Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 265865, 11 pages
http://dx.doi.org/10.1155/2014/265865

Research Article

Generalized Common Fixed Point Results via Greatest Lower Bound Property

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Received 26 March 2014; Accepted 16 June 2014; Published 10 July 2014

Academic Editor: Filomena Cianciaruso

The aim of this paper is to unify the concept of greatest lower bound (g.l.b) property and establish some generalized common fixed results. We support our results by a nontrivial example.

1. Introduction and Preliminaries

The Banach fixed point theorem is used to establish the existence of a unique solution for a nonlinear integral equation [1]. Moreover this theorem plays an important role in several branches of mathematics. For instance, it has been used to show the existence of solutions of nonlinear Volterr integral equations and nonlinear integrodifferential equations in Banach spaces and to show the convergence of algorithms in computational mathematics. Because of its importance and usefulness for mathematical theory, it has become a very popular tool of mathematical analysis in many directions. Nadler [2] introduced the concept of multivalued contraction mappings and obtained the fixed points results for multivalued mappings. Huang and Zhang [3] introduced the notion of cone metric space which is a generalization of metric37 space. They extended Banach contraction principle to cone metric spaces. Since then, Arshad et al. [4], Azam and Arshad [5], Cho and Bae [6], and many others obtained fixed point theorems in cone metric spaces.

Azam et al. in [7] introduced the notion of complex-valued metric space and obtained some common fixed points of a pair of mappings satisfying rational expressions contractive condition. Although complex-valued metric spaces form a special class of cone metric space, yet this idea is intended to define rational expressions which are not meaningful in cone metric spaces. Subsequently, Rouzkard and Imdad [8] and Abbas et al. [9, 10] established some common fixed point theorems satisfying certain rational expressions in complex-valued metric spaces which generalize, unify, and complement the results of Azam et al. [7]. Sintunavarat et al. [11, 12] obtained common fixed point results by replacing constant of contractive condition with control functions. Klin-eam and Suanoom [13] established a common fixed point result for two single valued mappings in complex-valued metric spaces. Abbas et al. [14] introduced complex-valued generalized metric space and obtained common fixed point results in this space. For more details in fixed point theory, we refer the reader to [15–23]. Very recently, Ahmad et al. [24] introduced the notion of greatest lower bound (g.l.b.) property of the multivalued mappings and obtained some common fixed point results in the context of complex-valued metric spaces. Then Azam et al. [25] extend the concept of greatest lower bound (g.l.b.) property and proved some new common fixed point theorems in the setting of complex-valued metric spaces. In this paper, we present some new common fixed results and generalized the results of [24, 25].

Let $\mathbb{C}$ denote the set of complex numbers. Let $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \quad \text{iff } \operatorname{Re}(z_1) \leqslant \operatorname{Re}(z_2), \qquad \operatorname{Im}(z_1) \leqslant \operatorname{Im}(z_2). \tag{1}$$

It follows that

$$z_1 \precsim z_2, \tag{2}$$

if one of the following conditions is satisfied:

(i) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$, $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,

(ii) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$, $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$,

(iii) $\operatorname{Re}(z_1) < \operatorname{Re}(z_2)$, $\operatorname{Im}(z_1) < \operatorname{Im}(z_2)$,

(iv) $\operatorname{Re}(z_1) = \operatorname{Re}(z_2)$, $\operatorname{Im}(z_1) = \operatorname{Im}(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|,$$
$$z_1 \precsim z_2,\ z_2 \prec z_3 \Longrightarrow z_1 \prec z_3. \tag{3}$$

Definition 1. Let X be a nonempty set. Suppose that a mapping $d_c : X \times X \to \mathbb{C}$ satisfies the following:

(1) $0 \precsim d_c(\zeta,\eta)$, for all $\zeta,\eta \in X$ and $d_c(\zeta,\eta) = 0$ if and only if $\zeta = \eta$;

(2) $d_c(\zeta,\eta) = d_c(\eta,\zeta)$ for all $\zeta,\eta \in X$;

(3) $d_c(\zeta,\eta) \precsim d_c(\zeta,\varsigma) + d_c(\varsigma,\eta)$, for all $\zeta,\eta,\varsigma \in X$.

Then d_c is called a complex-valued metric on X, and (X, d_c) is called a complex-valued metric space. A point $\zeta \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(\zeta, r) = \{\eta \in X : d_c(\zeta,\eta) \prec r\} \subseteq A. \tag{4}$$

A point $\zeta \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$

$$B(\zeta, r) \cap (A \setminus \{\zeta\}) \neq \phi. \tag{5}$$

A is called open set whenever each element of A is an interior point of A and a subset $B \subseteq X$ is called closed set whenever each limit point of B belongs to B. The family

$$F = \{B(\zeta, r) : \zeta \in X, 0 \prec r\} \tag{6}$$

is a subbase for a Hausdorff topology τ on X. Let $\{\zeta_n\}$ be a sequence in X and $\zeta \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_c(\zeta_n, \zeta) \prec c$, then $\{\zeta_n\}$ is said to be convergent, $\{\zeta_n\}$ converges to ζ, and ζ is the limit point of $\{\zeta_n\}$. We denote this by $\lim_{n\to\infty}\zeta_n = \zeta$, or $\zeta_n \to \zeta$ as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d_c(\zeta_n, \zeta_{n+m}) \prec c$, then $\{\zeta_n\}$ is called a Cauchy sequence in (X, d_c). If every Cauchy sequence is convergent in (X, d_c), then (X, d_c) is called a complete complex-valued metric space. We require the following lemmas.

Lemma 2 (see [7]). *Let (X, d_c) be a complex-valued metric space and let $\{\zeta_n\}$ be a sequence in X. Then $\{\zeta_n\}$ converges to ζ if and only if $|d_c(\zeta_n, \zeta)| \to 0$ as $n \to \infty$.*

Lemma 3 (see [7]). *Let (X, d_c) be a complex-valued metric space and let $\{\zeta_n\}$ be a sequence in X. Then $\{\zeta_n\}$ is a Cauchy sequence if and only if $|d_c(\zeta_n, \zeta_m)| \to 0$ as $n, m \to \infty$.*

2. Main Result

Let (X, d_c) be a complex-valued metric space. In the sequel of [24], we denote nonempty, closed, and bounded subsets of X by $CB(X)$, respectively.

