \equiv 0$.

Definition 3 (see [27]). For $(t, x) \in [t_{k-1}, t_k) \times \mathbb{R}^n$, the right and upper Dini's derivatives of $V \in \mathscr{V}_0$ are defined as

$$D^+ V(t, x) \equiv \lim_{h \to 0^+} \sup \frac{1}{h} \{ V[t + h, x + hf(t, x)] - V(t, x) \}. \tag{8}$$

Lemma 4 (see [28]). *Assume that there exist $V \in \mathscr{V}_0$, $\omega_1, \omega_2 \in \mathscr{K}$, $\varphi \in \mathscr{K}^*$, and $H \in \Sigma$ such that*

(i) $\omega_1(\|x\|) \leq V(t, x) \leq \omega_2(\|x\|)$ *for* $(t, x) \in [t_0, \infty) \times S_\rho$;

(ii) *for all $x \in S_{\rho_1}$, $0 < \rho_1 \leq \rho$, and $k \in \mathbb{N}$, $V(t_k, J_k(x)) \leq \psi(V(t_k^-, x))$;*

(iii) *for any solution $x(t)$ of (7), $V(t + s, x(t + s)) \leq \psi^{-1}(V(t, x(t)))$, $-\tau \leq s \leq 0$ implies that $D^+(V(t, x(t))) \leq g(t)H(V(t, x(t)))$, where $g : [t_0, \infty) \to \mathbb{R}^+$ is locally integrable, ψ^{-1} is the inverse function of ψ;*

(iv) *H is nondecreasing and there exist constants $\lambda_2 \geq \lambda_1 > 0$ and $A > 0$ such that for all $k \in \mathbb{N}$ and $\mu > 0$, $\lambda_1 \leq t_k - t_{k-1} \leq \lambda_2$ and $\int_{\psi(\mu)}^\mu (ds/H(s)) - \int_{t_{k-1}}^{t_k} g(s)ds \geq A$;*

then the zero solution of (7) is uniformly asymptotically stable.

Lemma 5 (see [29]). *For any symmetric and positive definite matrix $P \in \mathbb{R}^{n \times n}$, the following inequality holds:*

$$\lambda_m(P) x^T x \leq x^T P x \leq \lambda_M(P) x^T x, \quad \forall x \in \mathbb{R}^n. \tag{9}$$

Lemma 6 (see [30]). *If X, Y are real matrices with appropriate dimensions, then there exists a number $\varepsilon > 0$ such that*

$$X^T Y + Y^T X \leq \varepsilon X^T X + \frac{1}{\varepsilon} Y^T Y. \tag{10}$$

3. Main Results

In this section, we use the Lyapunov-like function

$$V(t, e(t)) = e^T(t) P e(t) \tag{11}$$

to derive the asymptotically stability conditions of the zero solution of the error system (6), which implies that the drive system (1) and the response system (5) can be asymptotically synchronized.

Theorem 7. *Assume that assumption (H1) holds. If there exist three $n \times n$ symmetric and positive definite matrices P, Q_1, Q_2, seven constants $0 < d < 1$, $\theta_1, \theta_2, \theta_5, \theta_6 > 0$, $0 < \theta_3, \theta_4 < 2$ such that the following inequalities hold:*

$$\Gamma \leq 0, \qquad \Omega_2 - \theta_5 P \leq 0, \qquad \Omega_3 - dP \leq 0,$$
$$\ln d + \theta_6 \sup \{ t_k - t_{k-1} \} < 0, \quad k \in \mathbb{N}, \tag{12}$$

where

$$\Gamma = \begin{bmatrix} \Omega_1 & PB & PC \\ \star & -\theta_1 I & 0 \\ \star & \star & -\theta_2 I \end{bmatrix}, \qquad \theta_5 P \leq Q_1, \qquad \theta_6 P \geq Q_2,$$

$$\Omega_1 = -A^T P - PA + \frac{\theta_1}{2\theta_3 - \theta_3^2} L_1 L_1^T + \frac{Q_1}{d} - Q_2 - \frac{2\theta_1}{2 - \theta_3} L_2,$$

$$\Omega_2 = \frac{\theta_2}{2\theta_4 - \theta_4^2} L_1 L_1^T - \frac{2\theta_2}{2 - \theta_4} L_2,$$

$$\Omega_3 = (I + W_k)^T P (I + W_k), \tag{13}$$

then the systems (1) and (5) are asymptotically synchronized.

Proof. From (11) and Lemma 5, we get

$$\lambda_m(P) e^T(t) e(t) \leq V(t, e(t)) \leq \lambda_M(P) e^T(t) e(t). \tag{14}$$

Let $\omega_1(\|e(t)\|) = \lambda_m(P) e^T(t) e(t)$, $\omega_2(\|e(t)\|) = \lambda_M(P) e^T(t) e(t)$, and then $\omega_1, \omega_2 \in \mathscr{K}$.

