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Research Article

Henry-Gronwall Integral Inequalities with "Maxima" and Their Applications to Fractional Differential Equations

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Some new weakly singular Henry-Gronwall type integral inequalities with "maxima" are established in this paper. Applications to Caputo fractional differential equations with "maxima" are also presented.

1. Introduction

It is well known that Gronwall-Bellman type integral inequalities play a dominant role in the study of quantitative properties of solutions of differential and integral equations [1–5]. Usually, the integrals concerning these type inequalities have regular or continuous kernels, but some problems of theory and practicality require us to solve integral inequalities with singular kernels. For example, Henry [6] proposed a method to find solutions and proved some results concerning linear integral inequalities with weakly singular kernel. Moreover, Medved' [7, 8] presented a new approach to solve integral inequalities of Henry-Gronwall type and their Bihari version and obtained global solutions of semilinear evolution equations. Ye and Gao [9] considered the integral inequalities of Henry-Gronwall type and their applications to fractional differential equations with delay. Ma and Pečarić [10] established some weakly singular integral inequalities of Gronwall-Bellman type and used them in the analysis of various problems in the theory of certain classes of differential equations, integral equations, and evolution equations. Shao and Meng [11] studied a certain class of nonlinear inequalities of Gronwall-Bellman type, which is used to a qualitative analysis to certain fractional differential equations. For other results on the subject we refer to [12–18] and references cited therein.

Differential equations with "maxima" are a special type of differential equations that contain the maximum of the unknown function over a previous interval. Several integral inequalities have been established in the case when maxima of the unknown scalar function are involved in the integral; see [19, 20] and references cited therein.

Recently in [21] some new types of integral inequalities on time scales with "maxima" are established, which can be used as a handy tool in the investigation of making estimates for bounds of solutions of dynamic equations on time scales with "maxima." In this paper we establish some Henry-Gronwall type integral inequalities with "maxima." The significance of our work lies in the fact that "maxima" are taken on intervals $[\beta t, t]$ which have nonconstant length, where $0 < \beta < 1$. Most of the papers take the "maxima" on $[t-h, t]$, where $h > 0$ is a given constant. We apply our results to demonstrate the bound of solutions and the dependence of solutions on the orders with initial conditions for Caputo fractional differential equations with "maxima"

$$D^{\alpha} x(t) = f\left(t, x(t), \max_{s\in[\beta t, t]} x(s)\right), \quad t \in I = [t_0, T),$$
$$x(t) = \phi(t), \quad t \in [\beta t_0, t_0]. \tag{1}$$

The paper is organized as follows. In Section 2 we recall some results from [21] in the special case $\mathbb{T} = \mathbb{R}$ which are used to prove our main results which are presented in Section 3. In the last Section 4 we give applications of our results for an initial value problem for a Caputo fractional differential equation with "maxima."

2. Preliminaries

For convenience we let throughout $t_0 > 0$. The following results in Lemmas 1 and 3 are obtained by reducing the time scales $\mathbb{T} = \mathbb{R}$, $f(t) = g(t) \equiv 1$, and $a(t) = b(t) \equiv 0$ for all $t \in [t_0, T)$ in Theorems 3.3 and 3.2 ([21], page 8 and page 6), respectively.

Lemma 1 (see [21]). *Let the following conditions be satisfied:*

(H_1) *the functions p and $q \in C([t_0, T), \mathbb{R}_+)$;*

(H_2) *the function $\phi \in C([\beta t_0, T), \mathbb{R}_+)$ with $\max_{s\in[\beta t_0,t_0]}\phi(s) > 0$, where $0 < \beta < 1$;*

(H_3) *the function $u \in C([\beta t_0, T), \mathbb{R}_+)$ and satisfies the inequalities*

$$u(t) \le \phi(t) + \int_{t_0}^{t} \left[p(s) u(s) + q(s) \max_{\xi\in[\beta s, s]} u(\xi) \right] ds, \quad t \in [t_0, T], \tag{2}$$
$$u(t) \le \phi(t), \quad t \in [\beta t_0, t_0].$$

Then

$$u(t) \le \phi(t) + h(t) \exp\left(\int_{t_0}^{t} [p(s) + q(s)]\, ds \right), \quad t \in [t_0, T) \tag{3}$$

holds, where

$$h(t) = \max_{s\in[\beta t_0, t_0]} \phi(s) + \int_{t_0}^{t} \left[p(s)\phi(s) + q(s) \max_{\xi\in[\beta s, s]} \phi(\xi) \right] ds, \quad t \in [t_0, T). \tag{4}$$

By splitting the initial function ϕ to be two functions, we deduce the following corollary.

