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Research Article

Certain Chebyshev Type Integral Inequalities Involving Hadamard's Fractional Operators

Sotiris K. Ntouyas,[1] Sunil D. Purohit,[2] and Jessada Tariboon[3]

[1] *Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece*
[2] *Department of Basic Sciences (Mathematics), College of Technology and Engineering, M.P. University of Agriculture and Technology, Udaipur 313001, India*
[3] *Nonlinear Dynamic Analysis Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand*

Correspondence should be addressed to Jessada Tariboon; jessadat@kmutnb.ac.th

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We establish certain new fractional integral inequalities for the differentiable functions whose derivatives belong to the space $L_p([1,\infty))$, related to the weighted version of the Chebyshev functional, involving Hadamard's fractional integral operators. As an application, particular results have been also established.

1. Introduction

In 1882, Chebyshev [1] gave the following inequality.

If $f, g : [a,b] \to \mathbb{R}^+$ are absolutely continuous functions, whose first derivatives f' and g' are bounded and

$$\begin{aligned} T(f,g) = {} & \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt \\ & - \left(\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt\right), \end{aligned} \tag{1}$$

then

$$\left|T(f,g)\right| \le \frac{1}{12}(b-a)^2\left\|f'\right\|_\infty\left\|g'\right\|_\infty, \tag{2}$$

where $\|\cdot\|_\infty$ denotes the norm in $L_\infty[a,b]$.

The Chebyshev functional (1) has many applications in numerical quadrature, transform theory, probability, study of existence for solutions of differential equations, and in statistical problems. Moreover, in the theory of approximations, under various assumptions (Chebyshev inequalities, Grüss inequality, etc.), Chebyshev functionals are useful to give lower bound or upper bounds for the functions. Therefore, in the literature, we found several extensions and generalizations of these classical integral inequalities, including fractional calculus and q-calculus operators also (see [2–14]).

Our work in the present paper is based on a weighted version of the Chebyshev functional (see [1]):

$$\begin{aligned} T(f,g,p) = {} & \int_a^b p(t)\,dt\int_a^b f(t)\,g(t)\,p(t)\,dt \\ & - \int_a^b f(t)\,p(t)\,dt\int_a^b g(t)\,p(t)\,dt, \end{aligned} \tag{3}$$

where f and g are two integrable functions on $[a,b]$ and $p(t)$ is a positive and integrable function on $[a,b]$. In 2000, Dragomir [15] derived the following inequality:

$$\begin{aligned} & 2\left|T(f,g,p)\right| \\ & \quad \le \left\|f'\right\|_r\left\|g'\right\|_s\left[\iint_a^b |x-y|\,p(x)\,p(y)\,dx\,dy\right], \end{aligned} \tag{4}$$

where f and g are two differentiable functions and $f' \in L_r(a,b)$, $g' \in L_s(a,b)$, $r > 1$, and $r^{-1} + s^{-1} = 1$. Recently, Dahmani et al. [16] added one more dimension to this study by introducing generalization of inequality (4), involving Riemann-Liouville fractional integrals. Moreover, Purohit and Raina [17–19] and Baleanu et al. [20, 21] introduced certain generalized integral inequalities for synchronous functions, involving the various fractional hypergeometric integral operators, while Tariboon et al. [22] studied Riemann-Liouville fractional integral inequalities.

In 1892, Hadamard [23] introduced a fractional derivative, which differs from the Riemann-Liouville and Caputo derivatives in the sense that the kernel of the integral contains logarithmic function of arbitrary exponent. For details and fundamental properties of Hadamard fractional derivative and integral can be found in [24–28]. Recently, some results on fractional integral inequalities have been derived by using Hadamard fractional integrals (see [29–32]).

In this paper, we establish certain integral inequalities related to the weighted Chebyshev's functional (3) in the case of differentiable functions whose derivatives belong to the space $L_p([1,\infty))$, involving Hadamard fractional integral operators [23]. We also develop some integral inequalities for the fractional integrals by suitably choosing the function $p(t)$, as special cases of our findings.

