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Research Article

Nonlinear Sum Operator Equations with a Parameter and Application to Second-Order Three-Point BVPs

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A class of nonlinear sum operator equations with a parameter on order Banach spaces were considered. The existence and uniqueness of positive solutions for this kind of operator equations and the dependence of solutions on the parameter have been obtained by using the properties of cone and nonlinear analysis methods. The critical value of the parameter was estimated. Further, the application to some nonlinear three-point boundary value problems was given to show the significance of the discussion.

1. Introduction and Preliminaries

The aim of this paper is to investigate the existence and uniqueness of positive solution for the following operator equations:

$$L(\lambda, x) = x, \tag{1}$$

where $L(\lambda, x) = Ax + \lambda Bx$, A is an operator with concavity, B is a pseudo subhomogeneous operator, and λ is a parameter. In addition, by applying our results to the second-order three-point boundary value problem (BVP),

$$\begin{gathered} -x''(t) = \lambda f(t, x(t)) + g(t), \quad t \in (0,1), \\ x(0) = \tau x'(0), \qquad x(1) - \sigma x(\xi) = \int_0^1 h(x(s))\, ds, \end{gathered} \tag{2}$$

where $f \in C([0,1] \times R)$, $g \in C[0,1]$, $h \in C(R)$, $\tau \geq 0$, $0 < \xi < 1$, and $0 < \sigma \leq 1$; we obtain that there exists a $\lambda^* > 0$, such that BVP(2) has a unique positive solution and no positive solution for $0 < \lambda < \lambda^*$ and $\lambda \geq \lambda^*$, respectively. In particular, such a positive solution $x_\lambda(t)$ of BVP(2) is increasing and continuous in λ for $\lambda \in [0, \lambda^*)$ and $\lim_{\lambda \to \lambda^* - 0} \|x_\lambda\|_c = +\infty$. Further, we estimate the critical value λ^*.

In recent years, many authors focus on multipoint boundary value problems for differential equations, since these problems arise in a variety of different areas of applied mathematics and physics (see [1]). For example, by using degree-theoretic arguments, Gupta [2] obtained the existence and uniqueness theorems for the following three-point boundary value problem:

$$\begin{gathered} x''(t) = f\left(t, x(t), x'(t)\right) - e(t), \quad t \in (0,1), \\ x(0) = 0, \qquad x(\xi) = x(1). \end{gathered} \tag{3}$$

Since then, the existence of solutions for nonlinear multipoint boundary value problems has been studied by many authors (see [3–8] and their references). The cases with special boundary value conditions in BVP(2) were discussed by [2–7], where $\tau = 0$ and $h(x) = 0$ or $\tau = 0$, $\sigma = 1$ and $h(x) = 0$. However, to the best of our knowledge, little has been done for the multipoint boundary value problem (2) with parameter, perturbed loading force, and nonlinear boundary conditions, especially on the existence and uniqueness of positive solution and the dependence of solutions on the parameter λ.

It is well known that fixed point theory is an effective tool in the treatment of existence results of boundary value problems for nonlinear differential equations. Many researchers were concerned with the existence and uniqueness of fixed point for concave operators and sum operators. For example, [9–15] investigated eigenvalue problems of concave operators, the existence and uniqueness of positive fixed point for concave operators, and the existence and uniqueness of positive fixed point for sum operator, respectively. However, to our knowledge, few of the results in literature can be applied to BVP(2) successfully. The above reasons stimulate us to do this work.

First, we consider the existence and property of positive solutions for nonlinear operator equations (1) on order Banach space E.

To be clear, some definitions, notations, and lemmas are presented as follows.

Let E be a real Banach space which is partially ordered by a cone P of E, that is, $x \leq y$ if and only if $y - x \in P$, and let θ be the zero element of E. If $x \leq y$ and $x \neq y$, then we denote $x < y$ or $y > x$. A nonempty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P$, $r \geq 0 \Rightarrow rx \in P$; (ii) $x \in P, -x \in P \Rightarrow x = \theta$. A cone P is said to be normal if there exists a positive number N, called the normal constant of P, such that, for all $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$.

Let $D \subseteq E$. An operator $T : D \to E$ is said to be increasing if, for $x_1, x_2 \in D$, $x_1 \leq x_2 \Rightarrow Tx_1 \leq Tx_2$. An element $x^* \in D$ is called a fixed point of T if $Tx^* = x^*$. For details on cone theory, see [16, 17].

Given $e > \theta$, let

$$\overline{P_e} = \{x \in E \mid \text{there exist } l = l(x) > 0 \text{ such that } 0 \leq x \leq le\}, \tag{4}$$

$$\begin{aligned} P_e = \{&x \in E \mid \text{there exist } l_1 = l_1(x) > 0, \\ &l_2 = l_2(x) > 0 \text{ such that } l_1 e \leq x \leq l_2 e\}. \end{aligned} \tag{5}$$

Then $P_e \subset \overline{P_e} \subset P$ and

$$\begin{aligned} &\forall x, y \in P_e, \quad \exists 0 < \mu_0 < 1 < l_0 < +\infty \\ &\text{such that } \mu_0 y \leq x \leq l_0 y. \end{aligned} \tag{6}$$

Lemma 1. *Let P be a normal cone in E and $T : P \to P$ an increasing operator. Suppose that*

(H1) $T(P_e) \subset P_e$;

(H2) *for any $r \in (0,1)$ and $[y, z] \subset P_e$ there exists $\eta(r, y, z) > 0$ such that*

$$T(rx) \geq r(1 + \eta(r, y, z))Tx, \quad \forall x \in [y, z], \ r \in (0,1). \tag{7}$$

Then T has a unique fixed point x^* in P_e if and only if there exist $u, v \in P_e$ such that $u \leq Tu \leq Tv \leq v$. Moreover, for any initial value $u_0 \in P_e$ and a sequence $u_n = Tu_{n-1}$ $(n = 1, 2, \ldots)$, one has $\lim_{n\to+\infty}\|u_n - x^*\| = 0$.