For all $e \in S_{\rho_1}$, $0 < \rho_1 \leq \rho$, $k \in \mathbb{N}$,

$$V(t_k, e(t_k)) = e^T(t_k) P e(t_k)$$
$$= e^T(t_k^-)(I + W_k)^T P (I + W_k) e(t_k^-)$$
$$= e^T(t_k^-) \left[(I + W_k)^T P (I + W_k) - dP \right] e(t_k^-)$$
$$\quad + de^T(t_k^-) P e(t_k^-)$$
$$= e^T(t_k^-)(\Omega_3 - dP) e(t_k^-) + dV(t_k^-, e(t_k^-))$$
$$\leq dV(t_k^-, e(t_k^-)). \tag{15}$$

Let $\psi(s) = ds$, and then $\psi \in \mathscr{K}^*$.

For any solution of (6), if

$$V(t + s, e(t + s)) \leq \psi^{-1}(V(t, e(t))), \quad \forall s \in [-\tau, 0], \tag{16}$$

that is,

$$e^T(t+s)Pe(t+s) \leq \frac{1}{d}e^T(t)Pe(t), \quad \forall s \in [-\tau, 0], \quad (17)$$

specially, for $s = -\tau(t)$, we have

$$e^T(t-\tau(t))Pe(t-\tau(t)) \leq \frac{1}{d}e^T(t)Pe(t). \quad (18)$$

For $t \in [t_{k-1}, t_k)$, $k \in \mathbb{N}$, the right and upper Dini's derivatives of $V(t, e(t))$ along the trajectory of system (6) are obtained as follows:

$$D^+V(t, e(t))$$

$$= e^T(t)\left(-A^T P - PA\right)e(t)$$

$$+ \left(e^T(t)PBh(e(t)) + h^T(e(t))B^T Pe(t)\right)$$

$$+ \left(e^T(t)PCh(e(t-\tau(t))) + h^T(e(t-\tau(t)))C^T Pe(t)\right). \quad (19)$$

It follows from Lemma 6 that

$$e^T(t)PBh(e(t)) + h^T(e(t))B^T Pe(t)$$
$$\leq \frac{1}{\theta_1}e^T(t)PBB^T Pe(t) + \theta_1 h^T(e(t))h(e(t)), \quad (20)$$

$$e^T(t)PCh(e(t-\tau(t))) + h^T(e(t-\tau(t)))C^T Pe(t)$$
$$\leq \frac{1}{\theta_2}e^T(t)PCC^T Pe(t) + \theta_2 h^T(e(t-\tau(t)))h(e(t-\tau(t))). \quad (21)$$

By assumption (H1), it is easy to see that

$$0 \leq \sum_{i=1}^n \left(l_i^+ e_i(t) - h_i(e_i(t))\right)\left(h_i(e_i(t)) - l_i^- e_i(t)\right)$$

$$= e^T(t)L_1 h(e(t)) - h^T(e(t))h(e(t)) - e^T(t)L_2 e(t), \quad (22)$$

$$0 \leq \sum_{i=1}^n \left(l_i^+ e_i(t-\tau(t)) - h_i(e_i(t-\tau(t)))\right)$$

$$\times \left(h_i(e_i(t-\tau(t))) - l_i^- e_i(t-\tau(t))\right)$$

$$= e^T(t-\tau(t))L_1 h(e(t-\tau(t)))$$

$$- h^T(e(t-\tau(t)))h(e(t-\tau(t)))$$

$$- e^T(t-\tau(t))L_2 e(t-\tau(t)). \quad (23)$$

Combining (20) and (22) together yields

$$e^T(t)PBh(e(t)) + h^T(e(t))B^T Pe(t)$$

$$\leq \frac{1}{\theta_1}e^T(t)PBB^T Pe(t)$$

$$+ \theta_1 \left(e^T(t)L_1 h(e(t)) - e^T(t)L_2 e(t)\right) \quad (24)$$

$$= e^T(t)\left(\frac{1}{\theta_1}PBB^T P - \theta_1 L_2\right)e(t)$$

$$+ \theta_1 e^T(t)L_1 h(e(t)).$$

From (21) and (23), we get

$$e^T(t)PCh(e(t-\tau(t))) + h^T(e(t-\tau(t)))C^T Pe(t)$$

$$\leq \frac{1}{\theta_2}e^T(t)PCC^T Pe(t)$$

$$+ \theta_2 \left(e^T(t-\tau(t))L_1 h(e(t-\tau(t)))\right.$$

$$\left. - e^T(t-\tau(t))L_2 e(t-\tau(t))\right) \quad (25)$$

$$= e^T(t)\left(\frac{1}{\theta_2}PCC^T P\right)e(t)$$

$$- e^T(t-\tau(t))\left(\theta_2 L_2\right)e(t-\tau(t))$$

$$+ \theta_2 e^T(t-\tau(t))L_1 h(e(t-\tau(t))).$$

Note that

$$e^T(t)L_1 h(e(t))$$

$$\leq \frac{1}{2\theta_3}e^T(t)L_1 L_1^T e(t) + \frac{\theta_3}{2}h^T(e(t))h(e(t))$$