Firstly, we mention below the basic definitions and notations of some well-known operators of fractional calculus, which shall be used in the sequel.

The Hadamard fractional integral of order $\alpha \in \mathbb{R}^+$ of a function $f(t)$, for all $t > 1$, is defined as [28]

$$ {}_HJ^{\alpha}\{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}. \tag{5}$$

Further, the Hadamard fractional derivative of order $\alpha \in [n-1, n)$, $n \in \mathbb{Z}^+$, of a function $f(t)$ is given by

$$ {}_HD^{\alpha}\{f(t)\} = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log \frac{t}{\tau}\right)^{n-\alpha-1} f(\tau) \frac{d\tau}{\tau}. \tag{6}$$

Our results in this paper are based on the following preliminary assertions giving composition formula of Hadamard fractional integral and derivatives with a power function ([4])

$$\begin{aligned} {}_HJ^{\alpha}(\log t)^{\mu-1} &= \frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)}(\log t)^{\mu+\alpha-1}, \\ {}_HD^{\alpha}(\log t)^{\mu-1} &= \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)}(\log t)^{\mu-\alpha-1}, \end{aligned} \tag{7}$$

where $0 < \alpha < 1$.

2. Main Results

Our results in this section are related to the Chebyshev's functional (3) in the case of differentiable mappings whose derivatives belong to the space $L_p([1,\infty))$ and satisfying Holder's inequality. Here, we obtain certain new integral inequalities which give an estimation for the fractional integral of a product in terms of the product of the individual function fractional integrals, involving Hadamard fractional integral operators.

Theorem 1. *Let p be a positive function and let f and g be two differentiable functions on $[1,\infty)$. If $f' \in L_r([1,\infty))$, $g' \in L_s([1,\infty))$, $r > 1$, $r^{-1} + s^{-1} = 1$, then, for all $t > 1$ and $\alpha > 0$,*

$$\begin{aligned} &2\left|{}_HJ^{\alpha}\{p(t)\}\,{}_HJ^{\alpha}\{p(t)f(t)g(t)\} - {}_HJ^{\alpha}\{p(t)f(t)\} \times {}_HJ^{\alpha}\{p(t)g(t)\}\right| \\ &\le \frac{\|f'\|_r\|g'\|_s}{\Gamma^2(\alpha)} \iint_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{p(\tau)p(\rho)}{\tau\rho} \times |\tau-\rho|\,d\tau\,d\rho \\ &\le \|f'\|_r\|g'\|_s t\left({}_HJ^{\alpha}\{p(t)\}\right)^2. \end{aligned} \tag{8}$$

Proof. We define

$$\mathscr{H}(\tau,\rho) = (f(\tau)-f(\rho))(g(\tau)-g(\rho)), \tag{9}$$

$$F(t,\tau) = \frac{(\log(t/\tau))^{\alpha-1}}{\tau\Gamma(\alpha)}, \quad \tau\in(1,t),\ t>1. \tag{10}$$

We observe that the function $F(t,\tau)$ remains positive, for all $\tau \in (1,t)$ $(t > 1)$. Multiplying both sides of (9) by $F(t,\tau)p(\tau)$ and integrating with respect to τ from 1 to t, we get

$$\begin{aligned} &\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{p(\tau)}{\tau}\mathscr{H}(\tau,\rho)\,d\tau \\ &= {}_HJ^{\alpha}\{p(t)f(t)g(t)\} - f(\rho)\,{}_HJ^{\alpha}\{p(t)g(t)\} \\ &\quad - g(\rho)\,{}_HJ^{\alpha}\{p(t)f(t)\} + f(\rho)g(\rho)\,{}_HJ^{\alpha}\{p(t)\}. \end{aligned} \tag{11}$$

