

Research Article

Nonlinear Instability for a Volume-Filling Chemotaxis Model with Logistic Growth

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This paper deals with a Neumann boundary value problem for a volume-filling chemotaxis model with logistic growth in a d -dimensional box $\mathbb{T}^d = (0, \pi)^d$ ($d = 1, 2, 3$). It is proved that given any general perturbation of magnitude δ , its nonlinear evolution is dominated by the corresponding linear dynamics along a finite number of fixed fastest growing modes, over a time period of the order $\ln(1/\delta)$. Each initial perturbation certainly can behave drastically different from another, which gives rise to the richness of patterns.

1. Introduction

An important variant of the chemotaxis model was initially proposed by Painter and Hillen in [1] to model the volume-filling effect. In the volume-filling effect, it is assumed that cells have a certain finite (nonzero) volume and that the occupation of an area limits other cells from penetrating it. A simple version of the volume-filling chemotaxis model (VF) is the following:

$$\begin{aligned} U_t &= \nabla \cdot (d_1 \nabla U - \chi U (1 - U) \nabla V), \\ V_t &= d_2 \nabla^2 V + \alpha U - \beta V, \end{aligned} \quad (1)$$

where d_1, d_2, χ, α , and β are given positive constants. $U(x, t)$ is the cell density and $V(x, t)$ denotes the density of the external chemical substance which is secreted by the cells themselves. d_1 and d_2 denote the cell and chemical diffusion coefficients, respectively. χ is called chemosensitivity. The term $U(1 - U)\nabla V$ denotes the chemotactic flux under a volume constraint 1 (called crowding capacity), meaning that the chemotactic movement will be inhibited at the aggregation location where the cell density reaches 1.

In recent years, the chemotaxis models with volume-filling effect have been studied extensively. Hillen and Painter [2]

firstly proved the global existence of solutions. Numerical simulations in one and two dimensions show interesting phenomena of pattern formation and formation of stable aggregates. Wrzosek [3] showed the existence of a compact global attractor in the space $W^{1,p}(\Omega, \mathbb{R}^2)$, $p > n$, $\Omega \subset \mathbb{R}^n$ for some cases. In [4], the structure of the attractor can be understood using Lyapunov functions. Stationary solutions which are inhomogeneous in space were investigated for a given range of parameters. In [1], a numerical exploration was conducted to determine the longtime patterning behaviour, revealing formation of multiple plateau type patterns which undergo a coarsening process with increasingly long transient times. Potapov and Hillen [5] investigated the metastability of steady states. The underlying bifurcation diagram was identified, revealing that the unstable eigenvalues are exponentially small. The plateau interactions were studied using asymptotic methods. In [6], it was obtained that a priori estimates for the classical chemotaxis model of Patlak, Keller, and Segel when a nonlinear diffusion or a nonlinear chemosensitivity was considered accounting for the finite size of the cells and how entropy estimates give natural conditions on the nonlinearities implying the absence of blow-up for the solutions were showed. Burger et al. [7] discussed

the effects of linear and nonlinear diffusion in the large time asymptotic behavior of the Keller-Segel model of chemotaxis with volume-filling effect. Moreover, the global existence of solutions and nontrivial steady states were also studied. Wang and Hillen [8] established the global existence of classical solutions to a generalized chemotaxis model, which includes the volume-filling effect expressed through a nonlinear squeezing probability. Necessary and sufficient conditions for spatial pattern formation were given and the underlying bifurcations were analyzed. In [9], the stationary solutions of the volume-filling chemotaxis model without a growth term were obtained by Jiang and Zhang.

Moreover, Wrzosek [10] considered various assumptions on nonlinear diffusion and chemotactic sensitivity function which lead to the existence of global in time solutions, thus preventing blow-up. In [11], Winkler proved that if certain conditions were fulfilled, then there were solutions that blow up in either finite or infinite time. In particular, in the framework of chemotaxis models incorporating a volume-filling effect in the sense of Painter and Hillen [1], his results indicated how strongly the cellular movement must be inhibited at large cell densities in order to rule out chemotactic collapse. Winkler and Djie [12] discussed boundedness and finite-time collapse for a chemotaxis system with volume-filling effect. Wang et al. [13] proved that for a wide range of nonlinear diffusion operators, including singular and degenerate ones, if the taxis force was strong enough with respect to diffusion and the initial data were chosen properly, then there was a classical solution which reaches the threshold at the maximal time of its existence; no matter whether the latter was finite or infinite. Zhang and Zheng [14] obtained the crucial uniform boundedness of the solution for a quasilinear nonuniform parabolic system modelling chemotaxis with volume-filling effect and the results on convergence to equilibrium and the decay rate using a suitable nonsmooth Simon-Łojasiewicz approach. In [15], the uniform boundedness, global in time existence and uniqueness of classical solution, were proved. With the help of a suitable nonsmooth Simon-Łojasiewicz approach, the results on convergence of the solution to equilibrium and the convergence rate were obtained. Li and Zhang [16] classified the existence or nonexistence of steady state solutions of a 1-D chemotaxis model with volume-filling effect. Their results provided insights on how the biological parameters affect pattern formation.

The chemotaxis models with logistic growth but without a volume-filling effect were studied (see [17–22]). The global attractor and traveling wave solutions of a volume-filling chemotaxis model with logistic growth were obtained in [3] and [23], respectively. Ma et al. [24] studied the existence of stationary solutions of a volume-filling chemotaxis model with logistic cell growth. Moreover, based on an explicit formula for the stationary solutions, which is derived by asymptotic bifurcation analysis, the stability criteria were established and a selection mechanism of the principal wave modes for the stable stationary solution by estimating the leading term of the principal eigenvalue was found. Quite recently, Ma et al. in [25] studied the nonexistence of nonconstant steady state (i.e., stationary pattern) for

a chemotaxis model with the volume-filling effect and logistic cell growth and established the critical value of the chemotactic coefficient between the existence and the nonexistence of stationary pattern.

Guo and Hwang [26] investigated nonlinear dynamics near an unstable constant equilibrium in the classical Keller-Segel model. Their results can be interpreted as a rigorous mathematical characterization for the early-stage pattern formation in the Keller-Segel model. Very recently, Fu and Liu in [22] and [27] studied instability in the Keller-Segel model with a logistic source and cubic source term, respectively. Their results indicated that chemotaxis-driven nonlinear instability occurs in these models.

In this paper, we mainly consider the nonlinear instability for the following chemotaxis model:

$$U_t = \nabla (d_1 \nabla U - \chi U (1 - U) \nabla V) + \mu U \left(1 - \frac{U}{U_c} \right),$$

$$x \in \mathbb{T}^d, \quad t > 0, \quad (2)$$

$$V_t = d_2 \nabla^2 V + \alpha U - \beta V, \quad x \in \mathbb{T}^d, \quad t > 0$$

which is subject to the Neumann boundary conditions

$$\frac{\partial U}{\partial x_i} = \frac{\partial V}{\partial x_i} = 0, \quad \text{at } x_i = 0, \pi, \quad 1 < i < d \quad (3)$$

and the nonnegative initial data

$$U(x, 0) = U_0(x) \geq 0, \quad V(x, 0) = V_0(x) \geq 0,$$

$$x \in \mathbb{T}^d, \quad (4)$$

where $\mathbb{T}^d = (0, \pi)^d$ ($d = 1, 2, 3$) is a d -dimensional box. The term $\mu U(1 - U/U_c)$ describes the logistic growth of cells with growth rate $\mu > 0$ and carrying capacity U_c fulfilling $0 < U_c \leq 1$. Our main result (see Theorem 6) indicates that the nonlinear dynamics near an unstable constant equilibrium points in the classical Keller-Segel model, the Keller-Segel model with a logistic source and cubic source term, respectively, and a volume-filling chemotaxis model with logistic source term are almost similar.

The organization of this paper is as follows: in Section 2, we show that the unique positive equilibrium point of (2) without chemotaxis is globally asymptotically stable and cross diffusion cannot induce the instability of the positive equilibrium. In Section 3, we consider the growing modes of (2). In Section 4, we present and prove the Bootstrap lemma, which was first introduced in [28]. In Section 5, for any given general perturbation of magnitude δ , we prove that its nonlinear evolution is dominated by the corresponding linear dynamics along a fixed finite number of fastest growing modes, over a time period of the order $\ln(1/\delta)$. Each initial perturbation certainly can behave drastically different from another, which gives rise to the richness of patterns.