$$\leq \frac{1}{2\theta_3}e^T(t)L_1 L_1^T e(t)$$

$$+ \frac{\theta_3}{2}\left(e^T(t)L_1 h(e(t)) - e^T(t)L_2 e(t)\right),$$

$$e^T(t-\tau(t))L_1 h(e(t-\tau(t)))$$

$$\leq \frac{1}{2\theta_4}e^T(t-\tau(t))L_1 L_1^T e(t-\tau(t))$$

$$+ \frac{\theta_4}{2}h^T(e(t-\tau(t)))h(e(t-\tau(t)))$$

$$\leq \frac{\theta_4}{2}\left(e^T(t-\tau(t))L_1 h(e(t-\tau(t)))\right.$$

$$\left. - e^T(t-\tau(t))L_2 e(t-\tau(t))\right)$$

$$+ \frac{1}{2\theta_4}e^T(t-\tau(t))L_1 L_1^T e(t-\tau(t)). \quad (26)$$

From (26), we can, namely, obtain

$$e^T(t) L_1 h(e(t)) \leq e^T(t) \left(\frac{1}{2\theta_3 - \theta_3^2} L_1 L_1^T - \frac{\theta_3}{2 - \theta_3} L_2 \right) e(t), \tag{27}$$

$$e^T(t - \tau(t)) L_1 h(e(t - \tau(t)))$$
$$\leq e^T(t - \tau(t)) \left(\frac{1}{2\theta_4 - \theta_4^2} L_1 L_1^T - \frac{\theta_4}{2 - \theta_4} L_2 \right) e(t - \tau(t)). \tag{28}$$

Introducing (24), (25), (27), and (28) to (19), from condition of Theorem 7 and Schur complement [31], we obtain

$$D^+ V(t, e(t))$$
$$\leq e^T(t) \left(-A^T P - PA + \frac{1}{\theta_1} PBB^T P \right.$$
$$\left. + \frac{1}{\theta_2} PCC^T P - \theta_1 L_2 \right) e(t)$$
$$+ e^T(t) \left(\frac{\theta_1}{2\theta_3 - \theta_3^2} L_1 L_1^T - \frac{\theta_1 \theta_3}{2 - \theta_3} L_2 \right) e(t)$$
$$+ e^T(t - \tau(t)) \left(\frac{\theta_2}{2\theta_4 - \theta_4^2} L_1 L_1^T - \frac{\theta_2 \theta_4}{2 - \theta_4} L_2 \right)$$
$$\times e(t - \tau(t))$$
$$- e^T(t - \tau(t)) (\theta_2 L_2) e(t - \tau(t))$$
$$= e^T(t) \left(-A^T P - PA + \frac{1}{\theta_1} PBB^T P + \frac{1}{\theta_2} PCC^T P \right.$$
$$\left. + \frac{\theta_1}{2\theta_3 - \theta_3^2} L_1 L_1^T - \frac{2\theta_1}{2 - \theta_3} L_2 \right) e(t)$$
$$+ e^T(t - \tau(t)) \left(\frac{\theta_2}{2\theta_4 - \theta_4^2} L_1 L_1^T - \frac{2\theta_2}{2 - \theta_4} L_2 \right) e(t - \tau(t))$$
$$\leq e^T(t) \left(-A^T P - PA + \frac{1}{\theta_1} PBB^T P \right.$$
$$\left. + \frac{1}{\theta_2} PCC^T P + \frac{\theta_1}{2\theta_3 - \theta_3^2} L_1 L_1^T - \frac{2\theta_1}{2 - \theta_3} L_2 \right) e(t)$$
$$+ e^T(t - \tau(t)) (\theta_5 P) e(t - \tau(t))$$
$$\leq e^T(t) \left(-A^T P - PA + \frac{1}{\theta_1} PBB^T P \right.$$
$$+ \frac{1}{\theta_2} PCC^T P + \frac{\theta_1}{2\theta_3 - \theta_3^2} L_1 L_1^T$$
$$\left. - \frac{2\theta_1}{2 - \theta_3} L_2 + \left(\frac{\theta_5}{d} - \theta_6 \right) P \right) e(t)$$
$$+ \theta_6 e^T(t) Pe(t)$$

$$\leq e^T(t) \left(-A^T P - PA + \frac{1}{\theta_1} PBB^T P + \frac{1}{\theta_2} PCC^T P \right.$$
$$+ \frac{\theta_1}{2\theta_3 - \theta_3^2} L_1 L_1^T - \frac{2\theta_1}{2 - \theta_3} L_2 + \frac{Q_1}{d} - Q_2 \right) e(t)$$
$$+ \theta_6 e^T(t) Pe(t)$$
$$\leq \theta_6 V(t, e(t)). \tag{29}$$