Next, multiplying both sides of (11) by $F(t,\rho)p(\rho)$ and integrating with respect to ρ from 1 to t, we can write

$$\begin{aligned} &\frac{1}{\Gamma^2(\alpha)}\iint_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{p(\tau)p(\rho)}{\tau\rho}\mathscr{H}(\tau,\rho)\,d\tau\,d\rho \\ &= 2\left({}_HJ^{\alpha}\{p(t)\}\,{}_HJ^{\alpha}\{p(t)f(t)g(t)\} - {}_HJ^{\alpha}\{p(t)f(t)\}\,{}_HJ^{\alpha}\{p(t)g(t)\}\right). \end{aligned} \tag{12}$$

In view of (9), we have

$$\mathscr{H}(\tau,\rho) = \iint_\tau^\rho f'(y)g'(z)\,dy\,dz. \tag{13}$$

Using the following Hölder's inequality for $r > 1$ and $r^{-1} + s^{-1} = 1$,

$$\begin{aligned} &\left|\iint_\tau^\rho f(y)g(z)\,dy\,dz\right| \\ &\le \left|\iint_\tau^\rho |f(y)|^r dy\,dz\right|^{r^{-1}}\left|\iint_\tau^\rho |g(z)|^s dy\,dz\right|^{s^{-1}}, \\ &\qquad (r^{-1}+s^{-1}=1), \end{aligned} \tag{14}$$

we obtain

$$\left|\mathscr{H}(\tau,\rho)\right| \le \left|\iint_{\tau}^{\rho}\left|f'(y)\right|^{r} dy\,dz\right|^{r^{-1}} \left|\iint_{\tau}^{\rho}\left|g'(z)\right|^{s} dy\,dz\right|^{s^{-1}}. \tag{15}$$

Since

$$\begin{aligned} \left|\iint_{\tau}^{\rho}\left|f'(y)\right|^{r} dy\,dz\right|^{r^{-1}} &= \left|\tau-\rho\right|^{r^{-1}} \left|\int_{\tau}^{\rho}\left|f'(y)\right|^{r} dy\right|^{r^{-1}}, \\ \left|\iint_{\tau}^{\rho}\left|g'(z)\right|^{s} dy\,dz\right|^{s^{-1}} &= \left|\tau-\rho\right|^{s^{-1}} \left|\int_{\tau}^{\rho}\left|g'(z)\right|^{s} dz\right|^{s^{-1}}, \end{aligned} \tag{16}$$

therefore, inequality (15) reduces to

$$\left|\mathscr{H}(\tau,\rho)\right| \le \left|\tau-\rho\right| \left|\int_{\tau}^{\rho}\left|f'(y)\right|^{r} dy\right|^{r^{-1}} \left|\int_{\tau}^{\rho}\left|g'(z)\right|^{s} dz\right|^{s^{-1}}. \tag{17}$$

It follows from (12) that

$$\begin{aligned} &\frac{1}{\Gamma^{2}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \left|\mathscr{H}(\tau,\rho)\right| d\tau\,d\rho \\ &\quad \le \frac{1}{\Gamma^{2}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \left|\tau-\rho\right| \\ &\qquad \times \left|\int_{\tau}^{\rho}\left|f'(y)\right|^{r} dy\right|^{r^{-1}} \left|\int_{\tau}^{\rho}\left|g'(z)\right|^{s} dz\right|^{s^{-1}} d\tau\,d\rho. \end{aligned} \tag{18}$$

Applying again Hölder's inequality on the right-hand side of (18), we get

$$\begin{aligned} &\frac{1}{\Gamma^{2}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \\ &\qquad \times \left|\mathscr{H}(\tau,\rho)\right| d\tau\,d\rho \\ &\quad \le \left[\frac{1}{\Gamma^{r}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \left|\tau-\rho\right| \right. \\ &\qquad \left. \times \left|\int_{\tau}^{\rho}\left|f'(y)\right|^{r} dy\right| d\tau\,d\rho\right]^{r^{-1}} \\ &\qquad \times \left[\frac{1}{\Gamma^{s}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \right. \\ &\qquad \left. \times \left|\tau-\rho\right| \left|\int_{\tau}^{\rho}\left|g'(z)\right|^{s} dz\right| d\tau\,d\rho\right]^{s^{-1}}. \end{aligned} \tag{19}$$