The proof of Lemma 1 is standard; we omit it here.

Definition 2 (see [14]). An operator $T : P \to P$ is said to be generalized α-concave if it satisfies (H1) and the following condition:

(H3) there exists an $\alpha : (0,1) \to (0,1)$ such that

$$T(rx) \geq r^{\alpha(r)} Tx, \quad \forall x \in P_e, \ r \in (0,1). \tag{8}$$

From Lemma 1 it is easy to show that the following lemma holds.

Lemma 3 (see [15]). *Let P be a normal cone of E and $T : P \to P$ increasing generalized α-concave. Then T has a unique fixed point x^* in P_e. Moreover, for any initial value $u_0 \in P_e$ and a sequence $u_n = Tu_{n-1}$ $(n = 1, 2, \ldots)$, one has $\lim_{n\to+\infty}\|u_n - x^*\| = 0$.*

In what follows, we introduce definitions of pseudo subhomogeneous operator and pseudo generalized α-concave operator.

Definition 4. An operator $T : P \to P$ is said to be pseudo subhomogeneous if it satisfies

(H4) $T(P_e) \subset \overline{P_e}$;

(H5) $T(rx) \geq rTx$, $\forall x \in P_e$, $r \in (0,1)$.

Definition 5. An operator $T : P \to P$ is said to be pseudo generalized α-concave if it satisfies (H3) and (H4).

Remark 6. An increasing pseudo subhomogeneous operator and an increasing pseudo generalized α-concave operator may have no fixed point in P_e. For example, Let $E = C[0,1]$, $P = \{x \in E \mid x(t) \geq 0, t \in [0,1]\}$, $e(t) \equiv 1$, and $T_1 x(t) = (1-t)x(t)/(1 + x(t))$. Obviously, $T_1 : P \to P$ is increasing pseudo subhomogeneous, but T_1 has no fixed point in P_e. Let

$$T_2 x(t) = \int_0^1 t(1-s)x^{\alpha}(s)\,ds, \quad x \in P, \tag{9}$$

$0 < \alpha < 1$. Clearly, $T_2 : P \to P$ is increasing pseudo generalized α-concave. Since $T_2 x(0) = 0$ for $x \in P_e$, then $T_2 x \notin P_e$ for all $x \in P_e$. Therefore, T_2 is not generalized α-concave and T_2 has no fixed point in P_e.

Remark 7. From Definitions 4 and 5, it is clear that a pseudo generalized α-concave operator is a pseudo subhomogeneous operator.

Remark 8. It is easy to show that (8) is equivalent to

$$T(sx) \leq s^{\alpha(1/s)} Tx, \quad x \in P_e, \ s > 1, \tag{10}$$

and (H5) is equivalent to

$$T(sx) \leq sTx, \quad x \in P_e, \ s > 1. \tag{11}$$

2. Positive Solutions of Operator Equation

In this section, we assume that E is a real Banach space, P is a normal cone in E with the normal constant N, $e > \theta$, and $A, B : P \to P$ are increasing operators.

Theorem 9. *Assume that A is a generalized α-concave operator and B is a pseudo subhomogeneous operator. Then the following four results are true.*

(i) *There exists a $\lambda^* > 0$ such that (1) has a unique solution x_λ in P_e for $\lambda \in [0, \lambda^*)$. For any initial value $u_0 \in P_e$, set $u_n = L(\lambda, u_{n-1})$ $(n = 1, 2, \ldots)$; then $\lim_{n\to+\infty}\|u_n - x_\lambda\| = 0$.*

(ii) *Equation (1) has no solution in P_e for $\lambda \geq \lambda^*$.*

(iii) *x_λ is increasing in λ for $\lambda \in [0, \lambda^*)$.*

(iv) *x_λ is continuous with respect to λ for $\lambda \in [0, \lambda^*)$.*

Proof. By Lemma 3, A has a unique fixed point $x_0 \in P_e$ and

$$L(\lambda, x_0) = Ax_0 + \lambda Bx_0 \geq Ax_0 = x_0, \quad \lambda \geq 0. \tag{12}$$

If $B|_{P_e} \equiv \theta$, $L(\lambda, \cdot)|_{P_e} = A$ for any $\lambda \geq 0$. Set $\lambda^* = +\infty$; it is obvious that conclusions (i)–(iv) hold.

If $B|_{P_e} \not\equiv \theta$, there exists $\overline{x} \in P_e$ such that $B\overline{x} > \theta$. By (6), for any $x \in P_e$, there exists $\mu_0 \in (0,1)$ such that $\mu_0 \overline{x} \leq x$. Moreover, Definition 4 shows that there exists $l_0 \in (1, +\infty)$ such that

$$\theta < \mu_0 B\overline{x} \leq Bx \leq l_0 x_0, \quad x \in P_e. \tag{13}$$

The facts that $A(P_e) \subset P_e$ and $B(P_e) \subset \overline{P_e}$ imply that

$$L(\lambda, P_e) \subset P_e, \quad \lambda \geq 0. \tag{14}$$

Next, we prove all statements by five steps.

Step 1. Existence of the critical value λ^*. Set

$$\Lambda = \{\lambda \geq 0 \mid \text{there exists } y_\lambda \in P_e \text{ such that } x_0 \leq y_\lambda, L(\lambda, y_\lambda) \leq y_\lambda\}, \tag{15}$$

and $\lambda^* = \sup \Lambda$. Now we show that $\Lambda = [0, \lambda^*)$.

Define a mapping $\rho : P_e \to [0, +\infty)$ by

$$\rho(x) = \inf\{\tau > 0 \mid Bx \leq \tau x_0\}, \quad x \in P_e. \tag{16}$$

By (13) it is obvious that $0 < \rho(x) < +\infty$. In addition, for any $x_1, x_2 \in P_e$, $x_1 \leq x_2$, we have $Bx_1 \leq Bx_2 \leq \rho(x_2)x_0$, which implies that $\rho(x_1) \leq \rho(x_2)$. That is, $\rho(x)$ is increasing in x for $x \in P_e$.