Corollary 2. *Let the following conditions be satisfied:*

(H_4) *the functions p, q, and $v \in C([t_0, T), \mathbb{R}_+)$;*

(H_5) *the function $w \in C([\beta t_0, t_0], \mathbb{R}_+)$ with $\max_{s\in[\beta t_0,t_0]} w(s) > 0$ and $w(t_0) = v(t_0)$, where $0 < \beta < 1$;*

(H_6) *the function $u \in C([\beta t_0, T), \mathbb{R}_+)$ and satisfies the inequalities*

$$u(t) \le v(t) + \int_{t_0}^{t} \left[p(s) u(s) + q(s) \max_{\xi\in[\beta s, s]} u(\xi) \right] ds, \quad t \in [t_0, T), \tag{5}$$
$$u(t) \le w(t), \quad t \in [\beta t_0, t_0].$$

Then

$$u(t) \le v(t) + h(t) \exp\left(\int_{t_0}^{t} [p(s) + q(s)]\, ds \right), \quad t \in [t_0, T) \tag{6}$$

holds, where

$$h(t) = \max_{s\in[\beta t_0, t_0]} w(s) + \int_{t_0}^{t} \left[p(s) v(s) + q(s) \max_{\xi\in[\beta s, s]} m(\xi) \right] ds, \quad t \in [t_0, T), \tag{7}$$

with

$$m(t) = \begin{cases} v(t), & t \in [t_0, T), \\ w(t), & t \in [\beta t_0, t_0]. \end{cases} \tag{8}$$

Lemma 3 (see [21]). *Let the condition (H_1) of Lemma 1 be satisfied. In addition, assume that*

(H_7) *the function $k \in C([t_0, T), (0, \infty))$ is nondecreasing;*

(H_8) *the function $\phi \in C([\beta t_0, t_0), \mathbb{R}_+)$, where $0 < \beta < 1$;*

(H_9) *the function $u \in C([\beta t_0, T), \mathbb{R}_+)$ and satisfies the inequalities*

$$u(t) \le k(t) + \int_{t_0}^{t} \left[p(s) u(s) + q(s) \max_{\xi\in[\beta s, s]} u(\xi) \right] ds, \quad t \in [t_0, T), \tag{9}$$
$$u(t) \le \phi(t), \quad t \in [\beta t_0, t_0].$$

Then

$$u(t) \le N k(t) \exp\left(\int_{t_0}^{t} [p(s) + q(s)]\, ds \right), \quad t \in [t_0, T) \tag{10}$$

holds, where

$$N = \max\left\{ 1, \frac{\max_{s\in[\beta t_0, t_0]} \phi(s)}{k(t_0)} \right\}. \tag{11}$$

The following lemma is a consequence of Jensen's inequality which can be found in [22].

Lemma 4 (see [22]). *Let $n \in N$, and let $x_1, \ldots, x_n$ be nonnegative real numbers. Then, for $\sigma > 1$,*

$$\left(\sum_{i=1}^{n} x_i \right)^{\sigma} \le n^{\sigma - 1} \sum_{i=1}^{n} x_i^{\sigma}. \tag{12}$$

3. Main Results

Theorem 5. *Suppose that the following conditions are satisfied:*

(H_{10}) *the functions p and $r \in C([t_0, T), \mathbb{R}_+)$;*

(H_{11}) *the function* $\phi \in C([\beta t_0, t_0], \mathbb{R}_+)$ *with* $\max_{s\in[\beta t_0,t_0]}\phi(s) > 0$, *where* $0 < \beta < 1$;

(H_{12}) *the function* $u \in C([\beta t_0, T), \mathbb{R}_+)$ *with*

$$u(t) \leq r(t) + \int_{t_0}^{t} (t-s)^{\alpha-1} p(s) \max_{\xi\in[\beta s,s]} u(\xi)\, ds, \quad t \in [t_0, T), \tag{13}$$

$$u(t) \leq \phi(t), \quad t \in [\beta t_0, t_0], \tag{14}$$

where $\alpha > 0$.

Then the following assertions hold.

(R_1) *Suppose* $\alpha > 1/2$; *then*

$$u(t) \leq e^t \left[c_1 r^2(t) + h_1(t) \exp\left(K_1 \int_{t_0}^{t} p^2(s)\, ds \right) \right]^{1/2}, \quad t \in [t_0, T), \tag{15}$$

where

$$c_1 = \max\left\{ 2e^{-2t_0}, e^{-2\beta t_0} \right\}, \tag{16}$$

$$K_1 = \frac{\Gamma(2\alpha-1)}{4^{\alpha-1}}, \tag{17}$$

$$h_1(t) = c_1 \max_{s\in[\beta t_0,t_0]} \phi^2(s) + c_1 K_1 \int_{t_0}^{t} p^2(s) \max_{\xi\in[\beta s,s]} m_1^2(\xi)\, ds, \quad t \in [t_0, T), \tag{18}$$

with

$$m_1(t) = \begin{cases} r(t), & t \in [t_0, T), \\ \phi(t), & t \in [\beta t_0, t_0]. \end{cases} \tag{19}$$

Moreover, if $r \in C([t_0, T), (0, \infty))$ *is a nondecreasing function, then*

$$u(t) \leq \sqrt{c_1 N_1}\, r(t) \exp\left(t + \frac{1}{2} K_1 \int_{t_0}^{t} p^2(s)\, ds \right), \quad t \in [t_0, T), \tag{20}$$

where

$$N_1 = \max\left\{ 1, \frac{\max_{s\in[\beta t_0,t_0]}\phi^2(s)}{r^2(t_0)} \right\}. \tag{21}$$