In view of the fact that

$$\left|\int_{\tau}^{\rho}\left|f(y)\right|^{p} dy\right| \le \left\|f\right\|_{p}^{p}, \tag{20}$$

we get

$$\begin{aligned} &\frac{1}{\Gamma^{2}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \left|\mathscr{H}(\tau,\rho)\right| d\tau\,d\rho \\ &\quad \le \left[\frac{\left\|f'\right\|_{r}^{r}}{\Gamma^{r}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \right. \\ &\qquad \left. \times \left|\tau-\rho\right| d\tau\,d\rho\right]^{r^{-1}} \\ &\qquad \times \left[\frac{\left\|g'\right\|_{s}^{s}}{\Gamma^{s}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \right. \\ &\qquad \left. \times \left|\tau-\rho\right| d\tau\,d\rho\right]^{s^{-1}}. \end{aligned} \tag{21}$$

From (21), we obtain

$$\begin{aligned} &\frac{1}{\Gamma^{2}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \left|\mathscr{H}(\tau,\rho)\right| d\tau\,d\rho \\ &\quad \le \frac{\left\|f'\right\|_{r}\left\|g'\right\|_{s}}{\Gamma^{2}(\alpha)} \left[\iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \right. \\ &\qquad \left. \times \left|\tau-\rho\right| d\tau\,d\rho\right]^{r^{-1}} \\ &\qquad \times \left[\iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\rho} \left|\tau-\rho\right| d\tau\,d\rho\right]^{s^{-1}}. \end{aligned} \tag{22}$$

Using the relation $r^{-1}+s^{-1}=1$, the above inequality yields to

$$\begin{aligned} &\frac{1}{\Gamma^{2}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \left|\mathscr{H}(\tau,\rho)\right| d\tau\,d\rho \\ &\quad \le \frac{\left\|f'\right\|_{r}\left\|g'\right\|_{s}}{\Gamma^{2}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \\ &\qquad \times \left|\tau-\rho\right| d\tau\,d\rho. \end{aligned} \tag{23}$$

On the other hand, (12) gives

$$\begin{aligned} &2\left| {}_{H}J^{\alpha}\{p(t)\}\, {}_{H}J^{\alpha}\{p(t)\,f(t)\,g(t)\} \right. \\ &\qquad \left. - {}_{H}J^{\alpha}\{p(t)\,f(t)\}\, {}_{H}J^{\alpha}\{p(t)\,g(t)\}\right| \\ &\quad \le \frac{1}{\Gamma^{2}(\alpha)} \iint_{1}^{t} \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)\,p(\rho)}{\tau\rho} \\ &\qquad \times \left|\mathscr{H}(\tau,\rho)\right| d\tau\,d\rho. \end{aligned} \tag{24}$$

On making use of (23) and (24), the left-hand side of inequality (8) follows very easily.

Now, to prove the right-hand side of inequality (8), we observe that $1 \le \tau \le t$, $1 \le \rho \le t$, and therefore,

$$0 \le |\tau - \rho| \le t. \tag{25}$$

Evidently, from (23), we get

$$\begin{aligned}
&\frac{1}{\Gamma^2(\alpha)} \iint_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \frac{p(\tau)p(\rho)}{\tau\rho} \\
&\qquad \times |\mathscr{H}(\tau,\rho)|\, d\tau\, d\rho \\
&\le \frac{\|f'\|_r \|g'\|_s t}{\Gamma^2(\alpha)} \iint_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\alpha-1} \\
&\qquad \times \frac{p(\tau)p(\rho)}{\tau\rho} d\tau\, d\rho \\
&= \|f'\|_r \|g'\|_s t \left({}_H J^\alpha \{p(t)\}\right)^2,
\end{aligned} \tag{26}$$

which completes the proof of Theorem 1. □

Now, we establish the following integral inequality, which may be regarded as a generalization of Theorem 1.