For a given $s_0 > 1$, set $y_0 = s_0 x_0$. Then $y_0 \in P_e$ and $x_0 \leq y_0$. It follows from Remark 8 that

$$\begin{gathered} L(\lambda, y_0) \leq s_0\left(s_0^{\alpha(1/s_0)-1} + \lambda\rho(x_0)\right)x_0 \leq y_0, \\ \lambda \in \left[0, \frac{1}{\rho(x_0)}\left(1 - s_0^{\alpha(1/s_0)-1}\right)\right]. \end{gathered} \tag{17}$$

This means that

$$\begin{gathered} \left[0, \frac{1}{\rho(x_0)}\left(1 - s_0^{\alpha(1/s_0)-1}\right)\right] \subset \Lambda, \\ \lambda^* \geq \frac{1}{\rho(x_0)}\left(1 - s_0^{\alpha(1/s_0)-1}\right) > 0. \end{gathered} \tag{18}$$

We assert that $\lambda^* \notin \Lambda$. If $\lambda^* = +\infty$, from (15) it is obvious that $\lambda^* \notin \Lambda$. Suppose that $\lambda^* < +\infty$ and $\lambda^* \in \Lambda$. Then again by (15) there exists $y_{\lambda^*} \in P_e$ with $x_0 \leq y_{\lambda^*}$ such that $L(\lambda^*, y_{\lambda^*}) \leq y_{\lambda^*}$. Set $v_0 = s_1 y_{\lambda^*}$ for a given $s_1 > 1$. Then $v_0 \in P_e$ and $x_0 \leq v_0$. Note that $(1/\rho(y_{\lambda^*}))(1 - s_1^{\alpha(1/s_1)-1}) > 0$; we can take a number $\delta > 0$ sufficiently small such that

$$\delta < \frac{1}{\rho(y_{\lambda^*})}\left(1 - s_1^{\alpha(1/s_1)-1}\right). \tag{19}$$

From (16)–(19) and the fact that $x_0 = Ax_0 \leq Ay_{\lambda^*}$, we obtain

$$\begin{aligned} L(\lambda^* + \delta, v_0) &= A(s_1 y_{\lambda^*}) + (\lambda^* + \delta) B(s_1 y_{\lambda^*}) \\ &\leq s_1 (Ay_{\lambda^*} + \lambda^* By_{\lambda^*}) + \left(s_1^{\alpha(1/s_1)} - s_1\right) Ay_{\lambda^*} \\ &\quad + \delta s_1 By_{\lambda^*} \\ &\leq s_1 L(\lambda^*, y_{\lambda^*}) - \left(s_1 - s_1^{\alpha(1/s_1)}\right) Ax_0 \\ &\quad + \delta s_1 \rho(y_{\lambda^*}) x_0 \\ &\leq s_1 y_{\lambda^*} + \left(\delta s_1 \rho(y_{\lambda^*}) - \left(s_1 - s_1^{\alpha(1/s_1)}\right)\right) x_0 \\ &\leq s_1 y_{\lambda^*} = v_0, \end{aligned} \tag{20}$$

which means that $\lambda^* + \delta \in \Lambda$. This contradicts the definition of λ^*. Therefore, we conclude that

$$\Lambda = [0, \lambda^*). \tag{21}$$

Step 2. Conclusion (i) holds. For given $\lambda \in [0, \lambda^*)$, consider (1). Since A and B are increasing, $L(\lambda, x)$ is increasing in x. Moreover, combining (12) and (15) gives

$$x_0 \leq L(\lambda, x_0) \leq L(\lambda, y_\lambda) \leq y_\lambda. \tag{22}$$

Besides, by (H2) and (H4) we obtain

$$\begin{aligned} L(\lambda, rx) &\geq r^{\alpha(r)} Ax + \lambda r Bx \\ &\geq rL(\lambda, x) + \left(r^{\alpha(r)} - r\right) Ax \\ &\geq r\left(1 + \frac{r^{\alpha(r)} - r}{r}\varphi(x, \lambda)\right) L(\lambda, x), \\ &\qquad x \in P_e,\ r \in (0,1), \end{aligned} \tag{23}$$

where

$$\varphi(\lambda, x) = \sup\{\tau > 0 \mid Ax \geq \tau L(\lambda, x)\}, \quad x \in P_e. \tag{24}$$

For any $u, v \in P_e$ with $u \leq v$, (H1) implies that $Au, Av \in P_e$. Further, there exists $0 < \epsilon \leq 1$ such that $\epsilon Av \leq Au$. Hence,

$$\begin{aligned} Ax &\geq Au \geq \epsilon Av \geq \epsilon\varphi(\lambda, v) L(\lambda, v) \\ &\geq \epsilon\varphi(\lambda, v) L(\lambda, x), \quad x \in [u, v]. \end{aligned} \tag{25}$$

Evidently,

$$\varphi(\lambda, x) \geq \epsilon\varphi(\lambda, v) > 0, \quad \forall x \in [u, v]. \tag{26}$$

Therefore, let $\eta(r,u,v) = (\epsilon(r^{\alpha(r)} - r)/r)\varphi(\lambda, v)$; then it follows from (23) and (26) that

$$L(\lambda, rx) \geq r(1 + \eta(r,u,v)) L(\lambda, x), \quad x \in [u, v], \ r \in (0,1). \tag{27}$$

The application of Lemma 1 concludes the proof of (i).

Step 3. Conclusion (ii) holds. Suppose that there exists $\lambda' \geq \lambda^*$ such that $L(\lambda', x) = x$ has a solution $x_{\lambda'}$ in P_e. Since $x_0, x_{\lambda'} \in P_e$, by (6) there exists $s' > 1$ such that $x_0 \leq s' x_{\lambda'}$. Set $y_{\lambda'} = s' x_{\lambda'}$. Then

$$\begin{aligned} L(\lambda', y_{\lambda'}) &= A(s' x_{\lambda'}) + \lambda' B(s' x_{\lambda'}) \\ &\leq s' \left((s')^{\alpha(1/s') - 1} A x_{\lambda'} + \lambda' \ B x_{\lambda'} \right) \leq s' x_{\lambda'} = y_{\lambda'}, \end{aligned} \tag{28}$$

which means that $\lambda' \in \Lambda$. Equation (21) implies that $\lambda' < \lambda^*$, which is a contradiction to the hypothesis $\lambda' \geq \lambda^*$.