Let $g(s) \equiv 1$ and $H(s) = \theta_6 s$. Then

$$\int_{\varphi(\mu)}^{\mu} \frac{ds}{H(s)} - \int_{t_{k-1}}^{t_k} g(s) \, ds = \int_{d\mu}^{\mu} \frac{ds}{\theta_6 s} - (t_k - t_{k-1})$$
$$= -\frac{\ln d}{\theta_6} - (t_k - t_{k-1})$$
$$> -\frac{\ln d}{\theta_6} - \sup \{t_k - t_{k-1}\} > 0. \tag{30}$$

From Lemma 4, the zero solution of system (6) is asymptotically stable. Thus, system (1) and system (5) are synchronized. The proof of Theorem 7 is completed. □

Remark 8. The stable impulsive interval is associated with the impulsive matrix W_k and the choice of parameter θ_6. In order to reduce the man-made misleading during the prediction of stable impulsive interval, here only fix parameter θ_3, θ_4 and the impulsive matrix W_k, and then let $\theta_5 P \leq Q_1, \theta_6 P \geq Q_2$. Firstly, P, Q_1, Q_2 can be obtained by Theorem 7 via MATLAB LMI toolbox. Finally, θ_5, θ_6 can be calculated by following algebraic equations:

$$\theta_5 = \sqrt[n]{\frac{\det(Q_1)}{\det(P)}}, \qquad \theta_6 = \sqrt[n]{\frac{\det(Q_2)}{\det(P)}}, \tag{31}$$

or linear matrix inequalities:

$$\theta_5 P < Q_1, \qquad \theta_6 P > Q_2. \tag{32}$$

In [22, 25], the parameters β_4 and r which are corresponding to parameter θ_6 had been selected subjectively, which may cause result to lack fidelity. In addition, parameters θ_3, θ_4 are selectable variables which can increase the flexibility of the possible outcomes. Therefore, our results is more objective in some situations.

From the control point of view, in order to obtain the synchronization between the drive system (1) and the controlled response system (5), we design (t_k, W_k), $k \in \mathbb{N}$ as the impulsive controller [32]. Let $W_k = bI$ in Theorem 7, the following corollary holds.