2. Stability of Positive Equilibrium Point of (2) without Chemotaxis

We first discuss the following corresponding kinetic equations of (2):

$$\begin{aligned} U_t &= \mu U \left(1 - \frac{U}{U_c} \right), \quad t > 0, \\ V_t &= \alpha U - \beta V, \quad t > 0. \end{aligned} \tag{5}$$

We use $[\cdot, \cdot]$ to denote a column vector. Evidently, (5) has the unique positive equilibrium point $\bar{\mathbf{W}} = [\bar{U}, \bar{V}] = [U_c, (\alpha/\beta)U_c]$. For simplicity, we denote $\mathbf{F}(\mathbf{W}) = [\mu U(1 - U/U_c), \alpha U - \beta V]$, and a direct calculation yields

$$\mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}}) = \begin{pmatrix} -\mu & 0 \\ \alpha & -\beta \end{pmatrix}. \tag{6}$$

The characteristic polynomial of $\mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}})$ is $\varphi(\lambda) = (\lambda + \mu)(\lambda + \beta)$; $-\mu$ and $-\beta$ are the two roots of $\varphi(\lambda) = 0$. Hence $[U_c, (\alpha/\beta)U_c]$ is locally stable. Define $E(t) = p(U - U_c - U_c \ln(U/U_c)) + (V - (\alpha/\beta)U_c)^2$, where $p = 2\alpha^2 U_c / \beta \mu$. By the Lyapunov-LaSalle invariance principle [29], $[U_c, (\alpha/\beta)U_c]$ is globally asymptotically stable.

We now consider system (2) without chemotaxis in the following form:

$$\begin{aligned} U_t &= d_1 \Delta U + \mu U \left(1 - \frac{U}{U_c} \right), \quad x \in \mathbb{T}^d \quad (d = 1, 2, 3), \\ & \quad t > 0, \\ V_t &= d_2 \Delta V + \alpha U - \beta V, \quad x \in \mathbb{T}^d \quad (d = 1, 2, 3), \\ & \quad t > 0, \\ \frac{\partial U}{\partial x_i} &= \frac{\partial V}{\partial x_i} = 0, \quad \text{at } x_i = 0, \pi, \quad \text{for } 1 \leq i \leq d. \end{aligned} \tag{7}$$

Let $0 = \mu_1 < \mu_2 < \mu_3 < \dots$ be the eigenvalues of the operator $-\Delta$ on \mathbb{T}^d ($d = 1, 2, 3$) with the homogeneous Neumann boundary condition and $E(\mu_i)$ the eigenspace corresponding to μ_i in $L^2(\mathbb{T}^d)$. Let $X = [L^2(\mathbb{T}^d)]^2$, $\{\phi_{ij} : j = 1, \dots, \dim E(\mu_i)\}$ be an orthonormal basis of $E(\mu_i)$, and $X_{ij} = \{\mathbf{c} \cdot \phi_{ij} \mid \mathbf{c} \in \mathbb{R}^2\}$. Then $X = \oplus_{i=1}^{\infty} X_i$, $X_i = \oplus_{j=1}^{\dim E(\mu_i)} X_{ij}$.

Let $\mathfrak{D} = \text{diag}(d_1, d_2)$ and $\mathfrak{L} = \mathfrak{D}\Delta + \mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}})$. The linearization of (7) at $[\bar{U}, \bar{V}]$ can be expressed by $\mathbf{W}_t = \mathfrak{L}(\mathbf{W} - \bar{\mathbf{W}})$. For each $i \geq 1$, X_i is invariant under the operator \mathfrak{L} , and λ is an eigenvalue of \mathfrak{L} on X_i if and only if it is an eigenvalue of the matrix

$$-\mu_i \mathfrak{D} + \mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}}) = \begin{pmatrix} -\mu_i d_1 - \mu & 0 \\ \alpha & -\mu_i d_2 - \beta \end{pmatrix}. \tag{8}$$

Thus, $-\mu_i \mathfrak{D} + \mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}})$ has two negative eigenvalues $-\mu_i d_1 - \mu$ and $-\mu_i d_2 - \beta$. It follows from Theorem 5.1.1 in [30] that $[U_c, (\alpha/\beta)U_c]$ is locally asymptotically stable.

Let $\mathbf{W} = [U, V]$ be the unique nonnegative global solution. The maximum principle gives

$$\begin{aligned} 0 &\leq U(x, t) \leq \max \left\{ U_c, \sup_{\mathbb{T}^d} U_0(x) \right\}, \\ 0 &\leq V(x, t) \leq \max \left\{ \frac{\alpha}{\beta} U_c, \frac{\alpha}{\beta} \sup_{\mathbb{T}^d} U_0(x), \sup_{\mathbb{T}^d} V_0(x) \right\}. \end{aligned} \tag{9}$$

Moreover, by the strong maximum principle [31], we know that if $U_0, V_0 \geq (\neq) 0$, then $U(x, t) > 0, V(x, t) > 0$ on $\bar{\mathbb{T}}^d$ for all $t > 0$.

We define the Lyapunov function

$$E(t) = \int_{\mathbb{T}^d} \left[p \left(U - U_c - U_c \ln \frac{U}{U_c} \right) + \left(V - \frac{\alpha}{\beta} U_c \right)^2 \right] dx, \tag{10}$$

where $p = 2\alpha^2 U_c / \beta \mu$. Then $E(t) \geq 0$ for all $t \geq 0$. Applying (7) and integration by parts, we have

$$\begin{aligned} E'(t) &= - \int_{\mathbb{T}^d} \left\{ \frac{p d_1 U_c}{U^2} |\nabla U|^2 + 2d_2 |\nabla V|^2 \right\} dx \\ &\quad - \int_{\mathbb{T}^d} \left\{ \frac{p \mu}{U_c} (U - U_c)^2 - 2\alpha (U - U_c) \left(V - \frac{\alpha}{\beta} U_c \right) \right. \\ &\quad \left. + 2\beta \left(V - \frac{\alpha}{\beta} U_c \right)^2 \right\} dx \\ &\leq - \frac{\alpha^2}{\beta} \int_{\mathbb{T}^d} (U - U_c)^2 dx \\ &\quad - \beta \int_{\mathbb{T}^d} \left(V - \frac{\alpha}{\beta} U_c \right)^2 dx. \end{aligned} \tag{11}$$

By (9), (11), the basic estimates for parabolic equations [31], and Lemma 2.1 in [22] (which is given in [32] in Chinese), we can conclude that

$$\begin{aligned} \lim_{t \rightarrow \infty} \|U(\cdot, t) - U_c\|_{L^2(\mathbb{T}^d)} &= 0, \\ \lim_{t \rightarrow \infty} \left\| V(\cdot, t) - \frac{\alpha}{\beta} U_c \right\|_{L^2(\mathbb{T}^d)} &= 0. \end{aligned} \tag{12}$$

The global asymptotic stability of $[U_c, (\alpha/\beta)U_c]$ follows from (12) together with the local stability of $[U_c, (\alpha/\beta)U_c]$.

Next, we consider that the cross diffusion model

$$\begin{aligned} U_t &= d_1 \Delta (U + d_3 UV) + \mu U \left(1 - \frac{U}{U_c} \right), \\ & \quad x \in \mathbb{T}^d, \quad t > 0, \\ V_t &= d_2 \Delta V + \alpha U - \beta V, \quad x \in \mathbb{T}^d, \quad t > 0, \end{aligned}$$

$$\begin{aligned} \frac{\partial U}{\partial x_i} &= \frac{\partial V}{\partial x_i} = 0, \quad \text{at } x_i = 0, \pi, \quad \text{for } 1 \leq i \leq d, \\ U(x, 0) &= U_0(x) \geq 0, \quad V(x, 0) = V_0(x) \geq 0, \\ x &\in \mathbb{T}^d. \end{aligned} \quad (13)$$

Let $\Phi(\mathbf{W}) = [d_1(U + d_3UV), d_2V]$. Then the linearizing system (13) at $\bar{\mathbf{W}}$ can be written as

$$\mathbf{W}_t = (\Phi_{\mathbf{W}}\Delta + \mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}})) \mathbf{W}, \quad (14)$$

where

$$\Phi_{\mathbf{W}}(\bar{\mathbf{W}}) = \Phi_{\mathbf{W}} = \begin{pmatrix} d_1(1 + d_3\bar{V}) & d_1d_3\bar{U} \\ 0 & d_2 \end{pmatrix}. \quad (15)$$

Then, for each $i \in \{1, 2, \dots\}$, X_i is invariant under the operator $\Phi_{\mathbf{W}}\Delta + \mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}})$, and ξ is an eigenvalue of $\Phi_{\mathbf{W}}\Delta + \mathbf{F}_{\mathbf{W}}(\bar{\mathbf{W}})$ on X_i if and only if ξ is an eigenvalue of the matrix

$$\mathcal{A}_i = \begin{pmatrix} -d_1(1 + d_3\bar{V})\mu_i - \mu & -d_1d_3\bar{U}\mu_i \\ \alpha & -d_2\mu_i - \beta \end{pmatrix}. \quad (16)$$

Notice that

$$\begin{aligned} \det \mathcal{A}_i &= d_1d_2(1 + d_3\bar{V})\mu_i^2 \\ &\quad + (d_2\mu + \beta d_1(1 + d_3\bar{V}) + d_1d_3\alpha\bar{U})\mu_i \\ &\quad + \mu\beta > 0, \end{aligned} \quad (17)$$

$$\text{Tr } \mathcal{A}_i = -((d_1 + d_2 + d_1d_3\bar{V})\mu_i + \mu + \beta) < 0.$$

Thus, the two eigenvalues ξ_i^+ and ξ_i^- of \mathcal{A}_i have negative real parts.