Step 4. Conclusion (iii) holds. Let $\lambda_1, \lambda_2 \in [0, \lambda^*)$ with $\lambda_1 \leq \lambda_2$. Then $L(\lambda_1, x_{\lambda_2}) \leq L(\lambda_2, x_{\lambda_2}) = x_{\lambda_2}$, and, further, $x_0 \leq L(\lambda_1, x_0) \leq L(\lambda_1, x_{\lambda_2}) \leq x_{\lambda_2}$. By the proof of conclusion (i), $x = L(\lambda_1, x)$ has a unique solution $x^*_{\lambda_1} \in [x_0, x_{\lambda_2}]$ in P_e, which implies that $x_{\lambda_1} = x^*_{\lambda_1}$. Thus, $x_{\lambda_1} \leq x_{\lambda_2}$.

Step 5. Conclusion (iv) holds. Let $\lambda_0 \in (0, \lambda^*)$. By conclusion (iii) we have

$$x_\lambda \leq x_{\lambda_0}, \quad \forall 0 < \lambda < \lambda_0. \tag{29}$$

Let

$$\tau_\lambda = \sup\left\{\tau > 0 x_\lambda \geq \tau x_{\lambda_0}\right\}, \quad 0 < \lambda < \lambda_0. \tag{30}$$

Then τ_λ is nondecreasing in λ for $\lambda \in [0, \lambda_0)$, and

$$0 < \tau_\lambda \leq 1, \quad x_\lambda \geq \tau_\lambda x_{\lambda_0}, \quad 0 < \lambda < \lambda_0. \tag{31}$$

We assert that

$$\lim_{\lambda \to \lambda_0^-} \tau_\lambda = 1, \tag{32}$$

if, otherwise, there exists a sequence $\{\lambda_n\}$

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \leq \lambda_0, \quad \lim_{n \to +\infty} \lambda_n = \lambda_0 \tag{33}$$

such that

$$\lim_{n \to +\infty} \tau_{\lambda_n} = \tau_0, \quad \tau_{\lambda_n} \leq \tau_0 < 1. \tag{34}$$

By (23), (24), and (30), we obtain

$$\begin{aligned} x_{\lambda_n} &= L(\lambda_n, x_{\lambda_n}) \geq L(\lambda_n, \tau_{\lambda_n} x_{\lambda_0}) \\ &= L\left(\lambda_n, \frac{\tau_{\lambda_n}}{\tau_0} \tau_0 x_{\lambda_0}\right) \geq \frac{\tau_{\lambda_n}}{\tau_0} L(\lambda_n, \tau_0 x_{\lambda_0}) \\ &\geq \frac{\lambda_n}{\lambda_0} \frac{\tau_{\lambda_n}}{\tau_0} L(\lambda_0, \tau_0 x_{\lambda_0}) \\ &\geq \frac{\lambda_n}{\lambda_0} \tau_{\lambda_n} \left(1 + \frac{\tau_0^{\alpha(\tau_0)} - \tau_0}{\tau_0} \varphi(x_{\lambda_0}, \lambda_0)\right) x_{\lambda_0}. \end{aligned} \tag{35}$$

Therefore,

$$\tau_{\lambda_n} \geq \frac{\lambda_n}{\lambda_0} \tau_{\lambda_n} \left(1 + \frac{\tau_0^{\alpha(\tau_0)} - \tau_0}{\tau_0} \varphi(x_{\lambda_0}, \lambda_0)\right). \tag{36}$$

That is,

$$\frac{\lambda_n}{\lambda_0} \left(1 + \frac{\tau_0^{\alpha(\tau_0)} - \tau_0}{\tau_0} \varphi(x_{\lambda_0}, \lambda_0)\right) \leq 1. \tag{37}$$

Taking the limit $n \to +\infty$, we get

$$1 + \frac{\tau_0^{\alpha(\tau_0)} - \tau_0}{\tau_0} \varphi(x_{\lambda_0}, \lambda_0) \leq 1, \tag{38}$$

which is a contradiction. So, (32) holds.

From (29), (31), and (32), we have

$$\left\|x_{\lambda_0} - x_\lambda\right\| \leq N(1 - \tau_\lambda) \left\|x_{\lambda_0}\right\|, \quad 0 < \lambda < \lambda_0, \tag{39}$$

which implies that $\|x_{\lambda_0} - x_\lambda\| \to 0$ as $\lambda \to \lambda_0^-$.

A similar argument shows that, for any $\lambda_0 \in [0, \lambda^*)$, $\|x_\lambda - x_{\lambda_0}\| \to 0$ as $\lambda \to \lambda_0^+$. Thus, conclusion (iv) holds. The proof of Theorem 9 is complete. □

Noting (18) and (21), we can easily obtain the following result.

Theorem 10. *Assume that the hypotheses in Theorem 9 hold. If* $\lim_{r \to 0^+} r^{1-\alpha(r)} = 0$, *then* λ^* *in Theorem 9 satisfies*

$$\lambda^* \geq \frac{1}{\rho(x_0)}, \tag{40}$$

where $\rho(x_0)$ *are defined by* (16).

Corollary 11. *Assume that* A *is a generalized* α_1*-concave operator and* B *is a pseudo generalized* α_2*-concave operator. Then*

(i) (1) *has a unique solution* $x_\lambda \in P_e$ *for* $\lambda \in [0, +\infty)$. *Moreover, for any* $u_0 \in P_e$, *set* $u_n = L(\lambda, u_{n-1})$ $(n = 1, 2, \ldots)$; *then* $\lim_{n \to +\infty} \|u_n - x_\lambda\| = 0$;

(ii) x_λ *is increasing in* λ *for* $\lambda \in [0, +\infty)$;

(iii) x_λ *is continuous with respect to* λ *for* $\lambda \in [0, +\infty)$;

(iv) *either* $x_\lambda = x_0$ *(the unique fixed point of* A *in* P_e*) or* $\lim_{\lambda\to+\infty}\|x_\lambda\| = +\infty$.