Throughout this paper, we denote $s(z_1) = \{z_2 \in \mathbb{C} : z_1 \precsim z_2\}$ for $z_1 \in \mathbb{C}$ and $s(a, B) = \cup_{b\in B} s(d_c(a, b)) = \cup_{b\in B}\{z \in \mathbb{C} : d_c(a, b) \precsim z\}$ for $a \in X$ and $B \in CB(X)$.

For $A, B \in CB(X)$ we denote

$$s(A, B) = \Big(\bigcap_{a\in A} s(a, B)\Big) \cap \Big(\bigcap_{b\in B} s(b, A)\Big). \tag{7}$$

Remark 4. Let (X, d_c) be a complex-valued metric space. If $\mathbb{C} = R$, then (X, d) is a metric space. Moreover for A, $B \in C(X)$, $H(A, B) = \inf s(A, B)$ is the Hausdorff distance induced by d_c.

Let (X, d_c) be a complex-valued metric space and $CB(X)$ be a collection of nonempty closed subsets of X. Let $S : X \to CB(X)$ be a multivalued map. For $\zeta \in X$ and $A \in CB(X)$, define

$$W_\zeta(A) = \{d_c(\zeta, a) : a \in A\}. \tag{8}$$

Thus for $\zeta, \eta \in X$,

$$W_\zeta(S\eta) = \{d_c(\zeta, u) : u \in S\eta\}. \tag{9}$$

Definition 5 (see [24]). Let (X, d_c) be a complex-valued metric space. A nonempty subset A of X is called bounded from below if there exists some $z \in \mathbb{C}$, such that $z \precsim a$ for all $a \in A$.

Definition 6 (see [24]). Let (X, d_c) be a complex-valued metric space. A multivalued mapping $F : X \to 2^{\mathbb{C}}$ is called bounded from below if for $\zeta \in X$ there exists $z_\zeta \in \mathbb{C}$ such that

$$z_\zeta \precsim u, \tag{10}$$

for all $u \in F\zeta$.

Definition 7 (see [24]). Let (X, d_c) be a complex-valued metric space. The multivalued mapping $S : X \to CB(X)$ is said to have lower bound property (l.b. property) on (X, d_c), if for any $\zeta \in X$ the multivalued mapping $F_\zeta : X \to 2^{\mathbb{C}}$ defined by

$$F_\zeta(S\eta) = W_\zeta(S\eta) \tag{11}$$

is bounded from below. That is, for $\zeta, \eta \in X$ there exists an element $l_\zeta(S\eta) \in \mathbb{C}$ such that

$$l_\zeta(S\eta) \precsim u, \tag{12}$$

for all $u \in W_\zeta(S\eta)$, where $l_\zeta(S\eta)$ is called lower bound of S associated with (ζ, η).

Definition 8 (see [24]). Let (X, d_c) be a complex-valued metric space. The multivalued mapping $S : X \to CB(X)$ is said to have greatest lower bound property (g.l.b. property) on (X, d_c), if greatest lower bound of $W_\zeta(S\eta)$ exists in $\mathbb{C}$ for all $\zeta, \eta \in X$. We denote $d_c(\zeta, S\eta)$ by the g.l.b. of $W_\zeta(S\eta)$. That is

$$d_c(\zeta, S\eta) = \inf\{d_c(\zeta, u) : u \in S\eta\}. \tag{13}$$

Theorem 9. *Let (X, d_c) be a complete complex-valued metric space and let $S, F : X \to CB(X)$ be multivalued mappings with g.l.b. property such that*

$$Ad_c(\zeta,\eta) + B\frac{d_c(\zeta,S\zeta)\,d_c(\eta,F\eta)}{1+d_c(\zeta,\eta)} + C\frac{d_c(\eta,S\zeta)\,d_c(\zeta,F\eta)}{1+d_c(\zeta,\eta)} + D\frac{d_c(\zeta,S\zeta)\,d_c(\zeta,F\eta)}{1+d_c(\zeta,\eta)} + E\frac{d_c(\eta,S\zeta)\,d_c(\eta,F\eta)}{1+d_c(\zeta,\eta)} \in s(S\zeta, F\eta), \tag{14}$$

for all $\zeta, \eta \in X$, where A, B, C, D, and E are nonnegative real numbers with $A + B + C + 2D + 2E < 1$. Then S, F have a common fixed point.

Proof. Let ζ_0 be an arbitrary point in X and $\zeta_1 \in S\zeta_0$. From (14), we have

$$Ad_c(\zeta_0,\zeta_1) + B\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + C\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + D\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + E\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} \in s(S\zeta_0, F\zeta_1). \tag{15}$$

This implies that

$$\begin{aligned} &Ad_c(\zeta_0,\zeta_1) + B\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + C\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + D\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + E\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} \in \left(\bigcap_{\zeta\in S\zeta_0} s(\zeta, F\zeta_1)\right), \\ &Ad_c(\zeta_0,\zeta_1) + B\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + C\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + D\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + E\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} \in s(\zeta, F\zeta_1), \end{aligned} \tag{16}$$

for all $\zeta \in S\zeta_0$. Since $\zeta_1 \in S\zeta_0$, we have

$$\begin{aligned} &Ad_c(\zeta_0,\zeta_1) + B\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + C\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + D\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + E\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} \in s(\zeta_1, F\zeta_1), \\ &Ad_c(\zeta_0,\zeta_1) + B\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + C\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + D\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + E\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} \in \bigcup_{\zeta\in F\zeta_1} s(d_c(\zeta_1,\zeta)). \end{aligned} \tag{17}$$

So there exists some $\zeta_2 \in F\zeta_1$, such that

$$Ad_c(\zeta_0,\zeta_1) + B\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + C\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + D\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + E\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} \in s(d_c(\zeta_1,\zeta_2)). \tag{18}$$

That is

$$d_c(\zeta_1,\zeta_2) \precsim Ad_c(\zeta_0,\zeta_1) + B\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + C\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + D\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)} + E\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{1+d_c(\zeta_0,\zeta_1)}. \tag{19}$$