From this, we see that adding the cross diffusion to the system (7), the positive constant solution is also locally stable, which means that Turing instability does not occur. The information above indicates that the aggregation of individuals does not occur in the absence of chemotactic effect. It is the purpose of the present paper to clarify the effect of the chemotaxis and nonlinear patterns created by chemotaxis for a volume-filling chemotaxis model with logistic growth.

3. Growing Modes in the System (2)

The nonlinear evolution of a perturbation $u(\mathbf{x}, t) = U(\mathbf{x}, t) - \bar{U}$, $v(\mathbf{x}, t) = V(\mathbf{x}, t) - \bar{V}$ around $[\bar{U}, \bar{V}]$ satisfies

$$\begin{aligned} u_t &= d_1\nabla^2 u - \chi U_c(1 - U_c)\nabla^2 v - \mu u \\ &\quad - \chi\nabla[u(1 - u - 2U_c)\nabla v] - \frac{\mu u^2}{U_c}, \end{aligned} \quad (18)$$

$$v_t = d_2\nabla^2 v + \alpha u - \beta v.$$

The corresponding linearized system is as follows:

$$\begin{aligned} u_t &= d_1\nabla^2 u - \chi U_c(1 - U_c)\nabla^2 v - \mu u, \\ v_t &= d_2\nabla^2 v + \alpha u - \beta v. \end{aligned} \quad (19)$$

Let $\mathbf{w}(\mathbf{x}, t) \equiv [u(\mathbf{x}, t), v(\mathbf{x}, t)]$, $\mathbf{q} = (q_1, \dots, q_d) \in \Omega = (\mathbb{N} \cup \{0\})^d$, and $e_{\mathbf{q}}(\mathbf{x}) = \prod_{i=1}^d \cos(q_i x_i)$. Then $\{e_{\mathbf{q}}(\mathbf{x})\}_{\mathbf{q} \in \Omega}$ forms a basis of the space of functions in \mathbb{T}^d that satisfies Neumann boundary conditions. We try to find a normal mode to the linear system (19) of the following form:

$$\mathbf{w}(\mathbf{x}, t) = \mathbf{r}_{\mathbf{q}} e^{\lambda_{\mathbf{q}} t} e_{\mathbf{q}}(\mathbf{x}), \quad (20)$$

where $\mathbf{r}_{\mathbf{q}}$ is a vector depending on \mathbf{q} . Substituting (20) into (19) yields

$$\lambda_{\mathbf{q}} \mathbf{r}_{\mathbf{q}} = \begin{pmatrix} -d_1 q^2 - \mu & \chi U_c(1 - U_c) q^2 \\ \alpha & -d_2 q^2 - \beta \end{pmatrix} \mathbf{r}_{\mathbf{q}}, \quad (21)$$

where $q^2 = \sum_{i=1}^d q_i^2$. We can obtain the following dispersion formula for $\lambda_{\mathbf{q}}$:

$$\begin{aligned} \lambda_{\mathbf{q}}^2 &+ ((d_1 + d_2)q^2 + \beta + \mu)\lambda_{\mathbf{q}} \\ &+ q^2 [d_1 d_2 q^2 + (d_1 \beta + d_2 \mu - \alpha \chi U_c(1 - U_c))] \\ &+ \mu \beta = 0. \end{aligned} \quad (22)$$

Thus, we deduce the following linear instability criterion by requiring that there exist a \mathbf{q} such that

$$q^2 [d_1 d_2 q^2 + (d_1 \beta + d_2 \mu - \alpha \chi U_c(1 - U_c))] + \mu \beta < 0 \quad (23)$$

to ensure that (22) has at least one positive root $\lambda_{\mathbf{q}}$. This means that

$$(q^2(d_1 - d_2) - \beta + \mu)^2 + 4\alpha \chi U_c(1 - U_c)q^2 > 0. \quad (24)$$

There exist two distinct real roots:

$$\begin{aligned} \lambda_{\mathbf{q}}^{\pm} &= \left(- (q^2(d_1 + d_2) + \beta + \mu) \right. \\ &\quad \left. \pm \sqrt{(q^2(d_1 - d_2) - \beta + \mu)^2 + 4\alpha \chi U_c(1 - U_c)q^2} \right) \\ &\quad \times (2)^{-1} \end{aligned} \quad (25)$$

for all $\mathbf{q} \neq \mathbf{0}$ to the quadratic equation (22), which are denoted by $\lambda_{\mathbf{q}}^- < \lambda_{\mathbf{q}}^+$. We denote the corresponding (linearly independent) eigenvectors by $\mathbf{r}_-(\mathbf{q})$ and $\mathbf{r}_+(\mathbf{q})$, such that

$$\mathbf{r}_{\pm}(\mathbf{q}) = \left[\frac{\lambda_{\mathbf{q}}^{\pm} + d_2 q^2 + \beta}{\alpha}, 1 \right]. \quad (26)$$

Clearly, for q large,

$$q^2 [d_1 d_2 q^2 + (d_1 \beta + d_2 \mu - \alpha \chi U_c(1 - U_c))] + \mu \beta > 0. \quad (27)$$

Thus, there are only finitely many \mathbf{q} such that $\lambda_{\mathbf{q}}^+ > 0$. We denote the largest eigenvalue by $\lambda_{\max}^+ > 0$ and define

$\Omega_{\max} \equiv \{\mathbf{q} \in \Omega \mid \lambda_{\mathbf{q}}^+ = \lambda_{\max}\}$. From (25) we can regard $\lambda_{\mathbf{q}}^+$ as a function of q^2 . Therefore, there is one q^2 (possibly two) having $\lambda^+(q^2) = \lambda_{\max}$. We also denote $\rho > 0$ to be the gap between the λ_{\max} and the rest; that is, $\rho = \min_{\mathbf{q} \in \Omega \setminus \Omega_{\max}} |\lambda_{\max} - \lambda_{\mathbf{q}}|$.

Given any initial perturbation $\mathbf{w}(\mathbf{x}, 0)$, we can expand it as

$$\mathbf{w}(\mathbf{x}, 0) = \sum_{\mathbf{q} \in \Omega} \mathbf{w}_{\mathbf{q}} e_{\mathbf{q}}(\mathbf{x}) = \sum_{\mathbf{q} \in \Omega} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q})\} e_{\mathbf{q}}(\mathbf{x}), \tag{28}$$

$$\mathbf{w}_{\mathbf{q}} = w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}). \tag{29}$$

In the sequel, denote by $\langle \cdot, \cdot \rangle$ and (\cdot, \cdot) the inner product of $[L^2(\mathbb{T}^d)]^2$ and the scalar product of \mathbb{R}^2 , respectively. For any $\mathbf{g}(\cdot, t) \in [L^2(\mathbb{T}^d)]^2$, we denote $\|\mathbf{g}(\cdot, t)\| \equiv \|\mathbf{g}(\cdot, t)\|_{L^2}$. Throughout this paper, we always denote universal constants by C_i ($i = 1, 2, \dots$).

Clearly,

$$\|\mathbf{w}(\mathbf{x}, 0)\|^2 = \left(\frac{\pi}{2}\right)^d \sum_{\mathbf{q} \in \Omega} |\mathbf{w}_{\mathbf{q}}|^2. \tag{30}$$

The unique solution $\mathbf{w}(\mathbf{x}, t) = [u(\mathbf{x}, t), v(\mathbf{x}, t)]$ of (19) is given by

$$\begin{aligned} \mathbf{w}(\mathbf{x}, t) &= \sum_{\mathbf{q} \in \Omega} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) e^{\lambda_{\mathbf{q}}^- t} + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e^{\lambda_{\mathbf{q}}^+ t}\} e_{\mathbf{q}}(\mathbf{x}) \\ &\equiv e^{2t} \mathbf{w}(\mathbf{x}, 0). \end{aligned} \tag{31}$$

Our main result in this section is the following lemma.