Proof. From Definitions 2 and 4, it is clear that $L(\lambda, P_e) \subset P_e$. Since A and B are increasing, $L(\lambda, x)$ is increasing in x for any $\lambda \geq 0$. Let $\alpha(r) = \max\{\alpha_1(r), \alpha_2(r)\}$; then

$$\begin{aligned} L(\lambda, rx) &= A(rx) + \lambda B(rx) \\ &\geq r^{\alpha_1(r)}Ax + \lambda r^{\alpha_2(r)}Bx \\ &\geq r^{\alpha(r)}L(\lambda, x), \quad x \in P_e,\ r \in (0,1). \end{aligned} \tag{41}$$

Thus, conclusion (i) follows from Lemma 3. Similar to the proofs of Theorem 9, the proofs of (ii) and (iii) can be completed.

Note that

$$\begin{aligned} x_\lambda &= Ax_\lambda = x_0, \quad \text{if } B|_{P_e} \equiv \theta, \\ x_\lambda &= Ax_\lambda + \lambda Bx_\lambda \geq \lambda Bx_\lambda \geq \lambda Bx_0, \quad \text{if } B|_{P_e} \not\equiv \theta, \end{aligned} \tag{42}$$

and, therefore, normality of P implies that conclusion (iv) holds. This ends the proof. □

Theorem 12. *Assume that* A *is a pseudo generalized* α_1*-concave operator and* B *is a generalized* α_2*-concave operator. Then,*

(i) (1) *has a unique solution* $x_\lambda \in P_e$ *for* $\lambda \in (0, +\infty)$. *Moreover, for any* $u_0 \in P_e$, *set* $u_n = L(\lambda, u_{n-1})$ $(n = 1, 2, \ldots)$; *then* $\lim_{n\to+\infty}\|u_n - x_\lambda\| = 0$;

(ii) x_λ *is increasing with respect to* λ *for* $\lambda \in (0, +\infty)$;

(iii) x_λ *is continuous with respect to* λ *for* $\lambda \in (0, +\infty)$;

(iv) $\lim_{\lambda\to 0}\|x_\lambda - Ax_\lambda\| = 0$; $\lim_{\lambda\to+\infty}\|x_\lambda\| = +\infty$.

Proof. Similar to the proof of Corollary 11, the proofs of (i), (ii), and (iii) can be completed. To prove (iv), let $x_1 \in P_e$ be the unique solution of (1) with $\lambda = 1$. From conclusion (ii) of this theorem, we obtain

$$\begin{aligned} \theta < x_\lambda - Ax_\lambda &= \lambda Bx_\lambda \leq \lambda Bx_1, \quad 0 < \lambda < 1, \\ x_\lambda = Ax_\lambda + \lambda Bx_\lambda &\geq \lambda Bx_\lambda \geq \lambda Bx_1 > \theta, \quad \lambda > 1. \end{aligned} \tag{43}$$

Therefore,

$$\begin{aligned} \|x_\lambda - Ax_\lambda\| &\leq N\lambda \|Bx_1\|, \quad 0 < \lambda < 1, \\ \|x_\lambda\| &\geq \frac{\lambda}{N}\|Bx_1\|, \quad \lambda > 1, \end{aligned} \tag{44}$$

which implies that $\lim_{\lambda\to 0}\|x_\lambda - Ax_\lambda\| = 0$ and $\lim_{\lambda\to+\infty}\|x_\lambda\| = +\infty$. This ends the proof. □

Next, we discuss the case of (1) with $L(\lambda, x) = x_0 + \lambda Bx$, $x_0 \in E$; that is,

$$x_0 + \lambda Bx = x \tag{45}$$

which can be widely applied to various problems for differential equations.

Theorem 13. *Assume that* $x_0 \in P_e$ *and* B *is a pseudo subhomogeneous operator. Then*

(i) *there exists a* $\lambda^* \geq (1/\rho_0) > 0$ *such that* (45) *has a unique solution* $x_\lambda \in P_e$ *for* $\lambda \in [0, \lambda^*)$. *Moreover, for any* $u_0 \in P_e$ *and a sequence* $u_n = x_0 + \lambda Bu_{n-1}$ $(n = 1, 2, \ldots)$, *one has* $\lim_{n\to+\infty}\|u_n - x_\lambda\| = 0$;

(ii) (45) *has no solution in* P_e *for* $\lambda \geq \lambda^*$;

(iii) x_λ *is increasing in* λ *for* $\lambda \in [0, \lambda^*)$;

(iv) x_λ *is continuous with respect to* λ *for* $\lambda \in [0, \lambda^*)$; *moreover,* $\lim_{\lambda\to 0}\|x_\lambda - x_0\| = 0$;

(v) *further, if* $Bx \in \overline{P_e}$ *for* $x \geq x_0$ *and* B *is completely continuous, then* $\lim_{\lambda\to\lambda^*-0}\|x_\lambda\| = +\infty$.

Here $\rho_0 = \inf\{\tau > 0 \mid Bx_0 \leq \tau x_0\}$.

Proof. Define an operator A by $Ax = x_0$ for $x \in P$. Then $A : P \to P_e$ is increasing and x_0 is the unique fixed point of A in P_e.

Conclusions (i)–(iv) can be proved similarly to the proof of Theorem 9. We only prove conclusion (v) by considering the following two cases.

Case 1 $(\lambda^* = +\infty)$. Note that $x_\lambda = x_0 + \lambda Bx_\lambda \geq \lambda Bx_\lambda \geq \lambda Bx_0$ for $\lambda \in [0, +\infty)$; it is clear that $\lim_{\lambda\to\lambda^*-0}\|x_\lambda\| = +\infty$.

