*Research Article*

# Fractional Killing-Yano Tensors and Killing Vectors Using the Caputo Derivative in Some One- and Two-Dimensional Curved Space

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The classical free Lagrangian admitting a constant of motion, in one- and two-dimensional space, is generalized using the Caputo derivative of fractional calculus. The corresponding metric is obtained and the fractional Christoffel symbols, Killing vectors, and Killing-Yano tensors are derived. Some exact solutions of these quantities are reported.

## 1. Introduction

The tool of the fractional calculus started to be successfully applied in many fields of science and engineering (see, e.g., [1–12] and the references therein). Fractals and its connection to local fractional vector calculus represents another interesting field of application (see, e.g., [13, 14] and the references therein). Several definitions of the fractional differentiation and integration exist in the literature. The most commonly used are the Riemann-Liouville and the Caputo derivatives. The Riemann-Liouville derivative of a constant is not zero while Caputo's derivative of a constant is zero. This property makes the Caputo definition more suitable in all problems involving the fractional differential geometry [15, 16]. The Caputo differential operator of fractional calculus is defined as [1–8]

$$
{}_a D_x^\alpha f(x) \equiv \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \\ \times \displaystyle\int_a^x (x-u)^{n-\alpha-1} \dfrac{d^n f(u)}{du^n} du, & n-1 < \alpha < n \\ \dfrac{d^n}{dx^n} f(x)\,, & \alpha = n, \end{cases}
$$

(1)

where $\Gamma(\cdot)$ is the Gamma function and $x > a$. In this work, we consider the case $a = 0, n-1 < \alpha \le n$. For the power function $x^p$, $p \in R$, the Caputo fractional derivative satisfies

$$
D_x^\alpha x^p = \begin{cases} \dfrac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha} \\ 0, & p = 0,1,2,\ldots,n-1. \end{cases}
$$

(2)

The role played by Killing and Killing-Yano tensors for the geodesic motion of the particle and the superparticle in a curved background was a topic subjected to an intense debate during the last decades [17–26]. In [27] a generalization of exterior calculus was presented. Besides, the quadratic Lagrangians are introduced by adding surface terms to a free-particle Lagrangian in [28].

Motivated by the above mentioned results in differential geometry, we discuss in this paper the hidden symmetries corresponding to the fractional Killing vectors and Killing-Yano tensors on curved spaces deeply related to physical systems.

The Caputo partial differential operator of fractional order $\alpha$ is defined as

$$
\begin{aligned}
&_a\partial_x^\alpha f(x,y) \\
&\equiv \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \\ \times \displaystyle\int_a^x (x-u)^{n-\alpha-1} \dfrac{\partial^n f(u,y)}{\partial u^n} du, & n-1 < \alpha < n \\ \dfrac{\partial^n}{\partial x^n} f(x,y). & \alpha = n \end{cases}
\end{aligned}
\tag{3}
$$

Again in this work we consider the case $a = 0$, $n-1 < \alpha \le n$, and we drop the term $a$ in the notation.

## 2. The Main Results

In the following, we present the Killing vectors and Killing-Yano tensors corresponding to some curved spaces with some physical significance.

*2.1. One-Dimensional Case.* Consider the one-dimensional free Lagrangian, admitting a constant of motion; that is, momentum [28]

$$
L = \frac{1}{2}\dot{x}^2 + \lambda_2 \dot{x}.
\tag{4}
$$

The Lagrangian can be rewritten as

$$
L = \frac{1}{2} g_{ij} \dot{u}^i \dot{u}^j,
\tag{5}
$$

where $g_{ij} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The fractional Lagrangian of order $q$ is given by

$$
L = \frac{1}{2} g_{ij} D_t^q u^i D_t^q u^j,
\tag{6}
$$

where we consider the Caputo fractional derivative.

We generalize the Christoffel symbols in the fractional case, of order $n-1 < q < n$, as

$$
{}^q\Gamma_{\beta\mu}^\gamma = \frac{1}{2} g^{\alpha\gamma} \left( \partial_\mu^q g_{\alpha\beta} + \partial_\beta^q g_{\alpha\mu} - \partial_\gamma^q g_{\beta\mu} \right),
\tag{7}
$$

where the partial derivatives of order $q$ are defined in the fractional case.