Lemma 1. *Suppose that the instability criterion (23) holds. Let $\mathbf{w}(\mathbf{x}, t) \equiv e^{2t} \mathbf{w}(\mathbf{x}, 0)$ be a solution to the linearized system (19) with initial condition $\mathbf{w}(\mathbf{x}, 0)$. Then there exists a constant $\widehat{C}_1 \geq 1$ depending on $d_1, d_2, U_c, \chi, \mu, \alpha$, and β , such that*

$$\|\mathbf{w}(\cdot, t)\| \leq \widehat{C}_1 e^{\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|, \quad \forall t \geq 0. \tag{32}$$

Proof. We divide the proof into the following two cases.

(1) $t \geq 1$. It follows from (25) that

$$\lim_{\mathbf{q} \rightarrow \infty} \frac{\lambda_{\mathbf{q}}^{\pm}}{q^2} = -d_1, -d_2, \tag{33}$$

respectively. Thus, $\lambda_{\mathbf{q}}^{\pm} \leq -\min\{d_1, d_2\}q^2$ and there exists a positive constant C_1 for all $\mathbf{q} > 0$, such that

$$\left| \frac{\lambda_{\mathbf{q}}^{\pm}}{q^2} \right| \leq C_1. \tag{34}$$

By the quadratic formula of (22), one can obtain

$$|\lambda_{\mathbf{q}}^+ - \lambda_{\mathbf{q}}^-| \geq 2q \sqrt{\alpha \chi U_c (1 - U_c)}. \tag{35}$$

It follows from (29) that

$$|w_{\mathbf{q}}^{\pm}| \leq \frac{|\mathbf{r}_{\pm}(\mathbf{q})| \times |\mathbf{w}_{\mathbf{q}}|}{|\det[\mathbf{r}_-(\mathbf{q}), \mathbf{r}_+(\mathbf{q})]|}. \tag{36}$$

From (26) and (34), we can conclude that there exists a positive constant C_2 such that

$$|\mathbf{r}_{\pm}(\mathbf{q})| = \sqrt{\left[\left(\frac{\lambda_{\mathbf{q}}^{\pm}}{\alpha q^2} + \frac{d_2}{\alpha} \right) q^2 + \frac{\beta}{\alpha} \right]^2 + 1} \leq C_2 q^2 \tag{37}$$

for all $q > 0$, where $C_2 = 2 \max\{C_1 + d_2/\alpha, 1 + \beta/\alpha\}$. By (34), one can deduce

$$\frac{1}{|\det[\mathbf{r}_-(\mathbf{q}), \mathbf{r}_+(\mathbf{q})]|} \leq \frac{1}{2q} \sqrt{\frac{\alpha}{\chi U_c (1 - U_c)}}. \tag{38}$$

Combining (37) and (38), we find that

$$|w_{\mathbf{q}}^{\pm}| \leq C_3 q |\mathbf{w}_{\mathbf{q}}|, \tag{39}$$

where $C_3 = (C_2/2) \sqrt{\alpha/\chi U_c (1 - U_c)}$. For $t \geq 1$ and q large, it is not difficult to verify by (33), (37), and (39) that

$$\begin{aligned} |w_{\mathbf{q}}^{\pm} \mathbf{r}_{\pm}(\mathbf{q}) e^{\lambda_{\mathbf{q}}^{\pm} t}| &\leq C_2 C_3 q^3 |\mathbf{w}_{\mathbf{q}}| e^{\lambda_{\mathbf{q}}^{\pm} t} \\ &\leq C_4 |\mathbf{w}_{\mathbf{q}}| \frac{q^3}{\exp(\min\{d_1, d_2\} q^2)} \leq C_5 |\mathbf{w}_{\mathbf{q}}|. \end{aligned} \tag{40}$$

In view of (30) and (40), we observe that

$$\|\mathbf{w}(\mathbf{x}, t)\| \leq 2C_5 e^{\lambda_{\max} t} \|\mathbf{w}(\mathbf{x}, 0)\|, \quad \text{for } t \geq 1. \tag{41}$$

(2) $t \leq 1$. Multiplying the first equation of (19) by u and the second by Kv , adding them together, and integrating the result in \mathbb{T}^d , we have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} \{ |u|^2 + K|v|^2 \} \mathbf{d}\mathbf{x} \\ &+ \int_{\mathbb{T}^d} \{ d_1 |\nabla u|^2 + K d_2 |\nabla v|^2 \\ &\quad - \chi U_c (1 - U_c) \nabla u \nabla v \} \mathbf{d}\mathbf{x} \\ &= -\mu \int_{\mathbb{T}^d} u^2 \mathbf{d}\mathbf{x} - K\beta \int_{\mathbb{T}^d} v^2 \mathbf{d}\mathbf{x} \\ &+ \alpha K \int_{\mathbb{T}^d} uv \mathbf{d}\mathbf{x}. \end{aligned} \tag{42}$$

Let

$$K = \frac{[\chi U_c (1 - U_c)]^2}{d_1 d_2}. \tag{43}$$

Then the integrand of the second integral can be estimated as follows:

$$\begin{aligned} & d_1 |\nabla u|^2 + K d_2 |\nabla v|^2 - \chi U_c (1 - U_c) \nabla u \nabla v \\ & \geq \frac{d_1}{2} |\nabla u|^2 + \frac{[\chi U_c (1 - U_c)]^2}{2d_1} |\nabla v|^2 \geq 0. \end{aligned} \quad (44)$$

Using Young inequality, we deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} \{|u|^2 + K|v|^2\} \mathbf{dx} \leq \frac{\alpha \sqrt{K}}{2} \int_{\mathbb{T}^d} \{|u|^2 + K|v|^2\} \mathbf{dx}. \quad (45)$$

It follows from Gronwall inequality that

$$\int_{\mathbb{T}^d} \{|u(\mathbf{x}, t)|^2 + K|v(\mathbf{x}, t)|^2\} \mathbf{dx} \quad (46)$$

$$\leq e^{\alpha \sqrt{K} t} \int_{\mathbb{T}^d} \{|u(\mathbf{x}, t)|^2 + K|v(\mathbf{x}, t)|^2\} \mathbf{dx}.$$

If $K \geq 1$ and $t \leq 1$, then it is clear from (46) that

$$\|\mathbf{w}(\mathbf{x}, t)\| \leq \sqrt{K e^{\alpha \sqrt{K}} e^{\lambda_{\max} t}} \|\mathbf{w}(\mathbf{x}, 0)\|. \quad (47)$$

Similarly, if $0 < K < 1$, $t \leq 1$, then from (46), one has

$$\|\mathbf{w}(\mathbf{x}, t)\| \leq \sqrt{e^{\alpha \sqrt{K}} K e^{\lambda_{\max} t}} \|\mathbf{w}(\mathbf{x}, 0)\|. \quad (48)$$

Let $\widehat{C}_1 = \max\{2C_5, \sqrt{e^{\alpha \sqrt{K}}}\} \geq 1$ if $0 < K < 1$, and let $\widehat{C}_1 = \max\{2C_5, \sqrt{K e^{\alpha \sqrt{K}}}\} \geq 1$ if $K \geq 1$. Then

$$\|\mathbf{w}(\mathbf{x}, t)\| \leq \widehat{C}_1 e^{\lambda_{\max} t} \|\mathbf{w}(\mathbf{x}, 0)\|. \quad (49)$$

This completes the proof of Lemma 1. \square

4. Bootstrap Lemma

By a standard PDE theory [31], we can establish the existence of local solutions for (18).

Lemma 2 (Local existence). *For $s \geq 1$ ($d = 1$) and $s \geq 2$ ($d = 2, 3$), there exists a $T_0 > 0$ such that (18) with $u(\cdot, 0), v(\cdot, 0) \in H^s$ has a unique solution $\mathbf{w}(\cdot, t)$ on $(0, T_0)$ which satisfies*

$$\|\mathbf{w}(t)\|_{H^s} \leq C \|\mathbf{w}(0)\|_{H^s}, \quad 0 < t < T_0, \quad (50)$$

where C is a positive constant depending on $d_1, d_2, U_c, \alpha, \beta$, and χ .

It is not difficult to verify the following result.