Case 2 $(\lambda^* < +\infty)$. In this case, suppose, to the contrary, that $\lim_{\lambda\to\lambda^*-0}\|x_\lambda\| \neq +\infty$. Then, there exists a nondecreasing sequence $\{\lambda_n\} \subset [0, \lambda^*)$ and a constant $M > 0$ such that $\lim_{n\to+\infty}\lambda_n = \lambda^*$ and $\|x_{\lambda_n}\| \leq M$.

Since B is completely continuous, there exist $x^* \in P$ and subsequence $\{\lambda_{n_i}\} \subset \{\lambda_n\}$ such that $\lim_{i\to+\infty}Bx_{\lambda_{n_i}} = x^*$. Taking the limitation $i \to +\infty$ to both sides of

$$x_{\lambda_{n_i}} = x_0 + \lambda_{n_i}Bx_{\lambda_{n_i}}, \tag{46}$$

we have $\lim_{i\to+\infty}x_{\lambda_{n_i}} = x_0 + \lambda^* x^*$. Equation (46) implies that

$$x_0 + \lambda^* x^* = x_0 + \lambda^* B(x_0 + \lambda^* x^*). \tag{47}$$

The relation $x_0 + \lambda^* x^* > x_0$ shows that $B(x_0 + \lambda^* x^*) \in \overline{P_e}$. Further, $x_0 + \lambda^* B(x_0 + \lambda^* x^*) \in P_e$. This means that $x_0 + \lambda^* x^* \in P_e$. Therefore, (47) gives rise to the contradiction $\lambda^* \in [0, \lambda^*)$. This finishes the proof. □

Corollary 14. *Assume that* $x_0 \in P_e$ *and* B *is pseudo generalized* α*-concave. Then,*

(i) (45) *has a unique solution* $x_\lambda \in P_e$ *for* $\lambda \in [0, +\infty)$. *Moreover, for any* $u_0 \in P_e$, *set* $u_n = x_0 + \lambda Bu_{n-1}$ $(n = 1, 2, \ldots)$; *then* $\lim_{n\to+\infty}\|u_n - x_\lambda\| = 0$;

(ii) x_λ *is increasing with respect to* λ *for* $\lambda \in [0, +\infty)$;

(iii) x_λ *is continuous with respect to* λ *for* $\lambda \in [0, +\infty)$;

(iv) *either* $x_\lambda = x_0, \lambda \geq 0$ *or* $\lim_{\lambda\to+\infty}\|x_\lambda\| = +\infty$.

Proof. It is obvious that $L(\lambda, x) = x_0 + \lambda Bx$ is increasing in x and $L(\lambda, P_e) \subset P_e$ for any $\lambda \geq 0$. In addition,

$$\begin{aligned} L(\lambda, rx) &= x_0 + \lambda B(rx) \geq x_0 \\ &\quad + \lambda r^{\alpha(r)} Bx \geq r^{\alpha(r)} L(\lambda, x), \quad x \in P_e,\ r \in (0,1). \end{aligned} \tag{48}$$

This means that $L(\lambda, \cdot)$ is generalized α-concave. Thus, conclusion (i) follows from Lemma 3. Similarly to the proof of Corollary 11, the proofs of (ii), (iii), and (iv) can be completed. □

From the proofs of Theorem 12, Theorem 13, and Corollary 14, we easily prove the following results.

Theorem 15. *Assume that $x_0 \in \overline{P_e}$ and B is a generalized α-concave operator. Then,*

(i) (45) *has a unique solution $x_\lambda \in P_e$ for $\lambda \in (0, +\infty)$. Moreover, for any $u_0 \in P_e$, set $u_n = x_0 + \lambda B u_{n-1}$ ($n = 1, 2, \ldots$); then $\lim_{n\to+\infty} \|u_n - x_\lambda\| = 0$;*

(ii) *x_λ is increasing in λ and $x_\lambda \geq x_0$ for $\lambda \in (0, +\infty)$;*

(iii) *x_λ is continuous with respect to λ for $\lambda \in (0, +\infty)$;*

(iv) $\lim_{\lambda\to 0} \|x_\lambda - x_0\| = 0$ *and* $\lim_{\lambda\to+\infty} \|x_\lambda\| = +\infty$.

Remark 16. Different from Theorem 9 and Corollary 11, even if (1) satisfies the conditions in Theorem 12, (1) may not have a fixed point in P_e when $\lambda = 0$.

3. Three-Point Nonlinear Boundary Value Problem

In this section, based on the discussion of the previous section, we study the existence and uniqueness of positive solutions for the three-point BVP(2) and the dependence of solutions on the parameter λ.

In what follows, set $E = C[0,1]$, the Banach space of continuous functions on $[0,1]$ with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$. Consider that $P = \{x \in C[0,1] \mid x(t) \geq 0, t \in [0,1]\}$. It is clear that P is a normal cone with the normality constant 1; P_e is given as in (5) with $e(t) = \tau + t$. Let

$$\begin{aligned} G(t,s) &= \frac{1}{1+\tau-\sigma(\xi+\tau)} \\ &\times \begin{cases} (1-\sigma\xi-(1-\sigma)t)(\tau+s), & 0 \leq s \leq \min\{\xi, t\}, \\ (1-t)(\tau+s)+\sigma(\xi+\tau)(t-s), & \xi \leq s \leq t \leq 1, \\ (\tau+t)(1-s-\sigma(\xi-s)), & 0 \leq t \leq s \leq \xi, \\ (\tau+t)(1-s), & \max\{\xi, t\} \leq s \leq 1. \end{cases} \end{aligned} \tag{49}$$

It is easy to prove that

$$\begin{aligned} 0 &\leq G(t,s) \\ &\leq \frac{1+\sigma}{1+\tau-\sigma(\xi+\tau)}(\tau+t)(1-s), \quad t,s \in [0,1]. \end{aligned} \tag{50}$$

Define two operators $A : P \to C[0,1]$ and $B : P \to C[0,1]$ by

$$\begin{aligned} Ax(t) &= \frac{\tau+t}{1+\tau-\sigma(\xi+\tau)} \int_0^1 h(x(s))\,ds \\ &\quad + \int_0^1 G(t,s) g(s)\,ds. \end{aligned} \tag{51}$$

$$Bx(t) = \int_0^1 G(t,s) f(s, x(s))\,ds. \tag{52}$$

It is clear that a positive solution of BVP(2) is equivalent to nontrivial solution of (1) in P.