(R_2) *Suppose* $0 < \alpha \leq 1/2$; *then*

$$u(t) \leq e^t \left[c_2 r^b(t) + h_2(t) \exp\left(2^{b-1} K_2^b \int_{t_0}^{t} p^b(s)\, ds \right) \right]^{1/b}, \quad t \in [t_0, T), \tag{22}$$

where

$$a = \alpha + 1, \tag{23}$$

$$b = 1 + \frac{1}{\alpha}, \tag{24}$$

$$c_2 = \max\left\{ 2^{b-1} e^{-bt_0}, e^{-b\beta t_0} \right\}, \tag{25}$$

$$K_2 = \left(\frac{\Gamma(\alpha^2)}{a^{\alpha^2}} \right)^{1/a}, \tag{26}$$

$$h_2(t) = c_2 \max_{s\in[\beta t_0,t_0]} \phi^b(s) + 2^{b-1} c_2 K_2^b \int_{t_0}^{t} p^b(s) \max_{\xi\in[\beta s,s]} m_1^b(\xi)\, ds, \quad t \in [t_0, T). \tag{27}$$

Moreover, if $r \in C([t_0, T), (0, \infty))$ *is a nondecreasing function, then*

$$u(t) \leq (c_2 N_2)^{1/b} r(t) \exp\left(t + \frac{2^{b-1}}{b} K_2^b \int_{t_0}^{t} p^b(s)\, ds \right), \quad t \in [t_0, T), \tag{28}$$

where

$$N_2 = \max\left\{ 1, \frac{\max_{s\in[\beta t_0,t_0]}\phi^b(s)}{r^b(t_0)} \right\}. \tag{29}$$

Proof. Consider (R_1) $\alpha > 1/2$. Using the Cauchy-Schwarz inequality with (13), we get for $t \in [t_0, T)$

$$\begin{aligned} u(t) &\leq r(t) + \int_{t_0}^{t} (t-s)^{\alpha-1} e^s p(s) e^{-s} \max_{\xi\in[\beta s,s]} u(\xi)\, ds \\ &\leq r(t) + \left[\int_{t_0}^{t} (t-s)^{2\alpha-2} e^{2s} ds \right]^{1/2} \\ &\quad \times \left[\int_{t_0}^{t} p^2(s) e^{-2s} \left(\max_{\xi\in[\beta s,s]} u(\xi) \right)^2 ds \right]^{1/2}. \end{aligned} \tag{30}$$

The first integral of (30) implies the estimate

$$\begin{aligned} \int_{t_0}^{t} (t-s)^{2\alpha-2} e^{2s} ds &= \int_{0}^{t-t_0} \tau^{2\alpha-2} e^{2(t-\tau)} d\tau \\ &\leq e^{2t} \int_{0}^{t} \tau^{2\alpha-2} e^{-2\tau} d\tau \\ &= \frac{2e^{2t}}{4^{\alpha}} \int_{0}^{2t} \sigma^{2\alpha-2} e^{-\sigma} d\sigma \\ &< \frac{2e^{2t}}{4^{\alpha}} \Gamma(2\alpha-1). \end{aligned} \tag{31}$$

Therefore, from (30) and (31), we obtain

$$u(t) \leq r(t) + \left[\frac{2e^{2t}}{4^{\alpha}}\Gamma(2\alpha-1)\right]^{1/2} \times \left[\int_{t_0}^{t} p^2(s)e^{-2s}\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^2 ds\right]^{1/2}. \quad (32)$$

Applying Lemma 4 with $n = 2$, $\sigma = 2$, we get

$$u^2(t) \leq 2r^2(t) + \frac{e^{2t}}{4^{\alpha-1}}\Gamma(2\alpha-1)\int_{t_0}^{t} p^2(s)e^{-2s}\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^2 ds, \quad t \in [t_0, T). \quad (33)$$

Now, taking $v(t) = [e^{-t}u(t)]^2$, we have

$$\begin{aligned} v(t) &\leq 2e^{-2t}r^2(t) + \frac{\Gamma(2\alpha-1)}{4^{\alpha-1}} \times \int_{t_0}^{t} p^2(s)e^{-2s}\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^2 ds \\ &\leq 2e^{-2t_0}r^2(t) + \frac{\Gamma(2\alpha-1)}{4^{\alpha-1}} \times \int_{t_0}^{t} p^2(s)\max_{\xi\in[\beta s,s]}\left(e^{-2\xi}u^2(\xi)\right) ds \\ &\leq c_1 r^2(t) + K_1\int_{t_0}^{t} p^2(s)\max_{\xi\in[\beta s,s]} v(\xi)\, ds, \quad t \in [t_0, T), \end{aligned} \quad (34)$$

and, for $t \in [\beta t_0, t_0]$,

$$v(t) \leq e^{-2t}\phi^2(t) \leq e^{-2\beta t_0}\phi^2(t) \leq c_1\phi^2(t), \quad (35)$$

where c_1 and K_1 are defined by (16) and (17), respectively.

Applying Corollary 2 for (34) and (35), we obtain

$$v(t) \leq c_1 r^2(t) + h_1(t)\exp\left(K_1\int_{t_0}^{t} p^2(s)\, ds\right), \quad t \in [t_0, T), \quad (36)$$

where h_1 is defined by (18). Therefore, we get the required inequality in (15).