Lemma 3. *Let $\mathbf{w}(\mathbf{x}, t)$ be a solution of (18). Then the even extension of $\mathbf{w}(\mathbf{x}, t)$ on $2\mathbb{T}^d = (-\pi, \pi)^d$ ($d = 1, 2, 3$) is also the solution of (18) which satisfies homogeneous Neumann boundary conditions and periodical boundary conditions on $2\mathbb{T}^d = (-\pi, \pi)^d$ ($d = 1, 2, 3$).*

Lemma 4. *Let $[u(\mathbf{x}, t), v(\mathbf{x}, t)]$ be a solution of (18). Then*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ |D^\alpha u|^2 + \frac{[\chi U_c (1 - U_c)]^2}{d_1 d_2} |D^\alpha v|^2 \right\} \mathbf{dx} \\ & + \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ \frac{d_1}{4} |\nabla D^\alpha u|^2 + \frac{[\chi U_c (1 - U_c)]^2}{2d_1} |\nabla D^\alpha v|^2 \right\} \mathbf{dx} \\ & + \mu \sum_{|\alpha|=2} \int_{\mathbb{T}^d} |D^\alpha u|^2 \mathbf{dx} + \frac{\beta [\chi U_c (1 - U_c)]^2}{2d_1 d_2} \\ & \times \sum_{|\alpha|=2} \int_{\mathbb{T}^d} |D^\alpha v|^2 \mathbf{dx} \\ & \leq \widehat{C}_2 (\|\mathbf{w}\|_{H^2} + \|\mathbf{w}\|_{H^2}^2) \|\nabla^3 \mathbf{w}\|^2 + \widehat{C}_3 \|u\|^2, \end{aligned} \quad (51)$$

where $\widehat{C}_3 = C_0^3 \alpha^3 [\chi U_c (1 - U_c)]^6 / 2\beta^3 d_1^5 d_2^3$.

Proof. It is known by Lemma 3 that

$$\begin{aligned} \tilde{u}_t &= d_1 \nabla^2 \tilde{u} - \chi U_c (1 - U_c) \nabla^2 \tilde{v} - \mu \tilde{u} \\ & - \chi \nabla [\tilde{u} (1 - \tilde{u} - 2U_c) \nabla \tilde{v}] - \frac{\mu \tilde{u}^2}{U_c}, \end{aligned} \quad (52)$$

$$x \in 2\mathbb{T}^d, \quad t > 0,$$

$$\tilde{v}_t = d_2 \nabla^2 \tilde{v} + \alpha \tilde{u} - \beta \tilde{v}, \quad x \in 2\mathbb{T}^d, \quad t > 0,$$

$$\frac{\partial \tilde{u}}{\partial x_i} = \frac{\partial \tilde{v}}{\partial x_j} = 0, \quad \text{at } x_i = -\pi, 0, \pi, \text{ for } 1 \leq i \leq d, \quad (53)$$

where $[\tilde{u}(\mathbf{x}, t), \tilde{v}(\mathbf{x}, t)]$ is the even extension of $[u(\mathbf{x}, t), v(\mathbf{x}, t)]$ on $2\mathbb{T}^d$. Taking the second-order derivative of (52) for x_i, x_j and making inner product with $\partial_{x_i x_j} \tilde{u}$ and $K \partial_{x_i x_j} \tilde{v}$, respectively, and adding them together, we deduce that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{2\mathbb{T}^d} \left\{ \left| \partial_{x_i x_j} \tilde{u} \right|^2 + K \left| \partial_{x_i x_j} \tilde{v} \right|^2 \right\} \mathbf{dx} \\ & + \int_{2\mathbb{T}^d} \left\{ d_1 \left| \nabla \partial_{x_i x_j} \tilde{u} \right|^2 + K d_2 \left| \nabla \partial_{x_i x_j} \tilde{v} \right|^2 \right. \\ & \quad \left. - \chi U_c (1 - U_c) \nabla \partial_{x_i x_j} \tilde{u} \cdot \nabla \partial_{x_i x_j} \tilde{v} \right\} \mathbf{dx} \\ & + \mu \int_{2\mathbb{T}^d} \left| \partial_{x_i x_j} \tilde{u} \right|^2 \mathbf{dx} + \beta K \int_{2\mathbb{T}^d} \left| \partial_{x_i x_j} \tilde{v} \right|^2 \mathbf{dx} \\ & = \chi (1 - 2U_c) \int_{2\mathbb{T}^d} \nabla \partial_{x_i x_j} \tilde{u} \cdot \partial_{x_i x_j} (\tilde{u} \cdot \nabla \tilde{v}) \mathbf{dx} \\ & - \chi \int_{2\mathbb{T}^d} \nabla \partial_{x_i x_j} \tilde{u} \cdot \partial_{x_i x_j} (\tilde{u}^2 \nabla \tilde{v}) \mathbf{dx} \\ & + K \alpha \int_{2\mathbb{T}^d} \partial_{x_i x_j} \tilde{u} \cdot \partial_{x_i x_j} \tilde{v} \mathbf{dx} \\ & - \frac{2\mu}{U_c} \int_{2\mathbb{T}^d} \left[\tilde{u} (\partial_{x_i x_j} \tilde{u})^2 + \partial_{x_i} \tilde{u} \cdot \partial_{x_j} \tilde{u} \cdot \partial_{x_i x_j} \tilde{u} \right] \mathbf{dx} \\ & \equiv I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (54)$$

We can apply Young inequality and (43) to get

$$\begin{aligned}
 & d_1 \left| \nabla \partial_{x_i x_j} \tilde{u} \right|^2 + K d_2 \left| \nabla \partial_{x_i x_j} \tilde{v} \right|^2 \\
 & - \chi U_c (1 - U_c) \nabla \partial_{x_i x_j} \tilde{u} \cdot \nabla \partial_{x_i x_j} \tilde{v} \\
 & \geq \frac{d_1}{2} \left| \nabla \partial_{x_i x_j} \tilde{u} \right|^2 + \frac{[\chi U_c (1 - U_c)]^2}{2d_1} \left| \nabla \partial_{x_i x_j} \tilde{v} \right|^2.
 \end{aligned} \tag{55}$$

Now we estimate each term on the right-hand sides of (54). By using Hölder inequality,

$$\begin{aligned}
 I_1 & \leq \chi |1 - 2U_c| \\
 & \times \left\{ \|\nabla \tilde{v}\|_{L^\infty} \|\nabla \partial_{x_i x_j} \tilde{u}\| \|\partial_{x_i x_j} \tilde{u}\| \right. \\
 & + 2 \sum_{i=1}^d \|\nabla \tilde{u}\|_{L^\infty} \|\partial_{x_i x_j} \tilde{v}\| \|\nabla \partial_{x_i x_j} \tilde{u}\| \\
 & \left. + \|\tilde{u}\|_{L^\infty} \|\nabla \partial_{x_i x_j} \tilde{u}\| \|\nabla \partial_{x_i x_j} \tilde{v}\| \right\}.
 \end{aligned} \tag{56}$$

Notice that

$$\begin{aligned}
 \|g\|_{L^\infty(2\mathbb{T}^d)} & \leq C_6 \|g\|_{H^2(2\mathbb{T}^d)}, \\
 \|g\|_{L^4(2\mathbb{T}^d)} & \leq C_7 \|g\|_{H^2(2\mathbb{T}^d)}, \\
 \|g\|_{L^6(2\mathbb{T}^d)} & \leq C_8 \|g\|_{H^2(2\mathbb{T}^d)},
 \end{aligned} \tag{57}$$

for $d \leq 3$. Clearly,

$$\begin{aligned}
 \int_{2\mathbb{T}^d} \nabla \tilde{u} \, d\mathbf{x} & = \int_{2\mathbb{T}^d} \nabla \tilde{v} \, d\mathbf{x} = 0, \\
 \int_{2\mathbb{T}^d} \partial_{x_i x_j} \tilde{u} \, d\mathbf{x} & = \int_{2\mathbb{T}^d} \partial_{x_i x_j} \tilde{v} \, d\mathbf{x} = 0.
 \end{aligned} \tag{58}$$

Using the Poincaré inequality, we have

$$\begin{aligned}
 \|g\| & \leq C_9 \|g\|_{L^4(2\mathbb{T}^d)} \leq C_{10} \|g\|_{L^6(2\mathbb{T}^d)} \\
 & \leq C_{11} \|g\|_{H^1} \leq C_{12} \|\nabla g\|, \quad d \leq 3.
 \end{aligned} \tag{59}$$

It follows from (58) and (59) that

$$\|\partial_{x_i} g\| \leq C_{12} \|\nabla \partial_{x_i} g\|, \quad \|\partial_{x_i x_j} g\| \leq C_{12} \|\nabla \partial_{x_i x_j} g\|, \tag{60}$$

$$\|\nabla g\| \leq C_{12} \left(\sum_{|\alpha|=2} \|D^\alpha g\|^2 \right)^{1/2} \leq C_{12}^2 \left(\sum_{|\alpha|=2} \|\nabla D^\alpha g\|^2 \right)^{1/2}. \tag{61}$$

Furthermore,

$$\|\nabla g\|_{H^2} \leq C_{13} \left(\sum_{|\alpha|=2} \|\nabla D^\alpha g\|^2 \right)^{1/2}, \tag{62}$$

$$C_{13} = (C_{12}^4 + C_{12}^2 + 1)^{1/2}.$$

Combining (56), (57), and (62), we observe that

$$\sum_{|\alpha|=2} I_1 \leq \chi |1 - 2U_c| C_{14} \|\tilde{\mathbf{w}}\|_{H^2} \|\nabla^3 \tilde{\mathbf{w}}\|^2, \tag{63}$$

where $C_{14} = C_6[(1 + 2d)C_{13} + 1]$.