Corollary 9. *Assume that assumption (H1) holds. If there exist three $n \times n$ symmetric and positive definite matrices P, Q_1', Q_2,*
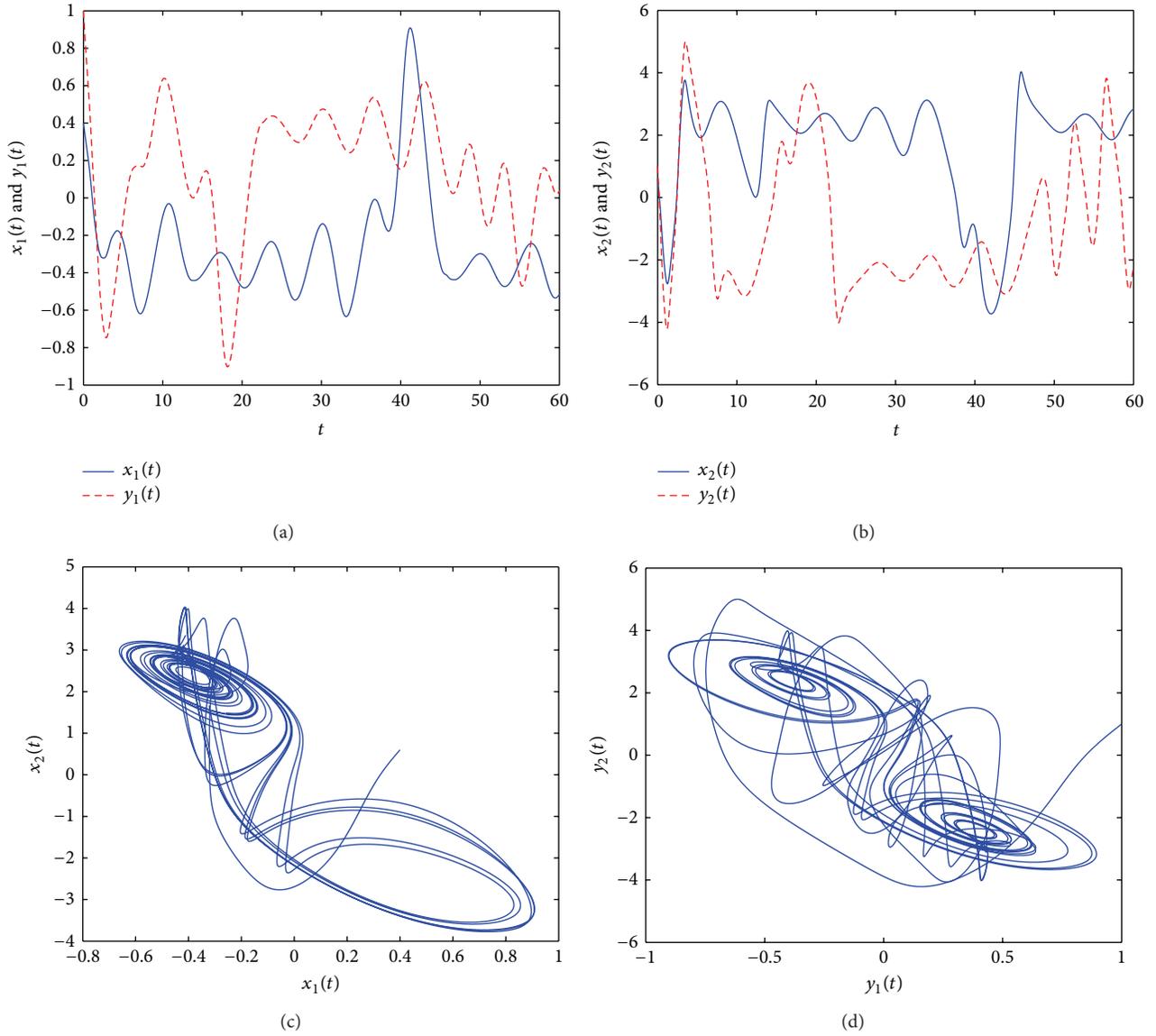
(a)

(b)

(c)

(d)

FIGURE 1: Time response of state variables and the phase plots of the drive system and response system in **Example 12** without impulsive control.

seven constants $-2 < b < 0$ $(b \neq -1)$, $\theta_1, \theta_2, \theta_5, \theta_6 > 0$, $0 < \theta_3, \theta_4 < 2$ *such that the following inequalities hold:*

$$\Gamma \leq 0,$$

$$\sup \{t_k - t_{k-1}\} < -\frac{2\ln|1+b|}{\theta_6}, \quad k \in \mathbb{N}, \tag{33}$$

where

$$\Gamma = \begin{bmatrix} \Omega_1' & PB & PC & 0 & 0 & 0 \\ \star & -\theta_1 I & 0 & 0 & 0 & 0 \\ \star & \star & -\theta_2 I & 0 & 0 & 0 \\ \star & \star & \star & \Omega_2 - \theta_5 P & 0 & 0 \\ \star & \star & \star & \star & \theta_5 P - Q_1 & 0 \\ \star & \star & \star & \star & \star & Q_2 - \theta_6 P \end{bmatrix},$$

$$\Omega_1' = -A^T P - PA + \frac{\theta_1}{2\theta_3 - \theta_3^2} L_1 L_1^T + \frac{Q_1}{(1+b)^2}$$

$$- Q_2 - \frac{2\theta_1}{2 - \theta_3} L_2,$$

$$\Omega_2 = \frac{\theta_2}{2\theta_4 - \theta_4^2} L_1 L_1^T - \frac{2\theta_2}{2 - \theta_4} L_2, \tag{34}$$

then the systems (1) *and* (5) *are asymptotically synchronized.*

In particular, when $l_i^- = -l_i^+ < 0$ in (H1), the following corollary holds.