Moreover, if $r \in C([t_0, T), (0, \infty))$ is a nondecreasing function, then, by applying Lemma 3 for (34) and (35), we obtain the estimate

$$v(t) \leq c_1 N_1 r^2(t)\exp\left(K_1\int_{t_0}^{t} p^2(s)\, ds\right), \quad t \in [t_0, T), \quad (37)$$

where N_1 is defined by (21). Thus, we get the desired inequality in (20). This completes the proof of the first part.

Consider (R_2) $0 < \alpha \leq 1/2$. Let a, b be defined by (23) and (24), respectively. It is obvious that $(1/a) + (1/b) = 1$. Using the Hölder inequality in (13), for $t \in [t_0, T)$, we have

$$\begin{aligned} u(t) &\leq r(t) + \int_{t_0}^{t} (t-s)^{\alpha-1}e^{s}p(s)e^{-s}\max_{\xi\in[\beta s,s]} u(\xi)\, ds \\ &\leq r(t) + \left[\int_{t_0}^{t} (t-s)^{a(\alpha-1)}e^{as} ds\right]^{1/a} \times \left[\int_{t_0}^{t} p^b(s)e^{-bs}\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^b ds\right]^{1/b}. \end{aligned} \quad (38)$$

Repeating the process to get (31), the first integral of (38) implies the estimate

$$\int_{t_0}^{t} (t-s)^{a(\alpha-1)}e^{as} ds < \frac{e^{at}}{a^{1-a(1-\alpha)}}\Gamma(1-a(1-\alpha)). \quad (39)$$

Obviously, $1 - a(1-\alpha) = \alpha^2 > 0$ and $\Gamma(1 - a(1-\alpha)) \in \mathbb{R}$. From (38) and (39), it follows that

$$u(t) \leq r(t) + K_2 e^{t}\left[\int_{t_0}^{t} p^b(s)e^{-bs}\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^b ds\right]^{1/b}, \quad (40)$$

where K_2 is defined by (26). Applying Lemma 4 with $n = 2$, $\sigma = b$, we have

$$u^b(t) \leq 2^{b-1}\left[r^b(t) + K_2^b e^{bt}\int_{t_0}^{t} p^b(s)e^{-bs}\left(\max_{\xi\in[\beta s,s]} u(\xi)\right)^b ds\right], \quad t \in [t_0, T). \quad (41)$$

By setting $v(t) = [e^{-t}u(t)]^b$, we get

$$v(t) \leq c_2 r^b(t) + 2^{b-1}K_2^b\int_{t_0}^{t} p^b(s)\max_{\xi\in[\beta s,s]} v(\xi)\, ds, \quad t \in [t_0, T), \quad (42)$$

and, for $t \in [\beta t_0, t_0]$,

$$v(t) \leq e^{-b\beta t_0}\phi^b(t) \leq c_2\phi^b(t), \quad (43)$$

where c_2 is defined by (25). Consequently, applying Corollary 2 with (42) and (43), we have

$$v(t) \leq c_2 r^b(t) + h_2(t)\exp\left(2^{b-1}K_2^b\int_{t_0}^{t} p^b(s)\, ds\right), \quad t \in [t_0, T), \quad (44)$$

where h_2 is defined by (27). Therefore, the desired inequality (22) is established.

Furthermore, if $r \in C([t_0, T), (0, \infty))$ is a nondecreasing function, then by applying Lemma 3 for (42) and (43) we deduce that

$$v(t) \le c_2 N_2 r^b(t) \exp\left(2^{b-1} K_2^b \int_{t_0}^t p^b(s)\, ds\right), \quad t \in [t_0, T). \tag{45}$$

Thus, inequality (28) is proved. This completes the proof. □

Theorem 6. *Assume that*

(H_{13}) *the conditions* (H_{10}), (H_{11}) *of Theorem 5 are satisfied;*

(H_{14}) *the function* $q \in C([t_0, T), \mathbb{R}_+)$;

(H_{15}) *the function* $u \in C([\beta t_0, T), \mathbb{R}_+)$ *with*

$$u(t) \le r(t) + \int_{t_0}^t (t-s)^{\alpha-1}\left[p(s)u(s) + q(s)\max_{\xi\in[\beta s, s]} u(\xi)\right] ds, \quad t \in [t_0, T), \tag{46}$$

$$u(t) \le \phi(t), \quad t \in [\beta t_0, t_0], \tag{47}$$

where $\alpha > 0$.

Then the following assertions hold.

(R_3) *Suppose* $\alpha > 1/2$; *then*

$$u(t) \le e^t\left[c_3 r^2(t) + h_3(t)\exp\left(K_3\int_{t_0}^t \left[p^2(s) + q^2(s)\right] ds\right)\right]^{1/2}, \quad t \in [t_0, T), \tag{48}$$

where

$$c_3 = \max\left\{3e^{-2t_0}, e^{-2\beta t_0}\right\}, \tag{49}$$

$$K_3 = \frac{6}{4^\alpha}\Gamma(2\alpha - 1), \tag{50}$$

$$h_3(t) = c_3 \max_{s\in[\beta t_0, t_0]} \phi^2(s) + c_3 K_3 \times \int_{t_0}^t \left[p^2(s) r^2(s) + q^2(s)\max_{\xi\in[\beta s, s]} m_1^2(\xi)\right] ds, \quad t \in [t_0, T), \tag{51}$$

with m_1 *being defined by* (19).