Theorem 2. *Let p be a positive function and let f and g be two differentiable functions on $[1,\infty)$. If $f' \in L_r([1,\infty))$, $g' \in L_s([1,\infty))$, $r > 1$, $r^{-1} + s^{-1} = 1$, then*

$$\begin{aligned}
&\Big| {}_H J^\alpha \{p(t)\}\, {}_H J^\beta \{p(t) f(t) g(t)\} \\
&\quad + {}_H J^\beta \{p(t)\}\, {}_H J^\alpha \{p(t) f(t) g(t)\} \\
&\quad - {}_H J^\alpha \{p(t) f(t)\}\, {}_H J^\beta \{p(t) g(t)\} \\
&\quad - {}_H J^\beta \{p(t) f(t)\}\, {}_H J^\alpha \{p(t) g(t)\} \Big| \\
&\le \frac{\|f'\|_r \|g'\|_s}{\Gamma(\alpha)\Gamma(\beta)} \iint_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\beta-1} \frac{p(\tau)p(\rho)}{\tau\rho} \\
&\qquad \times |\tau - \rho|\, d\tau\, d\rho \\
&\le \|f'\|_r \|g'\|_s t\, {}_H J^\alpha \{p(t)\}\, {}_H J^\beta \{p(t)\},
\end{aligned} \tag{27}$$

for all $t > 1$, $\alpha > 0$, and $\beta > 0$.

Proof. To prove the above theorem, we use inequality (11). Multiplying both sides of (11) by

$$\frac{(\log(t/\rho))^{\beta-1} p(\rho)}{\rho\Gamma(\beta)}, \quad \rho \in (1,t),\ t > 1, \tag{28}$$

which remains positive and integrating with respect to ρ from 1 to t, we get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \iint_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\beta-1} \frac{p(\tau)p(\rho)}{\tau\rho} \\
&\qquad \times \mathscr{H}(\tau,\rho)\, d\tau\, d\rho \\
&= {}_H J^\alpha \{p(t)\}\, {}_H J^\beta \{p(t) f(t) g(t)\} \\
&\quad + {}_H J^\beta \{p(t)\}\, {}_H J^\alpha \{p(t) f(t) g(t)\} \\
&\quad - {}_H J^\alpha \{p(t) f(t)\}\, {}_H J^\beta \{p(t) g(t)\} \\
&\quad - {}_H J^\beta \{p(t) f(t)\}\, {}_H J^\alpha \{p(t) g(t)\}.
\end{aligned} \tag{29}$$

Now making use of (17), (29) gives

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \iint_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\beta-1} \frac{p(\tau)p(\rho)}{\tau\rho} \\
&\qquad \times |\mathscr{H}(\tau,\rho)|\, d\tau\, d\rho \\
&\le \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \iint_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\beta-1} \frac{p(\tau)p(\rho)}{\tau\rho} \\
&\qquad \times |\tau - \rho| \left| \int_\tau^\rho |f'(y)|^r dy \right|^{r^{-1}} \\
&\qquad \times \left| \int_\tau^\rho |g'(z)|^s dz \right|^{s^{-1}} d\tau\, d\rho.
\end{aligned} \tag{30}$$

Applying Hölder's inequality on the right-hand side of (30), we get

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \iint_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\beta-1} \frac{p(\tau)p(\rho)}{\tau\rho} \\
&\qquad \times |\mathscr{H}(\tau,\rho)|\, d\tau\, d\rho \\
&\le \left[\frac{1}{\Gamma^r(\alpha)} \iint_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\beta-1} \frac{p(\tau)p(\rho)}{\tau\rho} |\tau - \rho| \right. \\
&\qquad \left. \times \left| \int_\tau^\rho |f'(y)|^r dy \right| d\tau\, d\rho \right]^{r^{-1}} \\
&\quad \times \left[\frac{1}{\Gamma^s(\beta)} \iint_1^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \left(\log\frac{t}{\rho}\right)^{\beta-1} \frac{p(\tau)p(\rho)}{\tau\rho} |\tau - \rho| \right. \\
&\qquad \left. \times \left| \int_\tau^\rho |g'(z)|^s dz \right| d\tau\, d\rho \right]^{s^{-1}}
\end{aligned} \tag{31}$$

or

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\iint_{1}^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\rho}\times\left|\mathscr{H}(\tau,\rho)\right|d\tau\,d\rho$$
$$\leq\frac{\left\|f'\right\|_r\left\|g'\right\|_s}{\Gamma(\alpha)\Gamma(\beta)}\iint_{1}^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\rho}\times\left|\tau-\rho\right|d\tau\,d\rho. \quad (32)$$