We notice that because the metric is constant, all the Christoffel symbols vanish,

$$
{}^q\Gamma_{\mu\nu}^\gamma = 0.
\tag{8}
$$

*2.1.1. Fractional Killing Vectors and Killing-Yano Tensors.* The Killing vectors can be calculated from the generalized equations, namely,

$$
V_{\alpha;\beta}^q + V_{\beta;\alpha}^q = 0,
\tag{9}
$$

where $V_{\alpha;\beta}^q$ is the fractional covariant derivative defined as

$$
V_{\alpha;\beta}^q = \partial_\beta^q V_\alpha + g_{\alpha\mu}{}^q\Gamma_{\delta\beta}^\mu g^{\delta\lambda} V_\lambda.
\tag{10}
$$

Because all the Christoffel symbols vanish, it is easy to show that

$$
\begin{aligned}
V_{1;1}^q &= \partial_1^q V_1 = 0, \\
V_{2;2}^q &= \partial_2^q V_2 = 0, \\
V_{1;2}^q + V_{2;1}^q &= V_{1,2}^q + V_{2,1}^q = \partial_2^q V_1 + \partial_1^q V_2 = 0,
\end{aligned}
\tag{11}
$$

For $0 < q \le 1$, a solution of the above equations is $V_1 = -cy^q$, $V_2 = cx^q$, where $c$ is a constant. While for $q > 1$, we have the general solution

$$
\begin{aligned}
V_1 &= -cy^q + \sum_{k=0}^{n-1} \left( a_k x^k + b_k y^k \right), \\
V_2 &= cx^q + \sum_{k=0}^{n-1} \left( a_k' x^k + b_k' y^k \right),
\end{aligned}
\tag{12}
$$

where $c$, $a_k$, $b_k$, $a_k'$, $b_k'$ are constants.

The fractional Killing-Yano antisymmetric tensor ${}^q f_{\mu\nu}$ can be calculated using the condition

$$
{}^q f_{\mu\nu;\lambda} + {}^q f_{\lambda\nu;\mu} = 0,
\tag{13}
$$

where ${}^q f_{\mu\nu;\lambda}$ is the fractional covariant derivative of the Killing-Yano tensor ${}^q f_{\mu\nu}$ defined as

$$
{}^q f_{\mu\nu;\lambda} = \partial_\lambda^q f_{\mu\nu} - f_{\alpha\nu}{}^q\Gamma_{\lambda\mu}^\alpha - f_{\mu\alpha}{}^q\Gamma_{\lambda\nu}^\alpha.
\tag{14}
$$

We find that

$$
\partial_\lambda^q f_{\mu\nu} = 0
\tag{15}
$$

for all values of $\lambda$, $\nu$, $\mu$. A solution is $f_{11} = f_{22} = 0$ and $f_{12} = c = -f_{21}$, where $c$ is a constant and for $0 < q \le 1$. While for $q > 1$, that is, $n \ge 2$, we have the general solution

$$
f_{12} = -f_{21} = \sum_{k=0}^{n-1} \left( a_k x^k + b_k y^k \right),
\tag{16}
$$

where $a_k$, $b_k$ are constants.

*2.2. Two-Dimensional Case.* Below we consider the classical free Lagrangian, in two dimensions, admitting a constant of motion; that is, angular momentum [28]

$$
L = \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 \right) + \lambda_3 \left( x\dot{y} - y\dot{x} \right).
\tag{17}
$$

The fractional Lagrangian is given by

$$
L = \frac{1}{2} g_{ij} D^\alpha q^i D^\alpha q^j,
\tag{18}
$$

where $g_{ij}$ is given by

$$
g_{ij} = \begin{bmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{bmatrix}.
\tag{19}
$$

The inverse matrix of the metric is

$$g^{ij} = \frac{1}{x^2+y^2}\begin{bmatrix} x^2 & xy & -y \\ xy & y^2 & x \\ -y & x & -1 \end{bmatrix}. \quad (20)$$

We generalize the Christoffel symbols in the fractional case, of order $n-1 < q < n$, as

$$^q\Gamma^\gamma_{\beta\mu} = \frac{1}{2}g^{\alpha\gamma}\left(\partial^q_\mu g_{\alpha\beta} + \partial^q_\beta g_{\alpha\mu} - \partial^q_\gamma g_{\beta\mu}\right). \quad (21)$$

One can show that

$$^q\Gamma^\gamma_{\mu\mu} = 0 \quad (22)$$

for $\gamma, \mu = 1, 2, 3$, while

$$^q\Gamma^\gamma_{12} = \frac{g^{3\gamma}}{2}\left(\partial^q_1 g_{32} + \partial^q_2 g_{31}\right),$$
$$^q\Gamma^\gamma_{13} = \frac{g^{2\gamma}}{2}\left(\partial^q_1 g_{32} - \partial^q_2 g_{31}\right), \quad (23)$$
$$^q\Gamma^\gamma_{23} = \frac{g^{1\gamma}}{2}\left(\partial^q_2 g_{13} + \partial^q_1 g_{23}\right).$$