Furthermore, if $r \in C([t_0, T), (0, \infty))$ *is a nondecreasing function, then*

$$u(t) \le \sqrt{c_3 N_1}\, r(t)\exp\left(t + \frac{1}{2}K_3\int_{t_0}^t \left[p^2(s) + q^2(s)\right] ds\right), \quad t \in [t_0, T), \tag{52}$$

where N_1 *is defined by* (21).

(R_4) *Suppose* $0 < \alpha \le 1/2$; *then*

$$u(t) \le e^t\left[c_4 r^b(t) + h_4(t)\exp\left(3^{b-1}K_2^b\int_{t_0}^t \left[p^b(s) + q^b(s)\right] ds\right)\right]^{1/b}, \quad t \in [t_0, T), \tag{53}$$

where a, b, and K_2 *are defined by* (23), (24), *and* (26), *respectively,*

$$c_4 = \max\left\{3^{b-1}e^{-bt_0}, e^{-b\beta t_0}\right\}, \tag{54}$$

$$h_4(t) = c_4 \max_{s\in[\beta t_0, t_0]} \phi^b(s) + c_4 3^{b-1} K_2^b \times \int_{t_0}^t \left[p^b(s) r^b(s) + q^b(s)\max_{\xi\in[\beta s, s]} m_1^b(\xi)\right] ds, \quad t \in [t_0, T). \tag{55}$$

Furthermore, if $r \in C([t_0, T), (0, \infty))$ *is a nondecreasing function, then*

$$u(t) \le (c_4 N_2)^{1/b} r(t)\exp\left(t + \frac{3^{b-1}}{b}K_2^b\int_{t_0}^t \left[p^b(s) + q^b(s)\right] ds\right), \quad t \in [t_0, T), \tag{56}$$

where N_2 *is defined by* (29).

Proof. Consider (R_3) $\alpha > 1/2$. By using the Cauchy-Schwarz inequality in (46), for $t \in [t_0, T)$, we have

$$\begin{aligned} u(t) &\le r(t) + \int_{t_0}^t (t-s)^{\alpha-1} e^s p(s) e^{-s} u(s)\, ds \\ &\quad + \int_{t_0}^t (t-s)^{\alpha-1} e^s q(s) e^{-s}\max_{\xi\in[\beta s, s]} u(\xi)\, ds \\ &\le r(t) + \left[\int_{t_0}^t (t-s)^{2\alpha-2} e^{2s} ds\right]^{1/2} \end{aligned}$$

$$\times\left[\int_{t_0}^{t} p^2(s)e^{-2s}u^2(s)ds\right]^{1/2}$$
$$+\left[\int_{t_0}^{t} (t-s)^{2\alpha-2}e^{2s}ds\right]^{1/2}$$
$$\times\left[\int_{t_0}^{t} q^2(s)e^{-2s}\left(\max_{\xi\in[\beta s,s]}u(\xi)\right)^2 ds\right]^{1/2}$$
$$\le r(t)+\left[\frac{2e^{2t}}{4^{\alpha}}\Gamma(2\alpha-1)\right]^{1/2}$$
$$\times\left\{\left[\int_{t_0}^{t} p^2(s)e^{-2s}u^2(s)ds\right]^{1/2}\right.$$
$$\left.+\left[\int_{t_0}^{t} q^2(s)e^{-2s}\left(\max_{\xi\in[\beta s,s]}u(\xi)\right)^2 ds\right]^{1/2}\right\}. \tag{57}$$

Applying Lemma 4 with $n=3$, $\sigma=2$, we get

$$u^2(t)\le 3r^2(t)+\frac{6e^{2t}}{4^{\alpha}}\Gamma(2\alpha-1)$$
$$\times\left[\int_{t_0}^{t} p^2(s)e^{-2s}u^2(s)ds\right.$$
$$\left.+\int_{t_0}^{t} q^2(s)e^{-2s}\left(\max_{\xi\in[\beta s,s]}u(\xi)\right)^2 ds\right], \tag{58}$$
$$t\in[t_0,T).$$

Taking $v(t)=[e^{-t}u(t)]^2$, we have

$$v(t)\le c_3r^2(t)+K_3\int_{t_0}^{t}p^2(s)v(s)ds$$
$$+K_3\int_{t_0}^{t}q^2(s)\max_{\xi\in[\beta s,s]}v(\xi)ds,\quad t\in[t_0,T), \tag{59}$$

$$v(t)\le c_3\phi^2(t),\quad t\in[\beta t_0,t_0], \tag{60}$$

where c_3 and K_3 are defined by (49) and (50), respectively. Using Corollary 2 for (59) and (60), it follows that

$$v(t)\le c_3r^2(t)+h_3(t)\exp\left(K_3\int_{t_0}^{t}\left[p^2(s)+q^2(s)\right]ds\right),$$
$$t\in[t_0,T), \tag{61}$$

where h_3 is defined by (51). Thus, we get the result in (48).