In view of (29) and (32) and the properties of modulus, one can easily arrive at the left-sided inequality of Theorem 2. Moreover, we have $1\leq\tau\leq t$, $1\leq\rho\leq t$; hence,

$$0\leq\left|\tau-\rho\right|\leq t. \quad (33)$$

Therefore, from (32), we get

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\iint_{1}^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\rho}\times\left|\mathscr{H}(\tau,\rho)\right|d\tau\,d\rho$$
$$\leq\frac{\left\|f'\right\|_r\left\|g'\right\|_s}{\Gamma(\alpha)\Gamma(\beta)}\iint_{1}^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\times\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{p(\tau)p(\rho)}{\tau\rho}\times\left|\tau-\rho\right|d\tau\,d\rho$$
$$\leq\left\|f'\right\|_r\left\|g'\right\|_s t\,{}_HJ^{\alpha}\{p(t)\}\,{}_HJ^{\beta}\{p(t)\}, \quad (34)$$

which completes the proof of Theorem 2. □

Remark 3. For $\beta=\alpha$, Theorem 2 immediately reduces to Theorem 1.

3. Special Cases

As implications of our main results, we consider some consequent results of Theorems 1 and 2 by suitably choosing the function $p(t)$. Other classes of no weighted inequalities are also obtained. To this end, let us set $p(t)=(\log t)^{\lambda}$ ($\lambda\in[0,\infty)$, $t\in(1,\infty)$); then Theorems 1 and 2 yield the following results.

Corollary 4. *Let f and g be two differentiable functions on $[1,\infty)$. If $f'\in L_r([1,\infty))$, $g'\in L_s([1,\infty))$, $r>1$, $r^{-1}+s^{-1}=1$, then, for all $t>1$, $\lambda\in[0,\infty)$ and $\alpha>0$,*

$$2\left|\frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+\alpha)}(\log t)^{\lambda+\alpha}\,{}_HJ^{\alpha}\left\{(\log t)^{\lambda}f(t)g(t)\right\}-{}_HJ^{\alpha}\left\{(\log t)^{\lambda}f(t)\right\}{}_HJ^{\alpha}\left\{(\log t)^{\lambda}g(t)\right\}\right|$$
$$\leq\frac{\left\|f'\right\|_r\left\|g'\right\|_s}{\Gamma^2(\alpha)}\iint_{1}^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{(\log\tau)^{\lambda}(\log\rho)^{\lambda}}{\tau\rho}\times\left|\tau-\rho\right|d\tau\,d\rho$$
$$\leq\left\|f'\right\|_r\left\|g'\right\|_s t\frac{\Gamma^2(1+\lambda)}{\Gamma^2(1+\lambda+\alpha)}(\log t)^{2\lambda+2\alpha}. \quad (35)$$

Corollary 5. *Let f and g be two differentiable functions on $[1,\infty)$. If $f'\in L_r([1,\infty))$, $g'\in L_s([1,\infty))$, $r>1$, $r^{-1}+s^{-1}=1$, then*