By Hölder inequality, it follows from (57) and (62) that

$$\sum_{|\alpha|=2} I_2 \leq \chi C_{15} \|\tilde{\mathbf{w}}\|_{H^2}^2 \|\nabla^3 \tilde{\mathbf{w}}\|^2, \tag{64}$$

where $C_{15} = C_6(2C_9^{-2}C_{12}^2C_{13} + 6C_6C_{13} + C_6)$.

Now we estimate I_3 . By interpolation for all $a > 0$, it can be proved that

$$\|\partial_{x_i x_j} \tilde{u}\|^2 \leq C_0 \left(a \|\nabla \partial_{x_i x_j} \tilde{u}\|^2 + \frac{\|\tilde{u}\|^2}{4a^2} \right). \tag{65}$$

Then it is easy to see that

$$\begin{aligned}
 \sum_{|\alpha|=2} I_3 & \leq \frac{K\beta}{2} \sum_{|\alpha|=2} \int_{2\mathbb{T}^d} |D^\alpha \tilde{v}|^2 \, d\mathbf{x} \\
 & + \frac{d_1}{4} \sum_{|\alpha|=2} \int_{2\mathbb{T}^d} |\nabla D^\alpha \tilde{u}|^2 \, d\mathbf{x} + \widehat{C}_3 \|\tilde{u}\|^2,
 \end{aligned} \tag{66}$$

where $\widehat{C}_3 = C_0^3 \alpha^3 [\chi U_c (1 - U_c)]^6 / 2\beta^3 d_1^5 d_2^3$ and $a = d_1^2 d_2 \beta / 2[\chi U_c (1 - U_c)]^2 C_0 \alpha$.

Finally, from (57) and (61), I_4 is bounded by

$$\sum_{|\alpha|=2} I_4 \leq \frac{4\mu}{U_c} C_{16} \|\tilde{\mathbf{w}}\|_{H^2} \|\nabla^3 \tilde{\mathbf{w}}\|^2. \tag{67}$$

Combining (54), (55), and (63)–(67), one can obtain

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \{|D^\alpha u|^2 + K|D^\alpha v|^2\} \, d\mathbf{x} \\
 & + \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ \frac{d_1}{4} |\nabla D^\alpha u|^2 + \frac{Kd_2}{2} |\nabla D^\alpha v|^2 \right\} \, d\mathbf{x} \\
 & + \mu \sum_{|\alpha|=2} \int_{\mathbb{T}^d} |D^\alpha u|^2 \, d\mathbf{x} + \frac{\beta K}{2} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} |D^\alpha v|^2 \, d\mathbf{x} \\
 & \leq \widehat{C}_2 (\|\mathbf{w}\|_{H^2} + \|\mathbf{w}\|_{H^2}^2) \|\nabla^3 \mathbf{w}\|^2 + \widehat{C}_3 \|u\|^2,
 \end{aligned} \tag{68}$$

where $\widehat{C}_2 = (\chi|1 - 2U_c| + \chi + (4\mu/U_c)) \max\{C_{14}, C_{15}, C_{16}\}$ and the proof is completed. \square

Lemma 5. Let $\mathbf{w}(\mathbf{x}, t)$ be a solution of (18) such that for $0 \leq t \leq T < T_0$,

$$\|\mathbf{w}(\cdot, t)\|_{H^2} + \|\mathbf{w}(\cdot, t)\|_{H^2}^2 \leq \frac{1}{\widehat{C}_2} \min \left\{ \frac{d_1}{4}, \frac{[\chi U_c (1 - U_c)]^2}{2d_1} \right\}, \tag{69}$$

$$\|\mathbf{w}(\cdot, t)\| \leq 2\widehat{C}_1 e^{\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|. \tag{70}$$

Then

$$\|\mathbf{w}(\cdot, t)\|_{H^2}^2 \leq \widehat{C}_4 \left\{ \|\mathbf{w}(\cdot, 0)\|_{H^2}^2 + e^{2\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|^2 \right\}, \quad (71)$$

$$0 \leq t \leq T,$$

where $\widehat{C}_4 = \max\{(1 + C_{12}^2)[\chi U_c(1 - U_c)]^2/d_1 d_2, 4\widehat{C}_1^2[1 + \widehat{C}_3(1 + C_{12}^2)/\lambda_{\max}]\}$ if $[\chi U_c(1 - U_c)]^2/d_1 d_2 \geq 1$ and $\widehat{C}_4 = \max\{(1 + C_{12}^2)d_1 d_2/[\chi U_c(1 - U_c)]^2, 4\widehat{C}_1^2[1 + \widehat{C}_3(1 + C_{12}^2)d_1 d_2/\lambda_{\max}[\chi U_c(1 - U_c)]^2]\}$ if $[\chi U_c(1 - U_c)]^2/d_1 d_2 < 1$.

Proof. It follows from (61) that

$$\|\nabla \mathbf{w}(\cdot, t)\|^2 \leq C_{12}^2 \sum_{|\alpha|=2} \|D^\alpha \mathbf{w}(\cdot, t)\|^2. \quad (72)$$

Thus,

$$\|\mathbf{w}(\cdot, t)\|_{H^2}^2 \leq \|\mathbf{w}(\cdot, t)\|^2 + (C_{12}^2 + 1) \sum_{|\alpha|=2} \|D^\alpha \mathbf{w}(\cdot, t)\|^2. \quad (73)$$

Now we estimate the second-order derivatives of $\mathbf{w}(\cdot, t)$. From Lemma 4 and (69), we can obtain

$$\frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ |D^\alpha u|^2 + \frac{[\chi U_c(1 - U_c)]^2}{d_1 d_2} |D^\alpha v|^2 \right\} \mathbf{d}\mathbf{x} \quad (74)$$

$$\leq \widehat{C}_3 \|u\|^2 \leq \widehat{C}_3 \|\mathbf{w}(\cdot, t)\|^2.$$

Integrating this from 0 to t , we deduce from (70) that

$$\sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ |D^\alpha u(\cdot, t)|^2 + \frac{[\chi U_c(1 - U_c)]^2}{d_1 d_2} |D^\alpha v(\cdot, t)|^2 \right\} \mathbf{d}\mathbf{x}$$

$$\leq \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \left\{ |D^\alpha u(\cdot, 0)|^2 + \frac{[\chi U_c(1 - U_c)]^2}{d_1 d_2} |D^\alpha v(\cdot, 0)|^2 \right\} \mathbf{d}\mathbf{x}$$

$$+ \frac{4\widehat{C}_1^2 \widehat{C}_3}{\lambda_{\max}} e^{2\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|^2. \quad (75)$$

We will proceed in the following two cases: $[\chi U_c(1 - U_c)]^2/d_1 d_2 \geq 1$, $[\chi U_c(1 - U_c)]^2/d_1 d_2 < 1$.

(1) If $[\chi U_c(1 - U_c)]^2/d_1 d_2 \geq 1$, then it is clear from (75) that

$$\sum_{|\alpha|=2} \|D^\alpha \mathbf{w}(\cdot, t)\|^2 \leq \frac{[\chi U_c(1 - U_c)]^2}{d_1 d_2} \sum_{|\alpha|=2} \|D^\alpha \mathbf{w}(\cdot, 0)\|^2 \quad (76)$$

$$+ \frac{4\widehat{C}_1^2 \widehat{C}_3}{\lambda_{\max}} e^{2\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|^2.$$

Using (73) and (76), we know that

$$\|\mathbf{w}(\cdot, t)\|_{H^2}^2 \leq \widehat{C}_4 \left\{ \|\mathbf{w}(\cdot, 0)\|_{H^2}^2 + \|\mathbf{w}(\cdot, 0)\|^2 e^{2\lambda_{\max} t} \right\}, \quad (77)$$

where $\widehat{C}_4 = \max\{(1 + C_{12}^2)[\chi U_c(1 - U_c)]^2/d_1 d_2, 4\widehat{C}_1^2[1 + \widehat{C}_3(1 + C_{12}^2)/\lambda_{\max}]\}$.