If $r\in C([t_0,T),(0,\infty))$ is a nondecreasing function, then Lemma 3 with (59) and (60) implies the estimate

$$v(t)\le c_3N_1r^2(t)\exp\left(K_3\int_{t_0}^{t}\left[p^2(s)+q^2(s)\right]ds\right),$$
$$t\in[t_0,T), \tag{62}$$

where N_1 is defined by (21). Thus, the required inequality (52) is established. This completes the proof of the first part.

Consider (R_4) $0<\alpha\le 1/2$. Let a, b be defined by (23) and (24), respectively. Applying the Hölder inequality in (46), we have that for $t\in[t_0,T)$

$$u(t)\le r(t)+\left[\int_{t_0}^{t}(t-s)^{a(\alpha-1)}e^{as}ds\right]^{1/a}$$
$$\times\left[\int_{t_0}^{t}p^b(s)e^{-bs}u^b(s)ds\right]^{1/b}$$
$$+\left[\int_{t_0}^{t}(t-s)^{a(\alpha-1)}e^{as}ds\right]^{1/a}$$
$$\times\left[\int_{t_0}^{t}q^b(s)e^{-bs}\left(\max_{\xi\in[\beta s,s]}u(\xi)\right)^b ds\right]^{1/b}$$
$$\le r(t)+K_2e^t\left\{\left[\int_{t_0}^{t}p^b(s)e^{-bs}u^b(s)ds\right]^{1/b}\right.$$
$$\left.+\left[\int_{t_0}^{t}q^b(s)e^{-bs}\left(\max_{\xi\in[\beta s,s]}u(\xi)\right)^b ds\right]^{1/b}\right\}, \tag{63}$$

where K_2 is defined by (26). By using Lemma 4 with $n=3$, $\sigma=b$, we obtain the estimate

$$u^b(t)\le 3^{b-1}r^b(t)+3^{b-1}K_2^be^{bt}$$
$$\times\left[\int_{t_0}^{t}p^b(s)e^{-bs}u^b(s)ds\right.$$
$$\left.+\int_{t_0}^{t}q^b(s)e^{-bs}\left(\max_{\xi\in[\beta s,s]}u(\xi)\right)^b ds\right], \tag{64}$$
$$t\in[t_0,T).$$

Substituting $v(t)=[e^{-t}u(t)]^b$, we get

$$v(t)\le c_4r^b(t)+3^{b-1}K_2^b$$
$$\times\left[\int_{t_0}^{t}p^b(s)v(s)ds\right.$$
$$\left.+\int_{t_0}^{t}q^b(s)\max_{\xi\in[\beta s,s]}v(\xi)ds\right],\quad t\in[t_0,T), \tag{65}$$

and, for $t \in [\beta t_0, t_0]$,

$$v(t) \leq c_4 \phi^b(t), \tag{66}$$

where c_4 is defined by (54). An application of Corollary 2 to (65) and (66) gives

$$v(t) \leq c_4 r^b(t) + h_4(t) \times \exp\left(3^{b-1} K_2^b \int_{t_0}^{t} \left[p^b(s) + q^b(s)\right] ds\right), \quad t \in [t_0, T), \tag{67}$$

where h_4 is defined by (55). Therefore, we deduce inequality (53).

As a special case, if $r \in C([t_0, T), (0, \infty))$ is a nondecreasing function, then, by Lemma 3 with (65) and (66), we get

$$v(t) \leq c_4 N_2 r^b(t) \times \exp\left(3^{b-1} K_2^b \int_{t_0}^{t} \left[p^b(s) + q^b(s)\right] ds\right), \quad t \in [t_0, T). \tag{68}$$

Therefore, the desired inequality (56) is established. This completes the proof of Theorem 6. □

4. Applications to Fractional Differential Equations with "Maxima"

In this section, we apply our results to demonstrate the bound of solutions and the dependence of solutions on the orders with initial conditions for Caputo fractional differential equations with "maxima." We consider the following fractional differential equations (FDEs) with "maxima"

$$D^\alpha x(t) = f\left(t, x(t), \max_{s\in[\beta t, t]} x(s)\right), \quad t \in I = [t_0, T), \tag{69}$$

and initial condition

$$x(t) = \phi(t), \quad t \in [\beta t_0, t_0], \tag{70}$$

where D^α represents the Caputo fractional derivative of order α $(\alpha > 0)$, $f \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, ϕ is a given continuously differentiable function on $[\beta t_0, t_0]$ up to order n $(n = -[-\alpha])$, and $0 < \beta < 1$. We denote $\phi^{(k)}(t_0) = \rho_k$, $k = 0, 1, 2, \ldots, n-1$. For more details on fractional differential equations, see [23, 24].

Theorem 7. *Assume that*

(H_{16}) *there exist functions $y, z \in C(I, \mathbb{R}_+)$ such that, for $t \in I$, $u_1, u_2 \in \mathbb{R}$,*

$$\left|f(t, u_1, u_2)\right| \leq y(t)\left|u_1\right| + z(t)\left|u_2\right|. \tag{71}$$

If x is solution of the initial value problem (69)-(70), then the following estimates hold.