By the greatest lower bound property (g.l.b. property) of S and F, we get

$$\begin{aligned} d_c(\zeta_1,\zeta_2) &\precsim Ad_c(\zeta_0,\zeta_1) + B\frac{d_c(\zeta_0,\zeta_1)\,d_c(\zeta_1,\zeta_2)}{1+d_c(\zeta_0,\zeta_1)} + C\frac{d_c(\zeta_1,\zeta_1)\,d_c(\zeta_0,\zeta_2)}{1+d_c(\zeta_0,\zeta_1)} + D\frac{d_c(\zeta_0,\zeta_1)\,d_c(\zeta_0,\zeta_2)}{1+d_c(\zeta_0,\zeta_1)} + E\frac{d_c(\zeta_1,\zeta_1)\,d_c(\zeta_1,\zeta_2)}{1+d_c(\zeta_0,\zeta_1)} \\ &= Ad_c(\zeta_0,\zeta_1) + B\frac{d_c(\zeta_0,\zeta_1)\,d_c(\zeta_1,\zeta_2)}{1+d_c(\zeta_0,\zeta_1)} + d_c\frac{d_c(\zeta_0,\zeta_1)\,d_c(\zeta_0,\zeta_2)}{1+d_c(\zeta_0,\zeta_1)}, \end{aligned} \tag{20}$$

which implies that

$$\begin{aligned}\left|d_c(\zeta_1,\zeta_2)\right| &\le A\left|d_c(\zeta_0,\zeta_1)\right| + B\frac{\left|d_c(\zeta_0,\zeta_1)\right|\left|d_c(\zeta_1,\zeta_2)\right|}{\left|1+d_c(\zeta_0,\zeta_1)\right|} \\ &\quad + D\frac{\left|d_c(\zeta_0,\zeta_1)\right|\left|d_c(\zeta_0,\zeta_2)\right|}{\left|1+d_c(\zeta_0,\zeta_1)\right|} \\ &= A\left|d_c(\zeta_0,\zeta_1)\right| + B\left|d_c(\zeta_1,\zeta_2)\right|\left|\frac{d_c(\zeta_0,\zeta_1)}{1+d_c(\zeta_0,\zeta_1)}\right| \\ &\quad + D\left|d_c(\zeta_0,\zeta_2)\right|\frac{\left|d_c(\zeta_0,\zeta_1)\right|}{\left|1+d_c(\zeta_0,\zeta_1)\right|}, \\ \left|d_c(\zeta_1,\zeta_2)\right| &\le A\left|d_c(\zeta_0,\zeta_1)\right| + B\left|d_c(\zeta_1,\zeta_2)\right| + D\left|d_c(\zeta_0,\zeta_2)\right|. \end{aligned} \tag{21}$$

Then

$$\left|d_c(\zeta_1,\zeta_2)\right| \le \frac{A+D}{(1-B-D)}\left|d_c(\zeta_0,\zeta_1)\right|. \tag{22}$$

Similarly, we get

$$\begin{aligned} &Ad_c(\zeta_1,\zeta_2) + B\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + C\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} + D\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + E\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \in s(F\zeta_1 S\zeta_2). \end{aligned} \tag{23}$$

This implies that

$$\begin{aligned} &Ad_c(\zeta_1,\zeta_2) + B\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + C\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} + D\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + E\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \in \left(\bigcap_{\zeta\in S\zeta_0} s(\zeta,S\zeta_2)\right), \\ &Ad_c(\zeta_1,\zeta_2) + B\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + C\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} + D\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + E\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \in s(\zeta,S\zeta_2), \end{aligned} \tag{24}$$

for all $\zeta \in F\zeta_1$. Since $\zeta_2 \in F\zeta_1$, we have

$$\begin{aligned} &Ad_c(\zeta_1,\zeta_2) + B\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + C\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} + D\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + E\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \in s(\zeta_2,S\zeta_2), \\ &Ad_c(\zeta_1,\zeta_2) + B\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + C\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} + D\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + E\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \in s(\zeta_2,S\zeta_2) \\ &\quad = \bigcup_{\zeta\in F\zeta_1} s(d_c(\zeta_2,\zeta)). \end{aligned} \tag{25}$$

So there exists some $\zeta_3 \in S\zeta_2$, such that

$$\begin{aligned} &Ad_c(\zeta_1,\zeta_2) + B\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + C\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} + D\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + E\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \in s(d_c(\zeta_2,\zeta_3)). \end{aligned} \tag{26}$$

That is,

$$\begin{aligned} d_c(\zeta_2,\zeta_3) &\preceq Ad_c(\zeta_1,\zeta_2) + B\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + C\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + D\frac{d_c(\zeta_2,F\zeta_1)\,d_c(\zeta_2,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + E\frac{d_c(\zeta_1,F\zeta_1)\,d_c(\zeta_1,S\zeta_2)}{1+d_c(\zeta_1,\zeta_2)}. \end{aligned} \tag{27}$$

By the greatest lower bound property (g.l.b. property) of S and F, we get

$$\begin{aligned} d_c(\zeta_2,\zeta_3) &\preceq Ad_c(\zeta_1,\zeta_2) + B\frac{d_c(\zeta_1,\zeta_2)\,d_c(\zeta_2,\zeta_3)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + C\frac{d_c(\zeta_2,\zeta_2)\,d_c(\zeta_1,\zeta_3)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + D\frac{d_c(\zeta_2,\zeta_2)\,d_c(\zeta_2,\zeta_3)}{1+d_c(\zeta_1,\zeta_2)} \\ &\quad + E\frac{d_c(\zeta_1,\zeta_2)\,d_c(\zeta_1,\zeta_3)}{1+d_c(\zeta_1,\zeta_2)}, \end{aligned} \tag{28}$$

which implies that

$$\begin{aligned} \left|d_c(\zeta_2,\zeta_3)\right| &\le A\left|d_c(\zeta_1,\zeta_2)\right| + B\left|d_c(\zeta_2,\zeta_3)\right|\left|\frac{d_c(\zeta_1,\zeta_2)}{1+d_c(\zeta_1,\zeta_2)}\right| \\ &\quad + E\left|d_c(\zeta_1,\zeta_3)\right|\left|\frac{d_c(\zeta_1,\zeta_2)}{1+d_c(\zeta_1,\zeta_2)}\right| \\ &\le A\left|d_c(\zeta_1,\zeta_2)\right| + B\left|d_c(\zeta_2,\zeta_3)\right| + E\left|d_c(\zeta_1,\zeta_3)\right|. \end{aligned} \tag{29}$$