Corollary 10. *Consider system* (6) *satisfying assumption* (H1) *with* $l_i^- = -l_i^+ < 0$. *If there exist three* $n \times n$ *symmetric and positive definite matrices* P, Q_1, Q_2, *seven constants*
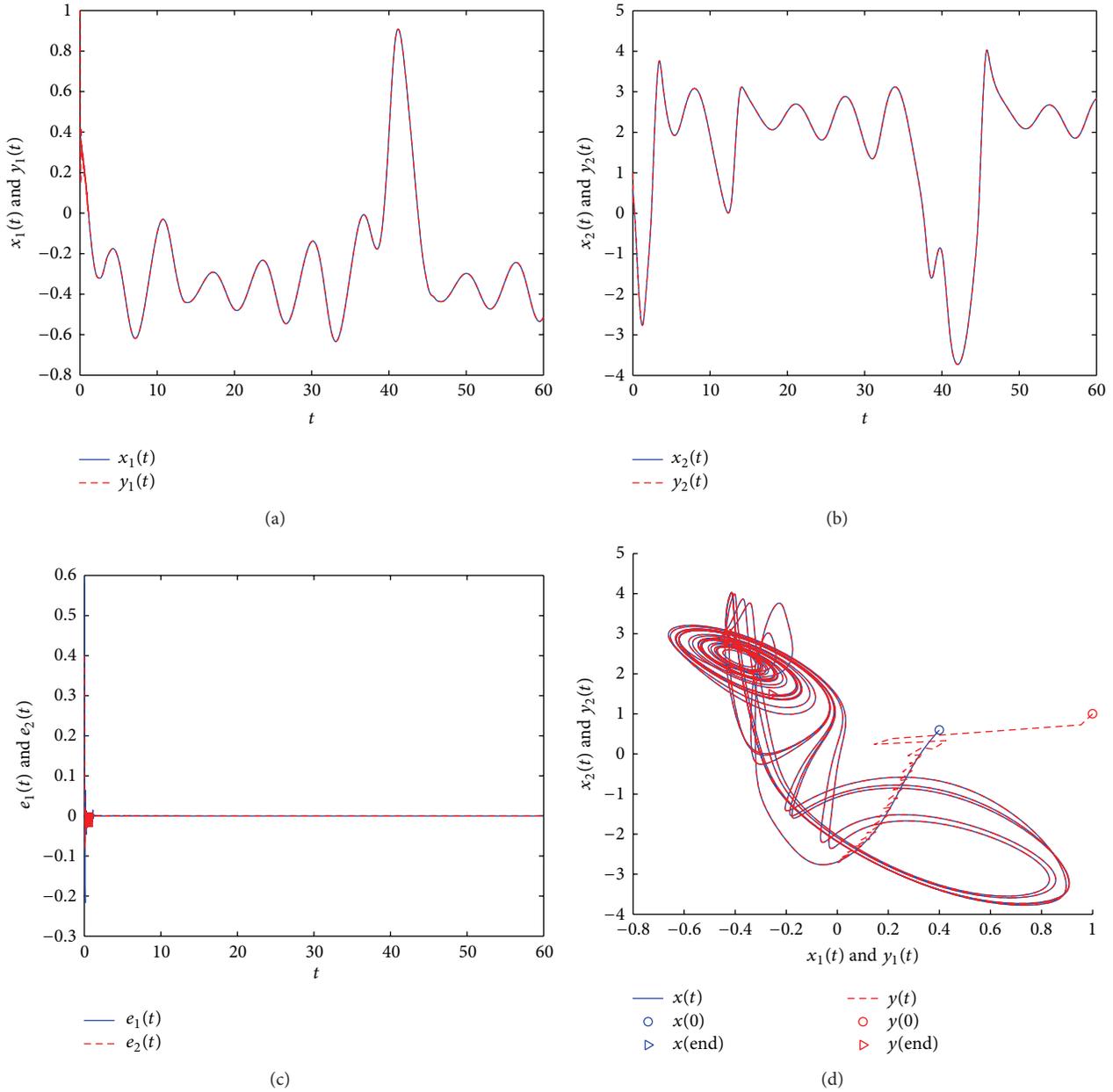
FIGURE 2: Time response of state variables, the synchronization errors, and the phase plots of the drive system and response system in Example 12 under the impulsive controller $(0.06k, -1.3I)$.

$-2 < b < 0$ $(b \neq -1)$, $\theta_1, \theta_2, \theta_5, \theta_6 > 0$, $0 < \theta_3, \theta_4 < 2$ *such that the following inequalities hold:*

$$\Gamma \le 0,$$

$$\sup \{t_k - t_{k-1}\} < -\frac{2 \ln |1 + b|}{\theta_6}, \quad k \in \mathbb{N}, \tag{35}$$

where

$$\Gamma = \begin{bmatrix} \Omega_1'' & PB & PC & 0 & 0 & 0 \\ \star & -\theta_1 I & 0 & 0 & 0 & 0 \\ \star & \star & -\theta_2 I & 0 & 0 & 0 \\ \star & \star & \star & \Omega_2' - \theta_5 P & 0 & 0 \\ \star & \star & \star & \star & \theta_5 P - Q_1 & 0 \\ \star & \star & \star & \star & \star & Q_2 - \theta_6 P \end{bmatrix},$$

$$\Omega_1'' = -A^T P - PA + \frac{Q_1}{(1+b)^2} - Q_2 - \frac{2\theta_1}{2-\theta_3} L_2,$$

