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Research Article

Exact Solutions for Nonlinear Wave Equations by the Exp-Function Method

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This paper elucidates the main advantages of the exp-function method in finding exact solutions of nonlinear wave equations. By the aid of some mathematical software, the solution process becomes extremely simple and accessible.

1. Introduction

One of the most important aspects in nonlinear science is how to solve an exact solution of a nonlinear equation. Recently many different methods have appeared, among which the homotopy perturbation method [1–4], the tanh-method [5], the sinh-method [6, 7], and the F-expansion method [8–11] have caught much attention; however, all these methods are valid for some special kinds of nonlinear equations. It is therefore very much needed to find a universal approach to nonlinear equations; this is very challenging indeed, and the exp-function method [12–15] meets this requirement. The exp-function method itself is mathematically beautiful and extremely accessible to nonmathematicians. The use of the method requires no special knowledge of advanced calculus, and it is especially effective for solitary solutions.

2. Exp-Function Method

The exp-function method was first proposed by He and Wu [16], and we consider a general partial differential equation (PED) in the form

$$P\left(u, u_t, u_x, u_y, u_{xx}, u_{tt}, u_{yy}\right) = 0 \tag{1}$$

to pick out the main solution process and its advantages.

Use a transformation [16]

$$\xi = kx + \omega t + ly, \tag{2}$$

where k, ω, and l are unknown constants and should be determined later. By (2), we can convert (1) to the following nonlinear ordinary differential equation:

$$G\left(u, u', u'', u''', \ldots\right) = 0. \tag{3}$$

According to the exp-function method, we assume that its solution can be expressed in the following form [16, 17]:

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_n \exp(m\xi)}, \tag{4}$$

where c, d, p, q are positive integers that could be freely chosen. To determine the value of c and p, we balance the linear term of highest order of (3) with the highest order of the nonlinear term. Similarly for determining the value of d and q, we balance the lowest orders of linear and nonlinear terms in (3). By substituting (4) into (3), collecting terms of the same term of $\exp(i\xi)$, and equating the coefficient of each power of exp to zero, we can get a set of algebraic equations for determining unknown constants.

3. Exact Solution for Nonlinear Wave Equation

In order to illustrate the basic solution process of the exp-function method, we use the Burgers-Huxley equation as an example, which can be expressed as [18]

$$u_t + u_{xx} + \frac{3}{k} u u_x + cu + u^2 + u^3 = 0, \tag{5}$$

where $u = u(x, t)$ is an unknown function, u_t, u_x are the partial derivatives of $u(x, t)$ with respect to t and x, respectively, and k and c are arbitrary constants.

According to the exp-function method [15–17], we introduce a complex variation ξ defined as

$$\xi = kx + \omega t. \tag{6}$$

Equation (5) thus becomes an ordinary differential equation as

$$k^2 u'' + (3u + \omega) u' + cu + u^2 + u^3 = 0. \tag{7}$$

We suppose that the solution of (7) can be expressed as

$$u(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \cdots + b_{-q} \exp(-q\xi)}. \tag{8}$$

Thus we have

$$\begin{aligned} u'' &= \frac{\gamma_1 \exp((c + 3p)\xi)}{\gamma_2 \exp(4\xi)}, \\ u^3 &= \frac{c_3 \exp(3c\xi) + \cdots}{c_4 \exp(3p\xi) + \cdots} = \frac{c_3 \exp((3c + p)\xi)}{c_4 \exp(4p\xi)}. \end{aligned} \tag{9}$$

Balancing highest order of exp-function in (9), we have $3c + p = c + 3p$, which leads to the result $p = c$. Similarly we balance the lowest orders of linear and nonlinear terms in (5) to determine values of d and q, and we can get $d = q$. For simplicity, we set $p = c = 1$ and $q = d = 1$; then (8) reduces to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{\exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{10}$$

Substituting (10) in to (5), we have

$$\begin{aligned} &\frac{1}{A} \left[E_3 \exp(3\xi) + E_2 \exp(2\xi) + E_1 \exp(\xi) + E_0 \right. \\ &\quad \left. + E_{-1} \exp(-\xi) + E_{-2} \exp(-2\xi) + E_{-3} \exp(-3\xi)\right] = 0, \end{aligned} \tag{11}$$