(2) If $[\chi U_c(1 - U_c)]^2/d_1 d_2 < 1$, then it is not hard to verify by (70), (73), and (75) that

$$\|\mathbf{w}(\cdot, t)\|_{H^2}^2 \leq \widehat{C}_4 \left\{ \|\mathbf{w}(\cdot, 0)\|_{H^2}^2 + \|\mathbf{w}(\cdot, 0)\|^2 e^{2\lambda_{\max} t} \right\}, \quad (78)$$

where $\widehat{C}_4 = \max\{(1 + C_{12}^2)(d_1 d_2/[\chi U_c(1 - U_c)]^2), 4\widehat{C}_1^2[1 + \widehat{C}_3(1 + C_{12}^2)d_1 d_2/\lambda_{\max}[\chi U_c(1 - U_c)]^2]\}$ and thereby completing the proof. \square

5. Main Result

Let $\theta > 0$ be a small fixed constant, and λ_{\max} the dominant eigenvalue which is the maximal growth rate. For $\delta > 0$ arbitrary small we define the escape time T^δ by

$$\theta = \delta e^{\lambda_{\max} T^\delta} \quad (79)$$

or equivalently

$$T^\delta = \frac{1}{\lambda_{\max}} \ln \frac{\theta}{\delta}. \quad (80)$$

Our main result in this paper reads as follows.

Theorem 6. Assume that the set of $q^2 = \sum_{i=1}^d q_i^2$ satisfying instability criterion (23) is not empty for given parameters $d_1, d_2, \alpha, \beta, \chi, \mu$, and U_c . Let

$$\mathbf{w}_0(\mathbf{x}) = \sum_{\mathbf{q} \in \Omega} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q})\} e_{\mathbf{q}}(\mathbf{x}) \in H^2, \quad (81)$$

such that $\|\mathbf{w}_0\| = 1$. Then there exist constants $\delta_0 > 0$, $\widehat{C} > 0$, and $\theta > 0$, depending on $d_1, d_2, \alpha, \beta, \chi, \mu$, and U_c , such that for all $0 < \delta \leq \delta_0$, if the initial perturbation of the steady state $[\bar{U}, \bar{V}]$ is $\mathbf{w}^\delta(\cdot, 0) = \delta \mathbf{w}_0$, then its nonlinear evolution $\mathbf{w}^\delta(\cdot, t)$ satisfies

$$\left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \right\| \quad (82)$$

$$\leq \widehat{C} \left\{ e^{-\rho t} + \delta \|\mathbf{w}_0\|_{H^2}^2 + \delta^2 \|\mathbf{w}_0\|_{H^2}^3 + \delta e^{\lambda_{\max} t} + \delta^2 e^{2\lambda_{\max} t} \right\} \delta e^{\lambda_{\max} t}$$

for $0 \leq t \leq T^\delta$, and $\rho > 0$ is the gap between λ_{\max} and the rest of $\lambda_{\mathbf{q}}$ in (82).

Proof. Let $\mathbf{w}^\delta(\mathbf{x}, t)$ be the solutions of (18) with initial data $\mathbf{w}^\delta(\cdot, 0) = \delta \mathbf{w}_0$. Define

$$T^* = \sup \left\{ t \mid \|\mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} t} \mathbf{w}_0\| \leq \frac{\widehat{C}_1}{2} \delta e^{\lambda_{\max} t} \right\},$$

$$T^{**} = \sup \left\{ t \mid \|\mathbf{w}^\delta(\cdot, t)\|_{H^2} + \|\mathbf{w}^\delta(\cdot, t)\|_{H^2}^2 \right\} \quad (83)$$

$$\leq \frac{1}{\widehat{C}_2^*} \min \left\{ \frac{d_1}{4}, \frac{[\chi U_c(1 - U_c)]^2}{2d_1} \right\}.$$

We recall (80) and choose θ such that

$$\widehat{C}_2^* \widehat{C}_4 \theta (1 + 2\widehat{C}_4^{1/2} \theta) < \min \left\{ \frac{\lambda_{\max}}{4}, \frac{d_1}{8}, \frac{[\chi U_c (1 - U_c)]^2}{4d_1} \right\}. \quad (84)$$

We first estimate H^2 norm of $\mathbf{w}^\delta(\mathbf{x}, t)$ for $0 \leq t \leq \min\{T^\delta, T^*, T^{**}\}$. By the definition of T^* and Lemma 1, for $0 < t \leq T^*$, we have

$$\|\mathbf{w}^\delta(\cdot, t)\| \leq \frac{3}{2} \widehat{C}_1 \delta e^{\lambda_{\max} t}. \quad (85)$$

From Lemma 5, direct computation gives

$$\|\mathbf{w}^\delta(\cdot, t)\|_{H^2} \leq \sqrt{\widehat{C}_4} \left\{ \delta \|\mathbf{w}_0\|_{H^2} + \delta e^{\lambda_{\max} t} \right\}. \quad (86)$$

It follows from (86) and $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ ($a, b \geq 0, p \geq 1$) that

$$\|\mathbf{w}^\delta(\cdot, t)\|_{H^2}^3 \leq 4(\widehat{C}_4)^{3/2} \left\{ \delta^3 \|\mathbf{w}_0\|_{H^2}^3 + \delta^3 e^{3\lambda_{\max} t} \right\}. \quad (87)$$

Secondly, we establish a sharper L^2 estimate for $\mathbf{w}^\delta(\mathbf{x}, t)$ for $0 \leq t \leq \min\{T^\delta, T^*, T^{**}\}$. Applying Duhamel's principle, we get

$$\begin{aligned} \mathbf{w}^\delta(\cdot, t) &= \delta e^{\mathfrak{A}t} \mathbf{w}_0 \\ &\quad - \int_0^t e^{\mathfrak{A}(t-\tau)} \left[\chi(1 - 2U_c) \nabla u^\delta(\tau) \nabla v^\delta(\tau) \right. \\ &\quad \quad - 2\chi u^\delta(\tau) \nabla u^\delta(\tau) \nabla v^\delta(\tau) \\ &\quad \quad + \chi(1 - 2U_c) u^\delta(\tau) \nabla^2 v^\delta(\tau) \\ &\quad \quad - \chi(u^\delta(\tau))^2 \nabla^2 v^\delta(\tau) \\ &\quad \quad \left. + \frac{\mu}{U_c} (u^\delta(\tau))^2, 0 \right] d\tau. \end{aligned} \quad (88)$$

By Lemma 1, (57), (59), and Lemma 5, for $0 \leq t \leq \min\{T^\delta, T^*, T^{**}\}$, we know

$$\begin{aligned} &\|\mathbf{w}^\delta(\cdot, t) - \delta e^{\mathfrak{A}t} \mathbf{w}_0\| \\ &\leq \widehat{C}_1 \widehat{C}_5 \int_0^t e^{\lambda_{\max}(t-\tau)} \left(\|\mathbf{w}^\delta(\tau)\|_{H^2}^2 + \|\mathbf{w}^\delta(\tau)\|_{H^2}^3 \right) d\tau, \end{aligned} \quad (89)$$

where $\widehat{C}_5 = \chi|1 - 2U_c|(C_{12}^2/C_9^2 + C_6) + \mu C_7/\chi|1 - 2U_c|U_c + \chi C_6(2C_{12}^2/C_9^2 + 1)$. By our choice of $t \leq \min\{T^\delta, T^*, T^{**}\}$, it is further bounded by

$$\begin{aligned} &\|\mathbf{w}^\delta(t) - \delta e^{\mathfrak{A}t} \mathbf{w}_0\| \\ &\leq \widehat{C}_1 \widehat{C}_4 \widehat{C}_5 \left\{ \frac{\delta \|\mathbf{w}_0\|_{H^2}^2 + 4\sqrt{\widehat{C}_4} \delta^2 \|\mathbf{w}_0\|_{H^2}^3}{\lambda_{\max}} \right. \\ &\quad \left. + \frac{\delta e^{\lambda_{\max} t} + 2\sqrt{\widehat{C}_4} \delta^2 e^{2\lambda_{\max} t}}{\lambda_{\max}} \right\} \delta e^{\lambda_{\max} t}. \end{aligned} \quad (90)$$

Next, We prove by contradiction that for δ sufficiently small, $T^\delta = \min\{T^\delta, T^*, T^{**}\}$. If T^{**} is the smallest, we can let $t = T^{**} \leq T^\delta$ in (86) and (87) to get

$$\begin{aligned} &\|\mathbf{w}^\delta(T^{**})\|_{H^2} + \|\mathbf{w}^\delta(T^{**})\|_{H^2}^2 \\ &\leq \sqrt{\widehat{C}_4} \delta \|\mathbf{w}_0\|_{H^2} + \widehat{C}_4 \delta^2 \|\mathbf{w}_0\|_{H^2}^2 \\ &\quad + \sqrt{\widehat{C}_4} \theta \left(1 + \sqrt{\widehat{C}_4} \theta \right) \\ &< \frac{1}{\widehat{C}_2^*} \min \left\{ \frac{d_1}{4}, \frac{[\chi U_c (1 - U_c)]^2}{2d_1} \right\}, \end{aligned} \quad (91)$$

where θ satisfies (84) with $\widehat{C}_4 \geq 1$ and δ is sufficiently small such that $\sqrt{\widehat{C}_4} \delta \|\mathbf{w}_0\|_{H^2} + \widehat{C}_4 \delta^2 \|\mathbf{w}_0\|_{H^2}^2 \leq (1/2\widehat{C}_2^*) \min\{d_1/4, [\chi U_c(1 - U_c)]^2/2d_1\}$. This is a contradiction to the definition of T^{**} .