$$\left|\frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+\alpha)}(\log t)^{\lambda+\alpha}\,{}_HJ^{\beta}\left\{(\log t)^{\lambda}f(t)g(t)\right\}+\frac{\Gamma(1+\lambda)}{\Gamma(1+\lambda+\beta)}(\log t)^{\lambda+\beta}\,{}_HJ^{\alpha}\left\{(\log t)^{\lambda}f(t)g(t)\right\}-{}_HJ^{\alpha}\left\{(\log t)^{\lambda}f(t)\right\}{}_HJ^{\beta}\left\{(\log t)^{\lambda}g(t)\right\}-{}_HJ^{\beta}\left\{(\log t)^{\lambda}f(t)\right\}{}_HJ^{\alpha}\left\{(\log t)^{\lambda}g(t)\right\}\right|$$
$$\leq\frac{\left\|f'\right\|_r\left\|g'\right\|_s}{\Gamma(\alpha)\Gamma(\beta)}\iint_{1}^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\beta-1}\frac{(\log\tau)^{\lambda}(\log\rho)^{\lambda}}{\tau\rho}\times\left|\tau-\rho\right|d\tau\,d\rho$$
$$\leq\left\|f'\right\|_r\left\|g'\right\|_s t\frac{\Gamma^2(1+\lambda)}{\Gamma(1+\lambda+\alpha)\Gamma(1+\lambda+\beta)}(\log t)^{2\lambda+\alpha+\beta}, \quad (36)$$

for all $t>1$, $\alpha>0$, $\beta>0$, and $\lambda\in[0,\infty)$.

Further, if we put $\lambda=0$ in Corollaries 4 and 5 (or set $p(t)=1$ in Theorems 1 and 2), we obtain the following results.

Corollary 6. *Let f and g be two differentiable functions on $[1,\infty)$. If $f'\in L_r([1,\infty))$, $g'\in L_s([1,\infty))$, $r>1$, $r^{-1}+s^{-1}=1$, then, for all $t>1$ and $\alpha>0$,*

$$2\left|\frac{(\log t)^{\alpha}}{\Gamma(1+\alpha)}\,{}_HJ^{\alpha}\{f(t)g(t)\}-{}_HJ^{\alpha}\{f(t)\}\,{}_HJ^{\alpha}\{g(t)\}\right|$$
$$\leq\frac{\left\|f'\right\|_r\left\|g'\right\|_s}{\Gamma^2(\alpha)}\iint_{1}^{t}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\left(\log\frac{t}{\rho}\right)^{\alpha-1}\frac{\left|\tau-\rho\right|}{\tau\rho}d\tau\,d\rho$$
$$\leq\left\|f'\right\|_r\left\|g'\right\|_s\frac{t(\log t)^{2\alpha}}{\Gamma^2(1+\alpha)}. \quad (37)$$

Corollary 7. *Let f and g be two differentiable functions on $[1,\infty)$. If $f'\in L_r([1,\infty))$, $g'\in L_s([1,\infty))$, $r>1$, $r^{-1}+s^{-1}=1$, then*

$$\left|\frac{(\log t)^{\alpha}}{\Gamma(1+\alpha)}\,{}_HJ^{\beta}\{f(t)g(t)\}+\frac{(\log t)^{\beta}}{\Gamma(1+\beta)}\,{}_HJ^{\alpha}\{f(t)g(t)\}-{}_HJ^{\alpha}\{f(t)\}\,{}_HJ^{\beta}\{g(t)\}-{}_HJ^{\beta}\{f(t)\}\,{}_HJ^{\alpha}\{g(t)\}\right|$$

$$\leq \frac{\|f'\|_r \|g'\|_s}{\Gamma(\alpha)\Gamma(\beta)} \iint_1^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \left(\log \frac{t}{\rho}\right)^{\beta-1} \frac{|\tau-\rho|}{\tau\rho} d\tau\, d\rho$$
$$\leq \|f'\|_r \|g'\|_s \frac{t(\log t)^{\alpha+\beta}}{\Gamma(1+\alpha)\Gamma(1+\beta)}, \tag{38}$$

for all $t > 1$, $\alpha > 0$, *and* $\beta > 0$.

We conclude our paper by remarking that we have introduced new general Chebyshev type inequalities involving Hadamard fractional integral operators. By suitably specializing the arbitrary function $p(t)$, one can further easily obtain additional fractional integral inequalities from our main results (Theorems 1 and 2).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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