where $= [b_0 + \exp(\xi) + b_{-1} \exp(-\xi)]^3$,

$$\begin{aligned}
E_3 &= a_1^3 + a_1^2 + ca_1, \\
E_2 &= ca_0 - a_0 a_1 - \omega a_0 + 3a_0 a_1^2 + 4a_1^2 b_0 \\
&\quad + k^2 a_0 + 2a_1 b_0 c + \omega a_1 b_0 - k^2 a_1 b_0, \\
E_0 &= a_0^3 + a_0^2 b_0 - 7a_0 a_{-1} + 6a_0 a_1 a_{-1} \\
&\quad + 11a_0 a_1 b_{-1} + 2b_0 a_1 a_{-1} + 2ca_0 b_{-1} + 2cb_0 a_{-1} \\
&\quad - 3\omega b_0 a_{-1} + ca_0 b_0^2 - 6k^2 a_0 b_{-1} + 3k^2 a_{-1} b_0 \\
&\quad + 3k^2 a_{-1} b_{-1} b_0 + 2ca_1 b_{-1} b_0 + 3\omega a_1 b_{-1} b_0, \\
E_1 &= ca_{-1} - 4a_1 a_{-1} - 2\omega a_{-1} + 3a_1 a_0^2 + 3a_1^2 a_{-1} \\
&\quad + 7a_1^2 b_{-1} + 4k^2 a_{-1} - 2a_0^2 + \left(k^2 + c\right) a_1 b_0^2 + 5a_0 a_1 b_0 \\
&\quad + 2ca_1 b_{-1} - \omega a_0 b_0 + 2\omega a_1 b_{-1} + 2ca_0 b_0, \\
E_{-1} &= 3a_0^2 a_{-1} + 3a_1 a_{-1}^2 + 4a_0^2 b_{-1} - 5a_{-1}^2 + a_{-1} b_0^2 k^2 \\
&\quad + 4a_1 b_{-1}^2 k^2 - a_0 a_{-1} b_0 + 8a_1 a_{-1} b_{-1} + (2c - 2\omega) a_{-1} b_{-1} \\
&\quad + ca_{-1} b_0^2 + ca_1 b_{-1}^2 - 4a_{-1} b_{-1}^2 k^2 - \omega a_{-1} b_0^2 + 2\omega a_1 b_{-1}^2 \\
&\quad + \left(2c + \omega - k^2\right) a_0 b_{-1} b_0, \\
E_{-2} &= 3a_0 a_{-1}^2 - 2a_{-1}^2 b_0 + k^2 a_0 b_{-1}^2 + 5a_0 a_{-1} b_{-1} \\
&\quad + (c + \omega) a_0 b_{-1}^2 + \left(2c - k^2 - \omega\right) a_{-1} b_{-1} b_0, \\
E_{-3} &= a_{-1}^3 + a_{-1}^2 b_{-1} + ca_{-1} b_{-1}^2.
\end{aligned} \tag{12}$$

Setting the coefficients of $\exp(i\xi)$, $(i = 0, \pm 1, \pm 2, \pm 3)$ to zero, we have

$$\begin{gathered} E_3 = 0, \qquad E_2 = 0, \qquad E_1 = 0 \\ E_0 = 0, \\ E_{-3} = 0, \qquad E_{-2} = 0, \qquad E_{-1} = 0. \end{gathered} \tag{13}$$

With the help of some mathematical software, we can solve the solutions of the algebraic equations.

Case 1. Consider

$$\begin{aligned}
a_1 &= \frac{\sqrt{1 - 4c} - 1}{2}, \\
b_0 &= \frac{-a_0 \left(3a_1^2 - a_1 + k^2 + c - \omega\right)}{a_1 \left(-k^2 + 4a_1 + 2c + \omega\right)}, \\
\omega &= \frac{ca_0 - a_0 a_1 + 3a_0 a_1^2 + 4a_1^2 b_0 + a_0 k^2 + 2a_1 b_0 c - a_1 b_0 k^2}{a_0 - a_1 b_0}, \\
a_{-1} &= \frac{b_{-1} \left(\sqrt{1 - 4c} - 1\right)}{2}.
\end{aligned} \tag{14}$$

This implies the following exact solution:

$$u(\xi) = \left(\frac{\sqrt{1-4c}-1}{2} \exp(\xi) + a_0 + \frac{b_2\left(\sqrt{1-4c}-1\right)}{2} \exp(-\xi) \right) \times \left(\exp(\xi) - \frac{\left(3a_0a_1^2 - a_0a_1 + a_0k^2 + ca_0 - \omega a_0\right)}{2ca_1 + \omega c - a_1k^2 + 4a_1^2} + b_2 \exp(-\xi) \right)^{-1}, \quad (15)$$

where

$$\xi = kx + \frac{ca_0 - a_0a_1 + 3a_0a_1^2 + 4a_1^2b_0 + a_0k^2 + 2a_1b_0c - a_1b_0k^2}{a_0 - a_1b_0} t, \quad (16)$$

a_0, b_2 are parameters, $b_2 \neq 0$, and k is a free real number.

Case 2. Consider

$$a_1 = \frac{1}{2}, \qquad b_0 = 0,$$
$$a_2 = -\frac{b_2}{2}, \qquad b_2 = b_2,$$
$$a_0 = \frac{\sqrt{2}}{2} + \sqrt{-16b_2k^2 + \frac{19}{2}b_2 + 2cb_2 + 8b_2\omega},$$
$$\omega = k^2 + c + \frac{1}{4}. \quad (17)$$

This case gives another exact solution as follows:

$$u(x,t) = \left(\frac{1}{2} \exp\left(kx + \left(k^2 + c + \frac{1}{4}\right)t\right) + \frac{\sqrt{2}}{2}\sqrt{10b_2c - 8b_2k^2 + \frac{19}{2}b_2 + 2b_2} - b_2 \exp\left(-kx - \left(k^2 + c + \frac{1}{4}\right)t\right) \right) \times \left(\exp\left(kx + \left(k^2 + c + \frac{1}{4}\right)t\right) + b_2 \exp\left(-kx - \left(k^2 + c + \frac{1}{4}\right)t\right) \right)^{-1}, \quad (18)$$

where b_2, k, are nonzero free parameters.

Case 3. Consider

$$a_1 = -\frac{\sqrt{1-4c}+1}{2}, \qquad a_0 = 0,$$
$$b_0 = 0, \qquad b_2 = b_2, \qquad a_2 = -\frac{\left(1+\sqrt{1-4c}\right)}{2} b_2. \quad (19)$$

This results in the following exact solution:

$$u(\xi) = \left(-\frac{\sqrt{1-4c}+1}{2} \exp(\xi) - \frac{\left(1+\sqrt{1-4c}\right)}{2} b_2 \exp(-\xi) \right) \times \left(\exp(\xi) + b_2 \exp(-\xi)\right)^{-1}, \quad (20)$$

where b_2 is nonzero free parameter.

4. Conclusion

By some mathematical software, the solution process is extremely simple and abundant solutions are predicted [19–21]. The exp-function method is a universal tool for nonlinear equations and can be easily extended to fractional calculus [22–27].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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