If T^* is the smallest, let $t = T^* \leq T^\delta$ in (88), we see that

$$\begin{aligned} &\|\mathbf{w}^\delta(\cdot, T^*) - \delta e^{\mathfrak{A}T^*} \mathbf{w}_0\| \\ &\leq \widehat{C}_1 \widehat{C}_4 \widehat{C}_5 \left\{ \frac{\delta \|\mathbf{w}_0\|_{H^2}^2 + 4\sqrt{\widehat{C}_4} \delta^2 \|\mathbf{w}_0\|_{H^2}^3}{\lambda_{\max}} \right. \\ &\quad \left. + \frac{\theta + 2\sqrt{\widehat{C}_4} \theta^2}{\lambda_{\max}} \right\} \delta e^{\lambda_{\max} T^*} \\ &< \frac{\widehat{C}_1}{2} \delta e^{\lambda_{\max} T^*}, \end{aligned} \quad (92)$$

for δ is small such that $\widehat{C}_4 \widehat{C}_5 (\delta \|\mathbf{w}_0\|_{H^2}^2 + 4\sqrt{\widehat{C}_4} \delta^2 \|\mathbf{w}_0\|_{H^2}^3) / \lambda_{\max} < 1/4$, by our choice of θ in (84) and let $\widehat{C}_5/\widehat{C}_2^* \leq 1$. This again contradicts the definition of T^* . Hence, T^δ is the smallest.

Finally, we can obtain from (31) that

$$\begin{aligned} &\left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \right\| \\ &\leq \|\mathbf{w}^\delta(\cdot, t) - \delta e^{\mathfrak{A}t} \mathbf{w}_0\| + \left\| \delta \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) e^{\lambda_{\mathbf{q}} t} e_{\mathbf{q}}(\mathbf{x}) \right\| \\ &\quad + \left\| \delta \sum_{\mathbf{q} \in \Omega \setminus \Omega_{\max}} \left\{ w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) e^{\lambda_{\mathbf{q}} t} + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e^{\lambda_{\mathbf{q}} t} \right\} e_{\mathbf{q}}(\mathbf{x}) \right\| \\ &= \|\mathbf{w}^\delta(\cdot, t) - \delta e^{\mathfrak{A}t} \mathbf{w}_0\| + I_1 + I_2. \end{aligned} \quad (93)$$

By (30), (37), (39), and the definition of ρ , we get

$$\begin{aligned}
 I_1^2 &\leq \delta^2 e^{2(\lambda_{\max}-\rho)t} \left(\frac{\pi}{2}\right)^d \sum_{\mathbf{q} \in \Omega_{\max}} |w_{\mathbf{q}}^-|^2 |\mathbf{r}_-(\mathbf{q})|^2 \\
 &\leq \delta^2 e^{2(\lambda_{\max}-\rho)t} C_2^2 C_4^2 \left(\frac{\pi}{2}\right)^d \sum_{\mathbf{q} \in \Omega_{\max}} q^6 |\mathbf{w}_{\mathbf{q}}|^2.
 \end{aligned}
 \tag{94}$$

From (25) we know that there is one (or two) q^2 satisfying $\lambda^+(q^2) = \lambda_{\max}$. If there is only one q^2 satisfying $\lambda^+(q^2) = \lambda_{\max}$, we denote it by q_{\max}^2 ; if there are q_1^2 and q_2^2 satisfying $\lambda^+(q_i^2) = \lambda_{\max}$ ($i = 1, 2$), we can let $q_{\max}^2 = \max\{q_1^2, q_2^2\}$. It follows from (30) and (94) that

$$I_1 \leq \widehat{C}_6 \delta e^{(\lambda_{\max}-\rho)t}, \tag{95}$$

where $\widehat{C}_6 = C_2 C_3 q_{\max}^3$. Furthermore, by (29), (30), and the definition of ρ , we know that

$$I_2 \leq \delta e^{(\lambda_{\max}-\rho)t}. \tag{96}$$

Substituting (90), (95), and (96) into (93), one can obtain

$$\begin{aligned}
 &\left\| \mathbf{w}^\delta(\cdot, t) - \delta e^{\lambda_{\max}t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(\mathbf{x}) \right\| \\
 &\leq \widehat{C} \left\{ e^{-\rho t} + \delta \|\mathbf{w}_0\|_{H^2}^2 + \delta^2 \|\mathbf{w}_0\|_{H^2}^3 \right. \\
 &\quad \left. + \delta e^{\lambda_{\max}t} + \delta^2 e^{2\lambda_{\max}t} \right\} \delta e^{\lambda_{\max}t},
 \end{aligned}
 \tag{97}$$

where $\widehat{C} = \max\{\widehat{C}_6 + 1, 4\widehat{C}_1 \widehat{C}_4^{3/2} \widehat{C}_5 / \lambda_{\max}\}$ with $\widehat{C}_4 \geq 1$ and thereby concluding the proof. \square

Theorem 6 implies that the dynamics of a general perturbation are characterized by such linear dynamics over a long time period of $\varepsilon T^\delta \leq t \leq T^\delta$, for any $\varepsilon > 0$. In particular, choose a fixed $\mathbf{q}_0 = (q_{01}, q_{02}, \dots, q_{0d}) \in \Omega_{\max}$ and let $\mathbf{w}_0(\mathbf{x}) = \kappa(\mathbf{r}_+(\mathbf{q}_0)/|\mathbf{r}_+(\mathbf{q}_0)|)e_{\mathbf{q}_0}(\mathbf{x})$, where $\kappa = 1/\|e_{\mathbf{q}_0}\| = \sqrt{(2/\pi)^d}$ so that $\|\mathbf{w}_0(\mathbf{x})\| = 1$. Then

$$\|\mathbf{w}_0(\mathbf{x})\|_{H^2} = (1 + |\mathbf{q}_0|^2 + |\mathbf{q}_0|^4)^{1/2}. \tag{98}$$

Therefore, if $t \leq T^\delta$, we can obtain from (97) and (98)

$$\begin{aligned}
 &\left\| \mathbf{w}^\delta(\cdot, T^\delta) - \delta e^{\lambda_{\max}T^\delta} \kappa \frac{\mathbf{r}_+(\mathbf{q}_0)}{|\mathbf{r}_+(\mathbf{q}_0)|} e_{\mathbf{q}_0}(\mathbf{x}) \right\| \\
 &\leq \widehat{C}_8 \left\{ \delta^{\rho/\lambda_{\max}} + \theta^2 + \theta^3 \right\},
 \end{aligned}
 \tag{99}$$

where $\widehat{C}_8 = \widehat{C} \max\{\theta^{1-(\rho/\lambda_{\max})}, (1 + |\mathbf{q}_0|^2 + |\mathbf{q}_0|^4)^{3/2} + 1\}$. If $0 < \theta < (1/\widehat{C}_8)^{3/2}$ and $\delta_0 = (\theta/2\widehat{C}_8 - \theta^2 - \theta^3)^{\lambda_{\max}/\rho}$, then

$$\begin{aligned}
 \|\mathbf{w}^\delta(\cdot, T^\delta)\| &\geq \theta - \widehat{C}_8 \left\{ \delta^{\rho/\lambda_{\max}} + \theta^2 + \theta^3 \right\} \geq \frac{\theta}{2} > 0, \\
 0 &< \delta \leq \delta_0.
 \end{aligned}
 \tag{100}$$

This means nonlinear instability as $\delta \rightarrow 0$.

Respecting the above-mentioned facts, the models (7) and (13) have no nonconstant positive steady state no matter what the self-diffusion coefficients d_1, d_2 and cross diffusion d_3 are; in other words, self-diffusion and cross diffusion (without chemotaxis) cannot drive instability and cannot generate patterns for the corresponding model. This means that chemotaxis-driven instability occurs. We prove that for any given general perturbation of magnitude δ , linear fastest growing modes determine the nonlinear evolution for the model (2), over a time period of the order $\ln 1/\delta$. Therefore, our results indeed provide a rigorous mathematical description for the nonlinear pattern formation in a volume-filling chemotaxis model with logistic growth.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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