$$\Omega_2' = -\frac{2\theta_2}{2-\theta_4} L_2, \tag{36}$$

then the systems (1) *and* (5) *are asymptotically synchronized.*

Remark 11. The sufficient conditions in Theorem 7 and Corollaries 9 and 10 are all independent of the delay parameter but rely on the maximum impulsive interval $\sup\{t_k - t_{k-1}\}$ and the impulsive matrix W_k, which plays a fundamental role when the size of the delay is unknown.
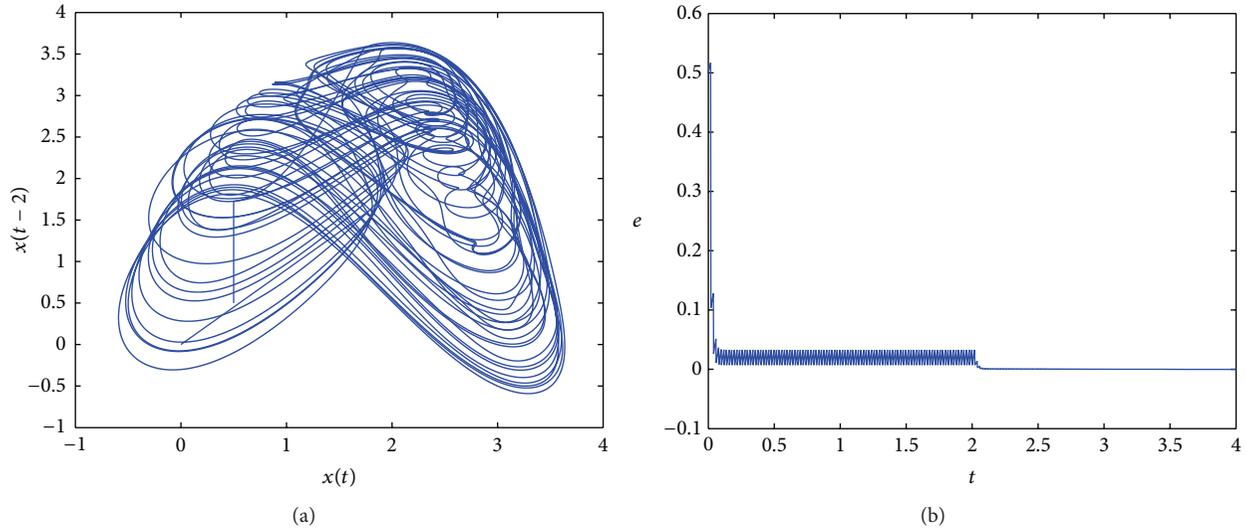
(a)

(b)

FIGURE 3: Chaotic attractor of Ikeda-type neural network in Example 14 and the synchronization error of the drive system and response system in Example 14.

TABLE 1: When the impulsive matrix W_k is given, $\sup_k\{t_k - t_{k-1}\}, k \in \mathbb{N}$ corresponding to different (θ_3, θ_4) and algorithms.

(W_k, Algorithms)	(θ_3, θ_4)				
	(0.1, 0.1)	(0.3, 0.3)	(0.5, 0.5)	(0.7, 0.7)	(0.9, 0.9)
($W_k = -1.1I$, (31))	0.0202	0.0288	0.0305	0.0356	0.0369
($W_k = -1.1I$, (32))	$2.8349e - 4$	$1.0980e - 4$	$5.3202e - 4$	$7.2925e - 4$	$8.9236e - 4$
($W_k = -1.3I$, (31))	0.0310	0.0401	0.0490	0.0718	0.0701
($W_k = -1.3I$, (32))	0.0025	0.0163	0.0200	0.0259	0.0259
($W_k = -1.5I$, (31))	0.0175	0.0266	0.0335	0.0483	0.0516
($W_k = -1.5I$, (32))	0.0058	0.0094	0.0122	0.0023	0.0022
($W_k = -1.7I$, (31))	0.0088	0.0168	0.0213	0.0245	0.0261
($W_k = -1.7I$, (32))	0.0027	0.0020	$9.4549e - 04$	0.0056	0.0059
($W_k = -1.9I$, (31))	0.0029	0.0057	0.0074	0.0093	0.0089
($W_k = -1.9I$, (32))	$2.7931e - 04$	0.0011	0.0015	0.0019	0.0016

4. Numerical Results

In order to illustrate the feasibility of our above-established criteria in the preceding sections, we provide two concrete examples. Throughout the simulations, we use the IMEX implicit Euler method.