Then

$$\left|d_c(\zeta_2,\zeta_3)\right| \le \frac{A+E}{(1-B-E)}\left|d_c(\zeta_1,\zeta_2)\right|. \quad (30)$$

Putting $\lambda = \max\{((A+D)/(1-B-D)),((A+E)/(1-B-E))\}$, we obtain a sequence $\{\zeta_n\}$ in ζ such that for $n = 0,1,2,\ldots,\zeta_{2n+1} \in S\zeta_{2n}$ and $\zeta_{2n+2} \in F\zeta_{2n+1}$

$$\begin{aligned}\left|d_c(\zeta_n,\zeta_{n+1})\right| &\le \lambda\left|d_c(\zeta_{n-1},\zeta_n)\right| \\ &\le \lambda^2\left|d_c(\zeta_{n-2},\zeta_{n-1})\right|\cdots \le \lambda^n\left|d_c(\zeta_0,\zeta_1)\right|.\end{aligned} \quad (31)$$

Now for $m > n$, we get

$$\begin{aligned}\left|d_c(\zeta_n,\zeta_m)\right| &\le \left|d_c(\zeta_n,\zeta_{n+1})\right| + \left|d_c(\zeta_{n+1},\zeta_{n+2})\right| \\ &\quad + \cdots + \left|d_c(\zeta_{m-1},\zeta_m)\right| \\ &\le \left[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right]\left|d_c(\zeta_0,\zeta_1)\right| \\ &\le \left[\frac{\lambda^n}{1-\lambda}\right]\left|d_c(\zeta_0,\zeta_1)\right|,\end{aligned} \quad (32)$$

and so

$$\left|d_c(\zeta_n,\zeta_m)\right| \le \frac{\lambda^n}{1-\lambda}\left|d_c(\zeta_0,\zeta_1)\right| \longrightarrow 0 \quad \text{as } m,n \longrightarrow \infty. \quad (33)$$

This implies that $\{\zeta_n\}$ is a Cauchy sequence in ζ. As X is complete space, there exists $u \in X$ such that $\zeta_n \to u$ as $n \to \infty$. We now show that $u \in Fu$ and $u \in Su$. From (14), we have

$$\begin{aligned}&Ad_c(\zeta_{2k},u) + B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + D\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + E\frac{d_c(u,S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \in s(S\zeta_{2k},Fu).\end{aligned} \quad (34)$$

This implies that

$$\begin{aligned}&Ad_c(\zeta_{2k},u) + B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + D\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + E\frac{d_c(u,S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \in \left(\bigcap_{\zeta\in S\zeta_{2k}} s(\zeta,Fu)\right), \\ &Ad_c(\zeta_{2k},u) + B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + D\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + E\frac{d_c(u,S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \in s(\zeta,Fu),\end{aligned} \quad (35)$$

for all $\zeta \in S\zeta_{2k}$. Since $\zeta_{2k+1} \in S\zeta_{2k}$, we have

$$\begin{aligned}&Ad_c(\zeta_{2k},u) + B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + D\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + E\frac{d_c(u,S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \in s(\zeta_{2k+1},Fu).\end{aligned} \quad (36)$$

By definition

$$\begin{aligned}&Ad_c(\zeta_{2k},u) + B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + D\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + E\frac{d_c(u,S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \in s(\zeta_{2k+1},Fu) \\ &= \bigcup_{u'\in Fu} s\left(d_c\left(\zeta_{2k+1},u'\right)\right).\end{aligned} \quad (37)$$

There exists some $u_k \in Fu$ such that

$$\begin{aligned}&Ad_c(\zeta_{2k},u) + B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + D\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{1+d_c(\zeta_{2k},u)} \\ &\quad + E\frac{d_c(u,S\zeta_{2k})\,d_c(u,Fu)}{1+d_c(\zeta_{2k},u)} \in s(d_c(\zeta_{2k+1},u_k));\end{aligned} \quad (38)$$

that is,

$$d_c(\zeta_{2k+1}, u_k) \precsim A d_c(\zeta_{2k}, u) + B\frac{d_c(\zeta_{2k}, S\zeta_{2k})\, d_c(u, Fu)}{1 + d_c(\zeta_{2k}, u)} + C\frac{d_c(u, S\zeta_{2k})\, d_c(\zeta_{2k}, Fu)}{1 + d_c(\zeta_{2k}, u)} + D\frac{d_c(\zeta_{2k}, S\zeta_{2k})\, d_c(\zeta_{2k}, Fu)}{1 + d_c(\zeta_{2k}, u)} + E\frac{d_c(u, S\zeta_{2k})\, d_c(u, Fu)}{1 + d_c(\zeta_{2k}, u)}. \tag{39}$$

By the greatest lower bound property (g.l.b. property) of S and F, we have

$$d_c(\zeta_{2k+1}, u_k) \precsim A d_c(\zeta_{2k}, u) + B\frac{d_c(\zeta_{2k}, \zeta_{2k+1})\, d_c(u, u_k)}{1 + d_c(\zeta_{2k}, u)} + C\frac{d_c(u, \zeta_{2k+1})\, d_c(\zeta_{2k}, u_k)}{1 + d_c(\zeta_{2k}, u)} + D\frac{d_c(\zeta_{2k}, \zeta_{2k+1})\, d_c(\zeta_{2k}, u_k)}{1 + d_c(\zeta_{2k}, u)} + E\frac{d_c(u, \zeta_{2k+1})\, d_c(u, u_k)}{1 + d_c(\zeta_{2k}, u)}. \tag{40}$$