*2.2.1. Fractional Killing Vectors.* The Killing vectors can be calculated from the generalized equations

$$V^q_{\alpha;\beta} + V^q_{\beta;\alpha} = 0, \quad (24)$$

where $V^q_{\alpha;\beta}$ is the fractional covariant derivative defined as

$$V^q_{\alpha;\beta} = \partial^q_\beta V_\alpha + g_{\alpha\mu}\,^q\Gamma^\mu_{\delta\beta}g^{\delta\lambda}V_\lambda. \quad (25)$$

It is easy to show that

$$V^q_{1;1} = \partial^q_1 V_1 = 0,$$
$$V^q_{2;2} = \partial^q_2 V_2 = 0,$$
$$V^q_{3;3} = \partial^q_3 V_3 = 0,$$
$$V^q_{1;2} + V^q_{2;1} = V^q_{1,2} + V^q_{2,1} = \partial^q_2 V_1 + \partial^q_1 V_2 = 0, \quad (26)$$
$$V^q_{1;3} + V^q_{3;1} = \partial^q_3 V_1 + \partial^q_1 V_3 + g^{2\lambda}V_\lambda \partial^q_2 g_{13} = 0,$$
$$V^q_{2;3} + V^q_{3;2} = \partial^q_3 V_2 + \partial^q_2 V_3 + g^{1\lambda}V_\lambda \partial^q_1 g_{23} = 0.$$

A solution for $V_1$ and $V_2$ can be easily found for any fractional order $q$, that is, $n-1 < q < n$, namely,

$$V_1 = cy^q + \sum_{k=0}^{n-1}\left(a_k x^k + b_k y^k\right),$$
$$V_2 = -cx^q + \sum_{k=0}^{n-1}\left(c_k x^k + d_k y^k\right), \quad (27)$$

where $c, a_k, b_k, c_k, d_k$ are constants. The solution to $V_3$ is not easy to find for $0 < q \le 1$. However, for $n \ge 2$, that is, $1 < q$, the equations simplify because

$$\partial^q_2 g_{13} = \partial^q_1 g_{23} = 0. \quad (28)$$

In this case a general solution is obtained as

$$V_3 = \sum_{k=0}^{n-1}\left(a'_k x^k + b'_k y^k\right), \quad (29)$$

where $a'_k, b'_k$ are constants.

*2.2.2. Fractional Killing-Yano Tensors.* The fractional antisymmetric Killing-Yano tensors can be derived using the condition that

$$^q f_{\mu\nu;\lambda} + \,^q f_{\lambda\nu;\mu} = 0, \quad (30)$$

where $^q f_{\mu\nu;\lambda}$ is the fractional covariant derivative of the Killing-Yano tensor $^q f_{\mu\nu}$ defined as

$$^q f_{\mu\nu;\lambda} = \partial^q_\lambda f_{\mu\nu} - f_{\alpha\nu}\,^q\Gamma^\alpha_{\lambda\mu} - f_{\mu\alpha}\,^q\Gamma^\alpha_{\lambda\nu}. \quad (31)$$

For the fractional order $0 < q < 1$, it is difficult to find an analytic solution. However, for the order $q > 1$, the Christoffel symbols vanish; we find that

$$\partial^q_\lambda f_{\mu\nu} = 0 \quad (32)$$

for all values of $\lambda, \nu, \mu$. A solution is that $f_{11} = f_{22} = f_{33} = 0$ and $f_{12}, f_{13}, f_{23}$ are a linear combination of $x^k, y^k$ where $k = 0, 1, 2, \ldots, n-1$, namely,

$$f_{12} = -f_{21} = \sum_{k=0}^{n-1}\left(a_k x^k + b_k y^k\right),$$
$$f_{13} = -f_{31} = \sum_{k=0}^{n-1}\left(a'_k x^k + b'_k y^k\right), \quad (33)$$
$$f_{23} = -f_{32} = \sum_{k=0}^{n-1}\left(c_k x^k + d_k y^k\right),$$

where $a_k, b_k, a'_k, b'_k, c_k, d_k$ are constants.

## 3. Conclusion

In this work, we investigate the existence of fractional Killing vectors and Killing-Yano tensors for the geometry induced by fractionalizing the classical free Lagrangian admitting a constant of motion. We discuss the cases of one-dimensional and two-dimensional curved space. We use the Caputo definition of the fractional derivative to calculate the fractional Christoffel symbols and consequently we provide explicit solution to the fractional Killing vectors and Killing-Yano tensors.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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