The following lemma can be proved easily by the Ascoli-Arzela theorem.

Lemma 17. *$B : P \to C[0,1]$ is completely continuous.*

The following hypotheses are needed in this section.

(L1) $f : [0,1] \times [0, +\infty) \to [0, +\infty)$ is continuous.

(L2) $f(t,x)$ is increasing in $x \in [0, +\infty)$ for fixed $t \in [0,1]$ and

$$\begin{aligned} f(t, rx) \geq r f(t,x), &\quad \forall t \in [0,1], \\ r \in (0,1), &\quad x \in [0, +\infty). \end{aligned} \tag{53}$$

(L3) $h : [0, +\infty) \to [0, +\infty)$ is continuous and $h(x) \not\equiv 0$.

(L4) $h(x)$ is increasing in $x \in [0, +\infty)$ and there exists a function $\alpha(r) \in (0,1)$ such that

$$h(rx) \geq r^{\alpha(r)} h(x), \quad \forall r \in (0,1),\ x \in (0, +\infty). \tag{54}$$

(L5) $g : [0,1] \to [0, +\infty)$ is continuous.

(L6) $\int_0^1 (\tau+s)(1-s) g(s)\,ds > 0$.

Lemma 18. *Suppose that (L1) and (L2) hold. Then $B : P \to P$ is increasing and satisfies*

(i) $B(P \setminus \{\theta\}) \subset \overline{P_e}$;

(ii) $B(rx) \geq rBx,\ x \in P_e, r \in (0,1)$.

Proof. From (50), (52), and (L1), it is clear that $B(P) \subset P$. By (L2) we obtain that B is an increasing operator and, for any $x \in P \setminus \{\theta\}$,

$$\begin{aligned} Bx(t) &= \int_0^1 G(t,s) f(s, x(s))\,ds \\ &\leq \frac{(1+\sigma)\int_0^1 (1-s) f(s, x(s))\,ds}{1+\tau-\sigma(\xi+\tau)} (\tau+t) \\ &\leq \frac{(1+\sigma)\int_0^1 (1-s) f(s, \|x\|)\,ds}{1+\tau-\sigma(\xi+\tau)} e(t) \\ &\leq l(x) e(t), \end{aligned} \tag{55}$$

where $l(x) \geq \max\{((1+\sigma)\int_0^1(1-s)f(s,\|x\|)ds)/(1+\tau-\sigma(\xi+\tau)), 1\}$, and

$$\begin{aligned} B(rx)(t) &= \int_0^1 G(t,s) f(s, rx(s))\, ds \\ &\geq r\int_0^1 G(t,s) f(s,x(s))\, ds = rBx(t), \quad r \in (0,1). \end{aligned} \tag{56}$$

The proof is complete. □

Theorem 19. *Suppose that (L1)–(L5) hold. Then there exists $\lambda^* > 0$ such that BVP(2) has a unique positive solution $x_\lambda(t)$ in P_e for $\lambda \in [0,\lambda^*)$ and has no solution in P_e for $\lambda \geq \lambda^*$. Furthermore such a solution $x_\lambda(t)$ satisfies the following properties:*

(i) *for any $u_0 \in P_e$, set*

$$\begin{aligned} u_n(t) &= \frac{\tau+t}{1+\tau-\sigma(\xi+\tau)}\int_0^1 h(u_{n-1}(s))\, ds \\ &\quad + \int_0^1 G(t,s) g(s)\, ds \\ &\quad + \lambda\int_0^1 G(t,s) f(s,u_{n-1}(s))\, ds, \end{aligned} \tag{57}$$

and then $u_n(t)$ uniformly converges to $x_\lambda(t)$ on $[0,1]$;

(ii) *$x_\lambda(t)$ is nondecreasing in λ for $\lambda \in [0,\lambda^*)$;*

(iii) *$x_\lambda(t)$ is continuous with respect to λ for $\lambda \in [0,\lambda^*)$.*

Proof. Consider A and B defined by (51) and (52). By (L3)–(L5), $A : P \rightarrow P$ is increasing. For any $x \in P_e$, from (50) we have

$$\begin{aligned} &\frac{\int_0^1 h(x(s))\, ds}{1+\tau-\sigma(\xi+\tau)}(\tau+t) \\ &\quad \leq Ax(t) \\ &\quad \leq \frac{\int_0^1 h(x(s))\, ds + (1+\sigma)\int_0^1(1-s) g(s)\, ds}{1+\tau-\sigma(\xi+\tau)}(\tau+t), \end{aligned} \tag{58}$$

which means that $A(P_e) \subset P_e$. In addition, for any $r \in (0,1)$ and $x \in P_e$, by (L4) we obtain

$$\begin{aligned} A(rx)(t) &= \frac{\tau+t}{1+\tau-\sigma(\xi+\tau)} \\ &\quad \times \int_0^1 h(rx(s))\, ds + \int_0^1 G(t,s) g(s)\, ds \\ &\geq r^{\alpha(r)}\left(\frac{\tau+t}{1+\tau-\sigma(\xi+\tau)}\int_0^1 h(x(s))\, ds\right. \\ &\quad \left. + r^{-\alpha(r)}\int_0^1 G(t,s) g(s)\, ds\right) \\ &\geq r^{\alpha(r)} Ax(t). \end{aligned} \tag{59}$$

So, the operator A is generalized α-concave. Lemma 18 implies that B is pseudo subhomogeneous. The conclusion follows from Theorem 9. The proof is complete. □

In the following, we consider three special cases of BVP(2).