Since

$$d_c(u, u_k) \precsim d_c(u, \zeta_{2k+1}) + d_c(\zeta_{2k+1}, u_k), \tag{41}$$

we get

$$d_c(u, u_k) \precsim d_c(u, \zeta_{2k+1}) + A d_c(\zeta_{2k}, u) + B\frac{d_c(\zeta_{2k}, \zeta_{2k+1})\, d_c(u, u_k)}{1 + d_c(\zeta_{2k}, u)} + C\frac{d_c(u, \zeta_{2k+1})\, d_c(\zeta_{2k}, u_k)}{1 + d_c(\zeta_{2k}, u)} + D\frac{d_c(\zeta_{2k}, \zeta_{2k+1})\, d_c(\zeta_{2k}, u_k)}{1 + d_c(\zeta_{2k}, u)} + E\frac{d_c(u, \zeta_{2k+1})\, d_c(u, u_k)}{1 + d_c(\zeta_{2k}, u)}, \tag{42}$$

which implies that

$$|d_c(u, u_k)| \le |d_c(u, \zeta_{2k+1})| + A|d_c(u, \zeta_{2k+1})| + B\frac{|d_c(\zeta_{2k}, \zeta_{2k+1})|\,|d_c(u, u_k)|}{|1 + d_c(\zeta_{2k}, u)|} + C\frac{|d_c(u, \zeta_{2k+1})|\,|d_c(\zeta_{2k}, u_k)|}{|1 + d_c(\zeta_{2k}, u)|} + D\frac{|d_c(\zeta_{2k}, \zeta_{2k+1})|\,|d_c(\zeta_{2k}, u_k)|}{|1 + d_c(\zeta_{2k}, u)|} + E\frac{|d_c(u, \zeta_{2k+1})|\,|d_c(u, u_k)|}{|1 + d_c(\zeta_{2k}, u)|}. \tag{43}$$

Taking the limit as $k \to \infty$, we get $|d_c(u, u_{k)}| \to 0$ as $k \to \infty$. By lemma 1 [7], we have $u_k \to u$ as $k \to \infty$. Since Fu is closed, $u \in Fu$. Similarly, it follows that $u \in Su$. Hence S and F have a common fixed point and our theorem follows. □

Consequently, we have the following corollaries.

By setting $S = F$ in Theorem 9, we get the following corollary.

Corollary 10. *Let (X, d_c) be a complete complex-valued metric space and let $F : X \to CB(X)$ be multivalued mapping with g.l.b. property such that*

$$A d_c(\zeta, \eta) + B\frac{d_c(\zeta, F\zeta)\, d_c(\eta, F\eta)}{1 + d_c(\zeta, \eta)} + C\frac{d_c(\eta, F\zeta)\, d_c(\zeta, F\eta)}{1 + d_c(\zeta, \eta)} + D\frac{d_c(\zeta, F\zeta)\, d_c(\zeta, F\eta)}{1 + d_c(\zeta, \eta)} + E\frac{d_c(\eta, F\zeta)\, d_c(\eta, F\eta)}{1 + d_c(\zeta, \eta)} \in s(F\zeta, F\eta), \tag{44}$$

for all $\zeta, \eta \in X$, where A, B, C, D, and E are nonnegative real numbers with $A + B + C + 2D + 2E < 1$. Then F has a fixed point.

By choosing $E = 0$ in Theorem 9, we get the following corollary.

Corollary 11. *Let (X, d_c) be a complete complex-valued metric space and let $S, F : X \to CB(X)$ be multivalued mappings with g.l.b. property such that*

$$A d_c(\zeta, \eta) + B\frac{d_c(\zeta, S\zeta)\, d_c(\eta, F\eta)}{1 + d_c(\zeta, \eta)} + C\frac{d_c(\eta, S\zeta)\, d_c(\zeta, F\eta)}{1 + d_c(\zeta, \eta)} + D\frac{d_c(\zeta, S\zeta)\, d_c(\zeta, F\eta)}{1 + d_c(\zeta, \eta)} \in s(S\zeta, F\eta), \tag{45}$$

for all $\zeta, \eta \in X$, where A, B, C, and D are nonnegative real numbers with $A + B + C + 2D < 1$. Then S, F have a common fixed point.

By setting $S = F$ in Corollary 11, we get the following corollary.

Corollary 12. *Let (X, d_c) be a complete complex-valued metric space and let $F : X \to CB(X)$ be multivalued mapping with g.l.b. property such that*

$$\begin{aligned} & A d_c(\zeta,\eta) + B\frac{d_c(\zeta,F\zeta)\,d_c(\eta,F\eta)}{1+d_c(\zeta,\eta)} \\ & \quad + C\frac{d_c(\eta,F\zeta)\,d_c(\zeta,F\eta)}{1+d_c(\zeta,\eta)} + D\frac{d_c(\zeta,F\zeta)\,d_c(\zeta,F\eta)}{1+d_c(\zeta,\eta)} \\ & \quad \in s(F\zeta,F\eta), \end{aligned} \tag{46}$$

for all $\zeta,\eta \in X$, where A, B, C, and D are nonnegative real numbers with $A+B+C+2D<1$. Then F has a fixed point.

By choosing $D = 0$ in Theorem 9, we get the following corollary.

Corollary 13. *Let (X, d_c) be a complete complex-valued metric space and let $S, F : X \to CB(X)$ be multivalued mappings with g.l.b. property such that*

$$\begin{aligned} & A d_c(\zeta,\eta) + B\frac{d_c(\zeta,S\zeta)\,d_c(\eta,F\eta)}{1+d_c(\zeta,\eta)} \\ & \quad + C\frac{d_c(\eta,S\zeta)\,d_c(\zeta,F\eta)}{1+d_c(\zeta,\eta)} + E\frac{d_c(\eta,S\zeta)\,d_c(\eta,F\eta)}{1+d_c(\zeta,\eta)} \\ & \quad \in s(S\zeta,F\eta), \end{aligned} \tag{47}$$

for all $\zeta,\eta \in X$, where A, B, C, and E are nonnegative real numbers with $A+B+C+2E<1$. Then S, F have a common fixed point.

By setting $S = F$ in Corollary 13, we get the following corollary.