(R_5) *Suppose $1/2 < \alpha \leq 1$. Then*

$$|x(t)| \leq \sqrt{c_3} M \exp\left(t + \frac{K_3}{2\Gamma^2(\alpha)} \int_{t_0}^{t} \left[y^2(s) + z^2(s)\right] ds\right), \quad t \in I. \tag{72}$$

(R_6) *Suppose $0 < \alpha \leq 1/2$. Then*

$$|x(t)| \leq (c_4)^{1/b} M \exp\left(t + \frac{3^{b-1} K_2^b}{b\Gamma^b(\alpha)} \int_{t_0}^{t} \left[y^b(s) + z^b(s)\right] ds\right), \quad t \in I. \tag{73}$$

(R_7) *Suppose $\alpha > 1$. Then*

$$|x(t)| \leq \sqrt{c_3}\left(M + \sum_{j=1}^{n-1} \frac{|\rho_j|}{j!}(t - t_0)^j\right) \times \exp\left(t + \frac{K_3}{2\Gamma^2(\alpha)} \int_{t_0}^{t} \left[y^2(s) + z^2(s)\right] ds\right), \quad t \in I, \tag{74}$$

where

$$M = \max_{t\in[\beta t_0, t_0]} |\phi(t)| > 0 \tag{75}$$

and b, c_3, c_4, K_2, and K_3 are defined as in Theorems 5 and 6.

Proof. The solution x of the initial value problem (69)-(70) satisfies the following equations (see [23]):

$$x(t) = \sum_{j=0}^{n-1} \frac{\rho_j}{j!}(t - t_0)^j + \frac{1}{\Gamma(\alpha)} \times \int_{t_0}^{t} (t - s)^{\alpha-1} f\left(s, x(s), \max_{\xi\in[\beta s, s]} x(\xi)\right) ds, \quad t \in I, \tag{76}$$

$$x(t) = \phi(t), \quad t \in [\beta t_0, t_0]. \tag{77}$$

For $0 < \alpha \leq 1$, by using the assumption (H_{16}), it follows that

$$|x(t)| \leq M + \frac{1}{\Gamma(\alpha)} \times \int_{t_0}^{t} (t - s)^{\alpha-1} \left(y(s)|x(s)| + z(s) \max_{\xi\in[\beta s, s]} |x(\xi)|\right) ds, \quad t \in I,$$
$$|x(t)| \leq M, \quad t \in [\beta t_0, t_0]. \tag{78}$$

Hence, Theorem 6 yields the estimate inequalities (72) and (73).

For $\alpha > 1$, by using the assumption (H_{16}) in (76), we have

$$|x(t)| \le M + \sum_{j=1}^{n-1} \frac{|\rho_j|}{j!}(t-t_0)^j + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \times \left(y(s)|x(s)| + z(s) \max_{\xi\in[\beta s, s]} |x(\xi)| \right) ds, \quad t \in I,$$
$$|x(t)| \le M, \quad t \in [\beta t_0, t_0]. \tag{79}$$

Since $\sum_{j=1}^{n-1}(|\rho_j|/j!)(t-t_0)^j$ is a nondecreasing function, Theorem 6 yields the estimate inequality (74). This completes the proof. □

Theorem 8. *Let $\alpha > 0$ and $\delta > 0$ such that $0 \le n-1 < \alpha-\delta < \alpha \le n$. Also let $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following assumption:*

(H_{17}) *there exist constants $L_1, L_2 > 0$ such that $|f(t,u_1,u_2) - f(t,v_1,v_2)| \le L_1|u_1 - v_1| + L_2|u_2 - v_2|$, for each $t \in I$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$.*

If x and y are the solutions of the initial value problem (69)-(70) and

$$D^{\alpha-\delta} y(t) = f\left(t, y(t), \max_{s\in[\beta t, t]} y(s)\right), \quad t \in I, \tag{80}$$

with initial condition

$$y(t) = \overline{\phi}(t), \quad t \in [\beta t_0, t_0], \tag{81}$$

respectively, where $\overline{\phi}$ is a given continuous function on $[\beta t_0, t_0]$ such that $\phi(t) \not\equiv \overline{\phi}(t)$ for all $t \in [\beta t_0, t_0)$ up to order n ($n = -[-(\alpha-\delta)]$). we denote $\overline{\phi}^{(k)}(t_0) = \overline{\rho}_k$, $k = 0, 1, 2, \ldots, n-1$. Then the following estimates hold for $t_0 < t \le h < T$.

(R_8) *Suppose $\alpha - \delta > 1/2$. Then for $t \in I$*

$$|y(t) - x(t)| \le e^t \left[c_5 A^2(t) + h_5(t) \times \exp\left(\frac{6\Gamma(2\alpha-2\delta-1)\left(L_1^2 + L_2^2\right)(t-t_0)}{4^{\alpha-\delta}\Gamma^2(\alpha)} \right) \right]^{1/2}. \tag{82}$$