Case 1. BVP(2) has no perturbation; that is, $g(t) \equiv 0$.

From the proof of Theorem 19, we have the following result.

Theorem 20. *Suppose that (L1)–(L4) hold. Then there exists $\lambda^* > 0$ such that*

$$\begin{gathered} -x''(t) = \lambda f(t, x(t)), \quad t \in (0,1), \\ x(0) = \tau x'(0), \qquad x(1) - \sigma x(\xi) = \int_0^1 h(x(s))\, ds \end{gathered} \tag{60}$$

have a unique positive solution $x_\lambda(t) \in P_e$ for $\lambda \in [0,\lambda^*)$ and no solution in P_e for $\lambda \geq \lambda^*$. Furthermore such a solution $x_\lambda(t)$ satisfies the following properties:

(i) *for any $u_0 \in P_e$, set*

$$\begin{aligned} u_n(t) &= \frac{\tau+t}{1+\tau-\sigma(\xi+\tau)}\int_0^1 h(u_{n-1}(s))\, ds \\ &\quad + \lambda\int_0^1 G(t,s) f(s,u_{n-1}(s))\, ds, \end{aligned} \tag{61}$$

and then $u_n(t)$ uniformly converges to $x_\lambda(t)$ on $[0,1]$;

(ii) *$x_\lambda(t)$ is nondecreasing in λ for $\lambda \in [0,\lambda^*)$;*

(iii) *$x_\lambda(t)$ is continuous with respect to λ for $\lambda \in [0,\lambda^*)$.*

Case 2. The nonlinear boundary value control function $h(x)$ in BVP(2) reduces to the linear one $h_0(>0)$.

Theorem 21. *Suppose that (L1), (L2), and (L5) hold. Then, there exists $\lambda^* > 0$ such that*

$$\begin{gathered} -x''(t) = \lambda f(t,x(t)) + g(t), \quad t \in (0,1), \\ x(0) = \tau x'(0), \qquad x(1) - \sigma x(\xi) = h_0 \end{gathered} \tag{62}$$

have a unique positive solution $x_\lambda(t) \in P_e$ for $\lambda \in [0,\lambda^*)$ and no solution in P_e for $\lambda \geq \lambda^*$; furthermore such a solution $x_\lambda(t)$ satisfies the following properties:

(i) *for any $u_0 \in P_e$, set*

$$\begin{aligned} u_n(t) &= \frac{\tau+t}{1+\tau-\sigma(\xi+\tau)}h_0 + \int_0^1 G(t,s) g(s)\, ds \\ &\quad + \lambda\int_0^1 G(t,s) f(s,u_{n-1}(s))\, ds, \end{aligned} \tag{63}$$

and then $u_n(t)$ uniformly converges to $x_\lambda(t)$ on $[0,1]$;

(ii) *$x_\lambda(t)$ is nondecreasing in λ for $\lambda \in [0,\lambda^*)$;*

(iii) *$x_\lambda(t)$ is continuous with respect to λ for $\lambda \in [0,\lambda^*)$;*

(iv) $\lim_{\lambda\rightarrow\lambda^*-0}\|x_\lambda\| = +\infty$.

Proof. Define operator B as (52) and let

$$x_0(t) = \frac{\tau + t}{1 + \tau - \sigma(\xi + \tau)} h_0 + \int_0^1 G(t,s)\, g(s)\, ds. \quad (64)$$

Then $x_0 \in P_e$. By Lemmas 17 and 18, $B : P \to P$ is completely continuous and B is pseudo subhomogeneous, and $Bx \in \overline{P_e}$ for $x \geq x_0$. The application of Theorem 13 completes the proof. □

Case 3. The nonlinear boundary value control function $h(x)$ in BVP(2) vanishes; that is, $h(x) \equiv 0$ for $t \in [0,1]$.

Theorem 22. *Suppose that (L1), (L2), (L5), and (L6) hold. Then there exists $\lambda^* > 0$ such that*

$$\begin{gathered} -x''(t) = \lambda f(t, x(t)) + g(t), \quad t \in (0,1), \\ x(0) = \tau x'(0), \qquad x(1) - \sigma x(\xi) = 0 \end{gathered} \quad (65)$$

have a unique positive solution $x_\lambda(t) \in P_e$ for $\lambda \in [0, \lambda^*)$ and no solution in P_e for $\lambda \geq \lambda^*$; furthermore such a solution $x_\lambda(t)$ satisfies the following properties:

(i) *for any $u_0 \in P_e$, set*

$$u_n(t) = \int_0^1 G(t,s)\, g(s)\, ds + \lambda \int_0^1 G(t,s)\, f(s, u_{n-1}(s))\, ds, \quad (66)$$

and then $u_n(t)$ uniformly converges to $x_\lambda(t)$ on $[0,1]$;

(ii) *$x_\lambda(t)$ is nondecreasing in λ for $\lambda \in [0, \lambda^*)$;*

(iii) *$x_\lambda(t)$ is continuous with respect to λ for $\lambda \in [0, \lambda^*)$;*

(iv) $\lim_{\lambda \to \lambda^* - 0} \|x_\lambda\| = +\infty$.

Proof. Define operator B as (52) and let

$$x_0(t) = \int_0^1 G(t,s)\, g(s)\, ds. \quad (67)$$

Note the monotonicity of $G(t,s)$ in t; it follows that

$$\begin{aligned} G(t,s) \geq{}& \frac{\sigma(1-\xi)(\tau+\xi)}{(1+\tau-\sigma(\xi+\tau))(\tau+1)^2} \\ & \times (\tau+s)(1-s)(\tau+t), \quad t, s \in [0,1], \end{aligned} \quad (68)$$

which together with (50), (L5), and (L6) leads to $x_0 \in P_e$. By Lemma 18, $B : P \to P$ is pseudo subhomogeneous and $Bx \in \overline{P_e}$ for $x \geq x_0$. The application of Theorem 13 finishes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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