Corollary 14. *Let (X, d_c) be a complete complex-valued metric space and let $F : X \to CB(X)$ be multivalued mapping with g.l.b. property such that*

$$\begin{aligned} & A d_c(\zeta,\eta) + B\frac{d_c(\zeta,F\zeta)\,d_c(\eta,F\eta)}{1+d_c(\zeta,\eta)} \\ & \quad + C\frac{d_c(\eta,F\zeta)\,d_c(\zeta,F\eta)}{1+d_c(\zeta,\eta)} + E\frac{d_c(\eta,F\zeta)\,d_c(\eta,F\eta)}{1+d_c(\zeta,\eta)} \\ & \quad \in s(F\zeta,F\eta), \end{aligned} \tag{48}$$

for all $\zeta,\eta \in X$, where A, B, C, and E are nonnegative real numbers with $A+B+C+2E<1$. Then F has a fixed point.

By choosing $D = E = 0$ in Theorem 9, we get the following corollary.

Corollary 15 (see [24]). *Let (X, d_c) be a complete complex-valued metric space and let $S, F : X \to CB(X)$ be multivalued mappings with g.l.b. property such that*

$$\begin{aligned} & A d_c(\zeta,\eta) + B\frac{d_c(\zeta,S\zeta)\,d_c(\eta,F\eta)}{1+d_c(\zeta,\eta)} \\ & \quad + C\frac{d_c(\eta,S\zeta)\,d_c(\zeta,F\eta)}{1+d_c(\zeta,\eta)} \in s(S\zeta,F\eta), \end{aligned} \tag{49}$$

for all $\zeta,\eta \in X$, where A, B, and C are nonnegative real numbers with $A+B+C<1$. Then S, F have a common fixed point.

By setting $S = F$ in Corollary 15, we get the following corollary.

Corollary 16 (see [24]). *Let (X, d_c) be a complete complex-valued metric space and let $F : X \to CB(X)$ be multivalued mapping with g.l.b. property such that*

$$\begin{aligned} & A d_c(\zeta,\eta) + B\frac{d_c(\zeta,F\zeta)\,d_c(\eta,F\eta)}{1+d_c(\zeta,\eta)} \\ & \quad + C\frac{d_c(\eta,F\zeta)\,d_c(\zeta,F\eta)}{1+d_c(\zeta,\eta)} \in s(F\zeta,F\eta), \end{aligned} \tag{50}$$

for all $\zeta,\eta \in X$, where A, B, and C are nonnegative real numbers with $A+B+C<1$. Then F has a fixed point.

By choosing $C = D = E = 0$ in Theorem 9, we get the following corollary.

Corollary 17 (see [24]). *Let (X, d_c) be a complete complex-valued metric space and let $S, F : X \to CB(X)$ be multivalued mappings with g.l.b. property such that*

$$A d_c(\zeta,\eta) + B\frac{d_c(\zeta,S\zeta)\,d_c(\eta,F\eta)}{1+d_c(\zeta,\eta)} \in s(S\zeta,F\eta), \tag{51}$$

for all $\zeta,\eta \in X$, where A, B are nonnegative real numbers with $A+B<1$. Then S, F have a common fixed point.

By setting $S = F$ in Corollary 17, we get the following corollary.

Corollary 18. *Let (X, d_c) be a complete complex-valued metric space and let $F : X \to CB(X)$ be multivalued mapping with g.l.b. property such that*

$$A d_c(\zeta,\eta) + B\frac{d_c(\zeta,F\zeta)\,d_c(\eta,F\eta)}{1+d_c(\zeta,\eta)}, \tag{52}$$

for all $\zeta,\eta \in X$, where A, B are nonnegative real numbers with $A+B<1$. Then F has a fixed point.

By Remark 4, we get the following corollaries.

Corollary 19. *Let (X, d) be a complete metric space and let $S, T : X \to CB(X)$ be multivalued mappings such that*

$$H(Sx, Ty) \le Ad(x, y) + B\frac{d(x, Sx)\,d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Sx)\,d(x, Ty)}{1 + d(x, y)} + D\frac{d(x, Sx)\,d(x, Ty)}{1 + d(x, y)} + E\frac{d(y, Sx)\,d(y, Ty)}{1 + d(x, y)}, \tag{53}$$

for all $x, y \in X$ and A, B, C, D, and E are nonnegative real numbers with $A + B + C + 2D + 2E < 1$. Then S, T have a common fixed point.

Corollary 20. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be multivalued mappings such that*

$$H(Tx, Ty) \le Ad(x, y) + B\frac{d(x, Tx)\,d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Tx)\,d(x, Ty)}{1 + d(x, y)} + D\frac{d(x, Tx)\,d(x, Ty)}{1 + d(x, y)} + E\frac{d(y, Tx)\,d(y, Ty)}{1 + d(x, y)}, \tag{54}$$

for all $x, y \in X$ and A, B, C, D, and E are nonnegative real numbers with $A + B + C + 2D + 2E < 1$. Then T has a fixed point.

Remark 21. By equating A, B, C, D, and E to 0 in all possible combinations, one can derive a host of corollaries which include Banach fixed point theorem for multivalued mappings in complete metric space.

Theorem 22. *Let (X, d_c) be a complete complex-valued metric space and let $S, F : X \to CB(X)$ be multivalued mappings with g.l.b. property such that*

$$Ad_c(\zeta, \eta) + B\frac{d_c(\zeta, S\zeta)\,d_c(\eta, F\eta)}{d_c(\zeta, F\eta) + d_c(\eta, S\zeta) + d_c(\zeta, \eta)} + C\frac{d_c(\eta, S\zeta)\,d_c(\zeta, F\eta)}{d_c(\zeta, F\eta) + d_c(\eta, S\zeta) + d_c(\zeta, \eta)} \in s(S\zeta, F\eta), \tag{55}$$

for all $\zeta, \eta \in X$, where A, B, and C are nonnegative real numbers with $A + B + C < 1$. Then S, F have a common fixed point.