Example 12. Consider a two-dimensional chaotic delayed neural networks as the drive system (1), where the initial condition $\phi(s) = (0.6, 0.4)^T$, $s \in [-\tau, 0]$, $f(x) = \tanh(x)$, $\tau(t) = 1$, $J = (0,0)^T$; then $L_1 = (1,1)^T$, $L_2 = (0,0)^T$, and the parameter matrices A, B, C are given as follows:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2.0 & -0.1 \\ -5.0 & 3 \end{pmatrix},$$

$$C = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -2.5 \end{pmatrix}. \tag{37}$$

The corresponding response system is designed as (5), where the initial condition $\varphi(s) = (1,1)^T$, $s \in [-\tau, 0]$. From [33], we

know that the system (1) of Example 12 has a chaotic attractor which can be seen from Figure 1. Figures 1(a)–1(d) show the time response of state variables and the phase plots of the drive system and response system in Example 12 without impulsive control.

By choosing $\theta_3 = \theta_4 = 0.7$, $b = -1.3$ and then using MATLAB LMI toolbox and (31), we can obtain the following feasible solutions to LMIs and θ_5, θ_6 in Corollary 9:

$$P = \begin{pmatrix} 0.0339 & 0.0022 \\ 0.0022 & 0.0111 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 0.0255 & -0.0004 \\ -0.0004 & 0.0266 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0.6427 & -0.0023 \\ -0.0023 & 0.6488 \end{pmatrix},$$

$$\theta_1 = 0.2070, \qquad \theta_2 = 0.0175, \qquad \theta_5 = 1.3526,$$

$$\theta_6 = 33.5145. \tag{38}$$

Thus, from Corollary 9 we can calculate that the maximum impulsive interval satisfies $\sup\{t_k - t_{k-1}\} < 0.0718$.

With the same initial conditions as given above and $t_k - t_{k-1} = 0.06$, Figures 2(a)–2(d) depict the time response of state variables, the synchronization errors, and the phase plots of the drive system and response system in Example 12 under the impulsive controller $(0.06k, -1.3I)$.

Remark 13. In Table 1, we list $\sup_k\{t_k - t_{k-1}\}$ corresponding to different (θ_3, θ_4) and algorithms when the impulsive matrix W_k is given. For each (θ_3, θ_4), the $(\theta_6, \sup_k\{t_k - t_{k-1}\})$ is obtained by solving the LMIs in Corollary 9 and (31) or (32). From the results in Table 1, we can see that the obtained maximum stable impulsive interval $\sup_k\{t_k - t_{k-1}\}$ by using (31) is better than (32) when the impulsive matrix W_k is given. And when θ_3, θ_4 are chosen in interval $[1, 2)$, we will have the same results which are chosen as $2 - \theta_3, 2 - \theta_4$.

Example 14. Consider the Ikeda-type neural network [34] as the drive system (1); when $A = 1.0, B = 0, C = 4.0, f(x) = \sin(x), \tau(t) = 2$, the system (1) exhibits chaotic behavior (see Figure 3(a)). And the initial condition $\phi(s) = 0.5, \ s \in [-\tau, 0]$. It is easy to obtain that $L_1 = 0, \ L_2 = -1$.

The corresponding response system is designed as (5), where the initial condition $\varphi(s) = 1, \ s \in [-\tau, 0]$.

By choosing $\theta_3 = \theta_4 = 0.5, \ b = -0.8$ and then using MATLAB LMI toolbox and (31), we can obtain the following feasible solutions to LMIs and θ_5, θ_6 in Corollary 10:

$$P = 0.0068, \quad Q_1 = 0.0161, \quad Q_2 = 0.8861,$$

$$\theta_1 = 0.2283, \quad \theta_2 = 0.0083, \quad \theta_5 = 2.3538, \quad \theta_6 = 129.6849.$$
$$(39)$$

Hence, the designed impulsive controller is

$$W_k = -0.8, \qquad \sup\{t_k - t_{k-1}\} < 0.0248. \qquad (40)$$

With the same initial conditions as given above, the simulations of the synchronization error in Example 14 under the impulsive controller $(0.02k, -0.8)$ are shown in Figure 3(b).

5. Conclusion

In this paper, the impulsive synchronization problem of chaotic delayed neural networks has been investigated. Some new criterions which ensure that the coupled chaotic delayed neural networks can be asymptotically synchronized have been derived in terms of linear matrix inequalities (LMIs) by using Lyapunov stability theorem, impulsive control theory, and linear matrix inequality (LMI) technique. The desired impulsive controller which is with respect to the stable impulsive interval and the impulsive matrix is established, its existence can be verified effectively by the simulations. It is worthwhile to mention that the positive constants set (θ_3, θ_4) can increase the flexibility for the design of the impulsive controller. Moreover, the stable impulsive interval can be calculated combining MATLAB LMI toolbox and one of the two given equations objectively.

Finally, two illustrative examples are given to show the applicability and usefulness of the proposed results.

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