(R_9) *Suppose $0 < \alpha - \delta \le 1/2$. Then for $t \in I$*

$$|y(t) - x(t)| \le e^t \left[c_6 A^b(t) + h_6(t) \times \exp\left(\frac{\left[3\Gamma\left((\alpha-\delta)^2\right)\right]^{1/(\alpha-\delta)}\left(L_1^b + L_2^b\right)(t-t_0)}{(\alpha-\delta+1)^{\alpha-\delta}\Gamma^b(\alpha)} \right) \right]^{1/b}, \tag{83}$$

where

$$A(t) = \left| \sum_{j=0}^{n-1} \frac{\left(\overline{\rho}_j - \rho_j\right)(t-t_0)^j}{j!} \right| + \left| \frac{(t-t_0)^{\alpha-\delta}}{\Gamma(\alpha-\delta+1)} - \frac{(t-t_0)^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} \right| \|f\| + \left| \frac{(t-t_0)^{\alpha-\delta}}{(\alpha-\delta)\Gamma(\alpha)} - \frac{(t-t_0)^{\alpha}}{\Gamma(\alpha+1)} \right| \|f\|,$$
$$\|f\| = \sup_{t_0 \le t \le h} \left| f\left(t, y(t), \max_{s\in[\beta t, t]} y(s)\right) \right|,$$
$$b = 1 + \frac{1}{\alpha-\delta},$$
$$c_5 = \max\left\{3e^{-2t_0}, e^{-2\beta t_0}\right\},$$
$$c_6 = \max\left\{3^{1/(\alpha-\delta)}e^{-bt_0}, e^{-b\beta t_0}\right\}, \tag{84}$$
$$h_5(t) = c_5 \max_{s\in[\beta t_0, t_0]} \left|\overline{\phi}(s) - \phi(s)\right|^2 + \frac{6c_5\Gamma(2\alpha-2\delta-1)}{4^{\alpha-\delta}\Gamma^2(\alpha)} \times \int_{t_0}^{t} \left(L_1^2 A^2(s) + L_2^2 \max_{\xi\in[\beta s, s]} m_2^2(\xi) \right) ds,$$
$$h_6(t) = c_6 \max_{s\in[\beta t_0, t_0]} \left|\overline{\phi}(s) - \phi(s)\right|^b + \frac{c_6\left[3\Gamma\left((\alpha-\delta)^2\right)\right]^{1/(\alpha-\delta)}}{(\alpha-\delta+1)^{\alpha-\delta}\Gamma^b(\alpha)} \times \int_{t_0}^{t} \left(L_1^b A^b(s) + L_2^b \max_{\xi\in[\beta s, s]} m_2^b(\xi) \right) ds,$$

with

$$m_2(t) = \begin{cases} A(t), & t \in I, \\ \left|\overline{\phi}(t) - \phi(t)\right|, & t \in [\beta t_0, t_0]. \end{cases} \tag{85}$$

Proof. The solutions x and y of the initial value problems (69)-(70) and (80)-(81) satisfy the following equations:

$$x(t) = \sum_{j=0}^{n-1} \frac{\rho_j}{j!}(t-t_0)^j + \frac{1}{\Gamma(\alpha)} \times \int_{t_0}^{t} (t-s)^{\alpha-1} f\left(s, x(s), \max_{\xi\in[\beta s,s]} x(\xi)\right) ds,$$
$$y(t) = \sum_{j=0}^{n-1} \frac{\overline{\rho}_j}{j!}(t-t_0)^j + \frac{1}{\Gamma(\alpha-\delta)} \times \int_{t_0}^{t} (t-s)^{\alpha-\delta-1} f\left(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\right) ds, \quad (86)$$

respectively. So, using the assumption (H_{17}), it follows that

$$\begin{aligned}
|y(t)-x(t)| &\le \left|\sum_{j=0}^{n-1} \frac{\overline{\rho}_j}{j!}(t-t_0)^j - \sum_{j=0}^{n-1} \frac{\rho_j}{j!}(t-t_0)^j\right| \\
&+ \left|\frac{1}{\Gamma(\alpha-\delta)} \int_{t_0}^{t} (t-s)^{\alpha-\delta-1} \times f\left(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\right) ds - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-\delta-1} \times f\left(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\right) ds\right| \\
&+ \left|\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-\delta-1} \times f\left(s, y(s), \max_{\xi\in[\beta s,s]} y(\xi)\right) ds - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-\delta-1} \times f\left(s, x(s), \max_{\xi\in[\beta s,s]} x(\xi)\right) ds\right| \\
&+ \left|\frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-\delta-1} \times f\left(s, x(s), \max_{\xi\in[\beta s,s]} x(\xi)\right) ds - \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \times f\left(s, x(s), \max_{\xi\in[\beta s,s]} x(\xi)\right) ds\right| \\
&\le A(t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-\delta-1} \times \left(L_1 |y(s)-x(s)| + L_2 \left|\max_{\xi\in[\beta s,s]} y(\xi) - \max_{\xi\in[\beta s,s]} x(\xi)\right|\right) ds \\
&\le A(t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-\delta-1} \times \left(L_1 |y(s)-x(s)| + L_2 \max_{\xi\in[\beta s,s]} |y(\xi)-x(\xi)|\right) ds, \quad t \in I,
\end{aligned} \quad (87)$$

where $A(t)$ is defined by (84) and

$$|y(t)-x(t)| = \left|\overline{\phi}(t) - \phi(t)\right|, \quad t \in [\beta t_0, t_0]. \quad (88)$$

Applying Theorem 6 yields the desired inequalities (82) and (83). This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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