Proof. Let ζ_0 be an arbitrary point in X and $\zeta_1 \in S\zeta_0$. From (55), we have

$$Ad_c(\zeta_0, \zeta_1) + B\frac{d_c(\zeta_0, S\zeta_0)\,d_c(\zeta_1, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} + C\frac{d_c(\zeta_1, S\zeta_0)\,d_c(\zeta_0, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} \in s(S\zeta_0, F\zeta_1). \tag{56}$$

This implies that

$$Ad_c(\zeta_0, \zeta_1) + B\frac{d_c(\zeta_0, S\zeta_0)\,d_c(\zeta_1, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} + C\frac{d_c(\zeta_1, S\zeta_0)\,d_c(\zeta_0, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} \in \left(\bigcap_{\zeta \in S\zeta_0} s(\zeta, F\zeta_1)\right),$$
$$Ad_c(\zeta_0, \zeta_1) + B\frac{d_c(\zeta_0, S\zeta_0)\,d_c(\zeta_1, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} + C\frac{d_c(\zeta_1, S\zeta_0)\,d_c(\zeta_0, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} \in s(\zeta, F\zeta_1), \tag{57}$$

for all $\zeta \in S\zeta_0$. Since $\zeta_1 \in S\zeta_0$, we have

$$Ad_c(\zeta_0, \zeta_1) + B\frac{d_c(\zeta_0, S\zeta_0)\,d_c(\zeta_1, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} + C\frac{d_c(\zeta_1, S\zeta_0)\,d_c(\zeta_0, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} \in s(\zeta_1, F\zeta_1),$$
$$Ad_c(\zeta_0, \zeta_1) + B\frac{d_c(\zeta_0, S\zeta_0)\,d_c(\zeta_1, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} + C\frac{d_c(\zeta_1, S\zeta_0)\,d_c(\zeta_0, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} \in \bigcup_{\zeta \in F\zeta_1} s(d_c(\zeta_1, \zeta)). \tag{58}$$

So there exists some $\zeta_2 \in F\zeta_1$, such that

$$Ad_c(\zeta_0, \zeta_1) + B\frac{d_c(\zeta_0, S\zeta_0)\,d_c(\zeta_1, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} + C\frac{d_c(\zeta_1, S\zeta_0)\,d_c(\zeta_0, F\zeta_1)}{d_c(\zeta_0, F\zeta_1) + d_c(\zeta_1, S\zeta_0) + d_c(\zeta_0, \zeta_1)} \in s(d_c(\zeta_1, \zeta_2)). \tag{59}$$

That is,

$$\begin{aligned} &d_c(\zeta_1,\zeta_2)\\ &\quad\preceq Ad_c(\zeta_0,\zeta_1)+B\frac{d_c(\zeta_0,S\zeta_0)\,d_c(\zeta_1,F\zeta_1)}{d_c(\zeta_0,F\zeta_1)+d_c(\zeta_1,S\zeta_0)+d_c(\zeta_0,\zeta_1)}\\ &\qquad+C\frac{d_c(\zeta_1,S\zeta_0)\,d_c(\zeta_0,F\zeta_1)}{d_c(\zeta_0,F\zeta_1)+d_c(\zeta_1,S\zeta_0)+d_c(\zeta_0,\zeta_1)}. \end{aligned} \tag{60}$$

By using the greatest lower bound property (g.l.b. property) of S and F, we get

$$d_c(\zeta_1,\zeta_2)\preceq Ad_c(\zeta_0,\zeta_1)+B\frac{d_c(\zeta_0,\zeta_1)\,d_c(\zeta_1,\zeta_2)}{d_c(\zeta_0,\zeta_2)+d_c(\zeta_0,\zeta_1)}, \tag{61}$$

which implies that

$$\left|d_c(\zeta_1,\zeta_2)\right|\leq A\left|d_c(\zeta_0,\zeta_1)\right|+B\frac{\left|d_c(\zeta_0,\zeta_1)\right|\left|d_c(\zeta_1,\zeta_2)\right|}{\left|d_c(\zeta_0,\zeta_2)+d_c(\zeta_0,\zeta_1)\right|}. \tag{62}$$

As earlier, by the triangular inequality

$$\left|d_c(\zeta_1,\zeta_2)\right|\leq\left|d_c(\zeta_1,\zeta_0)+d_c(\zeta_0,\zeta_2)\right|, \tag{63}$$

we get

$$\left|d_c(\zeta_1,\zeta_2)\right|\leq(A+B)\left|d_c(\zeta_0,\zeta_1)\right|. \tag{64}$$

Similarly we can prove that

$$\left|d_c(\zeta_2,\zeta_3)\right|\leq(A+B)\left|d_c(\zeta_1,\zeta_2)\right|. \tag{65}$$

By putting $A+B=l<1$ and continuing in a similar way to the proof of Theorem 9, we obtain that $\{\zeta_n\}$ is a Cauchy sequence in X and $\zeta_n\to u$ as $n\to\infty$. We now show that $u\in Fu$ and $u\in Su$. From (55), we have

$$\begin{aligned} &Ad_c(\zeta_{2k},u)+B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad+C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad\in s(S\zeta_{2k},Fu). \end{aligned} \tag{66}$$

This implies that

$$\begin{aligned} &Ad_c(\zeta_{2k},u)+B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad+C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad\in\left(\bigcap_{\zeta\in S\zeta_{2k}}s(\zeta,Fu)\right), \end{aligned} \tag{67}$$

$$\begin{aligned} &Ad_c(\zeta_{2k},u)+B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad+C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad\in s(\zeta,Fu), \end{aligned} \tag{68}$$

for all $\zeta\in S\zeta_{2k}$. Since $\zeta_{2k+1}\in S\zeta_{2k}$, we have

$$\begin{aligned} &Ad_c(\zeta_{2k},u)+B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad+C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad\in s(\zeta_{2k+1},Fu). \end{aligned} \tag{69}$$

By definition

$$\begin{aligned} &Ad_c(\zeta_{2k},u)+B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad+C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad\in s(\zeta_{2k+1},Fu)=\bigcup_{u'\in Fu}s\left(d_c(\zeta_{2k+1},u')\right). \end{aligned} \tag{70}$$

There exists some $u_k\in Fu$ such that

$$\begin{aligned} &Ad_c(\zeta_{2k},u)+B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad+C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\quad\in s(d_c(\zeta_{2k+1},u_k)); \end{aligned} \tag{71}$$

that is,

$$\begin{aligned} &d_c(\zeta_{2k+1},u_k)\\ &\quad\preceq Ad_c(\zeta_{2k},u)+B\frac{d_c(\zeta_{2k},S\zeta_{2k})\,d_c(u,Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}\\ &\qquad+C\frac{d_c(u,S\zeta_{2k})\,d_c(\zeta_{2k},Fu)}{d_c(\zeta_{2k},Fu)+d_c(u,S\zeta_{2k})+d_c(\zeta_{2k},u)}. \end{aligned} \tag{72}$$

By the greatest lower bound property (g.l.b. property) of S and F, we have

$$d_c(\zeta_{2k+1}, u_k) \preceq A d_c(\zeta_{2k}, u) + B\frac{d_c(\zeta_{2k}, \zeta_{2k+1}) d_c(u, Fu)}{d_c(\zeta_{2k}, Fu) + d_c(u, S\zeta_{2k}) + d_c(\zeta_{2k}, u)} + C\frac{d_c(u, \zeta_{2k+1}) d_c(\zeta_{2k}, Fu)}{d_c(\zeta_{2k}, Fu) + d_c(u, S\zeta_{2k}) + d_c(\zeta_{2k}, u)}. \quad (73)$$

Since

$$d_c(u, u_k) \preceq d_c(u, \zeta_{2k+1}) + d_c(\zeta_{2k+1}, u_k), \quad (74)$$

we get

$$d_c(u, u_k) \preceq d_c(u, \zeta_{2k+1}) + A d_c(\zeta_{2k}, u) + B\frac{d_c(\zeta_{2k}, \zeta_{2k+1}) d_c(u, Fu)}{d_c(\zeta_{2k}, Fu) + d_c(u, S\zeta_{2k}) + d_c(\zeta_{2k}, u)} + C\frac{d_c(u, \zeta_{2k+1}) d_c(\zeta_{2k}, Fu)}{d_c(\zeta_{2k}, Fu) + d_c(u, S\zeta_{2k}) + d_c(\zeta_{2k}, u)}, \quad (75)$$

which implies that

$$|d_c(u, u_k)| \le |d_c(u, \zeta_{2k+1})| + A|d_c(\zeta_{2k}, u)| + B\frac{|d_c(\zeta_{2k}, \zeta_{2k+1})| |d_c(u, Fu)|}{|d_c(\zeta_{2k}, Fu) + d_c(u, S\zeta_{2k}) + d_c(\zeta_{2k}, u)|} + C\frac{|d_c(u, \zeta_{2k+1})| |d_c(\zeta_{2k}, Fu)|}{|d_c(\zeta_{2k}, Fu) + d_c(u, S\zeta_{2k}) + d_c(\zeta_{2k}, u)|}. \quad (76)$$

Taking the limit as $k \to \infty$, we get $|d_c(u, u_k)| \to 0$ as $k \to \infty$. By lemma 1 [7], we have $u_k \to u$ as $k \to \infty$. Since Fu is closed, so $u \in Fu$. Similarly, it follows that $u \in Su$. Hence S and F have a common fixed point and our theorem follows. □

Corollary 23. *Let (X, d) be a complete complex-valued metric space and let $T : X \to CB(X)$ be multivalued mapping with g.l.b. property such that*

$$Ad(x, y) + B\frac{d(x, Tx) d(y, Ty)}{d(x, Ty) + d(y, Tx) + d(x, y)} + C\frac{d(y, Tx) d(x, Ty)}{d(x, Ty) + d(y, Tx) + d(x, y)} \in s(Tx, Ty), \quad (77)$$

for all $x, y \in X$, where A, B, and C are nonnegative real numbers with $A + B + C < 1$. Then T has fixed point.

Now, let us consider the following example.

Example 24. Let $X = [0, 1]$; define $d_c : X \times X \to \mathbb{C}$ by

$$d_c(\zeta, \eta) = |\zeta - \eta| e^{i\theta}, \quad \theta = \tan^{-1}\left|\frac{\eta}{\zeta}\right|. \quad (78)$$

Then (X, d_c) is a complex-valued metric space. Let $S, F : X \to CB(X)$ be the mappings defined by

$$S\zeta = \left[0, \frac{1}{6}\zeta\right], \quad F\zeta = \left[0, \frac{1}{12}\zeta\right]. \quad (79)$$

The contractive condition of main theorem is trivial for the case when $\zeta = \eta = 0$. Suppose without any loss of generality that all ζ, η are nonzero and $\zeta < \eta$. Then

$$d_c(\zeta, \eta) = |\eta - \zeta| e^{i\theta}, \quad d_c(\zeta, S\zeta) = \left|\zeta - \frac{\zeta}{6}\right| e^{i\theta},$$
$$d_c(\eta, F\eta) = \left|\eta - \frac{\eta}{12}\right| e^{i\theta}, \quad d_c(\eta, S\zeta) = \left|\eta - \frac{\zeta}{6}\right| e^{i\theta},$$
$$d_c(\zeta, F\eta) = \left|\zeta - \frac{\eta}{12}\right| e^{i\theta},$$
$$s(S\zeta, F\eta) = s\left(\left|\frac{\zeta}{6} - \frac{\eta}{12}\right| e^{i\theta}\right). \quad (80)$$

It is clear that, for any value of B, C, D, E, and $A = 1/6$, we have

$$\left|\frac{\zeta}{6} - \frac{\eta}{12}\right| \le \frac{1}{6}|\eta - \zeta|. \quad (81)$$

Thus

$$Ad_c(\zeta, \eta) + B\frac{d_c(\zeta, S\zeta) d_c(\eta, F\eta)}{1 + d_c(\zeta, \eta)} + C\frac{d_c(\eta, S\zeta) d_c(\zeta, F\eta)}{1 + d_c(\zeta, \eta)} + D\frac{d_c(\zeta, S\zeta) d_c(\zeta, F\eta)}{1 + d_c(\zeta, \eta)} + E\frac{d_c(\eta, S\zeta) d_c(\eta, F\eta)}{1 + d_c(\zeta, \eta)} \in s(S\zeta, F\eta). \quad (82)$$

Hence all the conditions of our main Theorem 9 are satisfied and 0 is a common fixed point of S and F.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All the authors contributed equally and significantly to writing this paper. All the authors read and approved the final paper.

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