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Research Article

Zagreb Eccentricity Indices of the Generalized Hierarchical Product Graphs and Their Applications

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Let G be a connected graph. The first and second Zagreb eccentricity indices of G are defined as $M_1^*(G) = \sum_{v\in V(G)} \varepsilon_G^2(v)$ and $M_2^*(G) = \sum_{uv\in E(G)} \varepsilon_G(u)\varepsilon_G(v)$, where $\varepsilon_G(v)$ is the eccentricity of the vertex v in G and $\varepsilon_G^2(v) = (\varepsilon_G(v))^2$. Suppose that $G(U) \sqcap H (\emptyset \neq U \subseteq V(G))$ is the generalized hierarchical product of two connected graphs G and H. In this paper, the Zagreb eccentricity indices M_1^* and M_2^* of $G(U) \sqcap H$ are computed. Moreover, we present explicit formulas for the M_1^* and M_2^* of S-sum graph, Cartesian, cluster, and corona product graphs by means of some invariants of the factors.

1. Introduction

A topological index is a real number associated with chemical constitution purporting for correlation of chemical structure with various physical properties, chemical reactivity, or biological activity, which is used to understand properties of chemical compounds in theoretical chemistry [1].

Up to now, hundreds of topological indices have been defined in chemical literatures, various applications of these topological indices have been found, and many mathematical properties are also investigated. Wiener index W is the first topological index, introduced by American chemist Wiener, for investigating boiling points of alkanes in 1947 [2]. The well known degree-based topological indices are the first and second Zagreb indices M_1 and M_2, which have been introduced by Gutman and Trinajstić [3] and applied to study molecular chirality in quantitative structure-activity relationship (QSAR) and quantitative structure-property relationship (QSPR) analysis and so forth. Resently, the first and second Zagreb eccentricity indices M_1^* and M_2^* have been introduced by Ghorbani and Hosseinzadeh [4] and Vukičević and Graovac [5] as the revised version of the Zagreb indices M_1 and M_2, respectively. They computed the Zagreb eccentricity indices of some composite graphs and showed that $M_1^*(G)/|G| \geq M_2^*(G)/|E(G)|$ holds for all acyclic and unicyclic graphs and that neither this nor the opposite inequality holds for all bicyclic graphs. For further results of the Zagreb eccentricity indices, we encourage the reader to refer to [6–8].

In 2009, Spain mathematicians Barrière and coauthors [9] introduced a new composite graph, namely, hierarchical product graph. In the same year, this team also reported a generalization of both Cartesian and the hierarchical product of graphs, namely, the generalized hierarchical product of graphs in [10]. After that, many results for some topological indices of the (generalized) hierarchical product of graphs are reported; see [11–17].

In this paper, the Zagreb eccentricity indices of the generalized hierarchical product graph $G(U) \sqcap H$ are computed and as some special cases of $G(U)\sqcap H$, the Zagreb eccentricity indices of the Cartesian product graph $G\square H$, the S-sum graph $G +_S H$, and the cluster product graph $G\{H\}$ are determined, respectively. Moreover, as applications, we present explicit formulas for the M_1^* and M_2^* indices of the C_4 nanotorus $C_m\square C_n$, the C_4 nanotubes $P_m\square C_n$, the zig-zay polyhex nanotube $TUHC_6[2n, 2]$, the hexagonal chain L_n, and so forth.

2. Preliminaries

Throughout this paper, all graphs are simple, finite, and undirected. For terminology and notations that are not defined here, we refer the reader to West [18].

Let $G = (V(G), E(G), \psi_G)$ be a graph with the vertex set $V(G) \neq \emptyset$, the edge set $E(G)$, and an incidence function ψ_G that associates with each edge of G, an unordered pair of vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to join u and v, and the vertices u and v are called the ends of e. The cardinality of $V(G)$ and $E(G)$ is denoted by $|G|$ and $|E(G)|$, respectively. We denote the degree and the neighborhood of a vertex v of G by $d_G(v)$ and $N_G(v)$; then $d_G(v) = |N_G(v)|$. As usual, the distance between vertices u and v of a connected graph G, denoted by $d_G(u, v)$, is defined as the number of edges in a shortest path connecting the vertices u and v. Suppose that $d(x \mid G) = \sum_{v\in V(G)} d_G(x, v)$ and $d^2(x \mid G) = \sum_{v\in V(G)}(d_G(x, v))^2$. The eccentricity $\varepsilon_G(v)$ of a vertex v in G is the largest distance between v and any other vertex u of G; that is, $\varepsilon_G(v) = \max_{u\in V(G)} d_G(u, v)$. For two graphs G and H, if there exist two bijections $\theta : V(G) \to V(H)$ and $\varphi : E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\varphi(e)) = \theta(u)\theta(v)$, then we say that G and H are isomorphic, denoted by $G \cong H$. Let $\mathrm{Top}(G)$ denote a certain topological index of G. In general, if $G \cong H$, then $\mathrm{Top}(G) = \mathrm{Top}(H)$.

The total eccentricity and the eccentric connectivity indices $\zeta(G)$ and $\xi^c(G)$ of graph G are defined as $\zeta(G) = \sum_{v\in V(G)} \varepsilon_G(v)$ and $\xi^c(G) = \sum_{uv\in E(G)}[\varepsilon_G(u) + \varepsilon_G(v)] = \sum_{v\in V(G)} d_G(v)\varepsilon_G(v)$, respectively. The Zagreb indices of G are defined as $M_1(G) = \sum_{uv\in E(G)}[d_G(u) + d_G(v)] = \sum_{u\in V(G)}(d_G(u))^2$ and $M_2(G) = \sum_{uv\in E(G)} d_G(u)d_G(v)$. Very recently, the topological indices based on vertex eccentricities attracted some attention in chemistry. In an analogy with the Zagreb indices, the first and second Zagreb eccentricity indices M_1^* and M_2^* of a connected graph G are defined by [4, 5]. That is,

$$M_1^*(G) = \sum_{v\in V(G)} \varepsilon_G^2(v) = \sum_{v\in V(G)} (\varepsilon_G(v))^2,$$
$$M_2^*(G) = \sum_{uv\in E(G)} \varepsilon_G(u)\varepsilon_G(v). \quad (1)$$

Lemma 1 (see [4]). *Let K_n be the complete graph of order n; then $\zeta(K_n) = n$, $\xi^c(K_n) = n(n-1)$, $M_1^*(K_n) = n$, and $M_2^*(K_n) = \binom{n}{2}$.*

Lemma 2 (see [4]). *Let C_n be the cycle of length n; then $\zeta(C_n) = n\lfloor n/2\rfloor$, $\xi^c(C_n) = 2n\lfloor n/2\rfloor$, and $M_1^*(C_n) = M_2^*(C_n) = n\lfloor n/2\rfloor^2$.*

Lemma 3 (see [4]). *Let P_n be the path on $n \geq 2$ vertices. Then*

$$\zeta(P_n) = \begin{cases} \frac{1}{4}n(3n-2), & 2 \mid n, \\ \frac{1}{4}(n-1)(3n+1), & 2 \nmid n, \end{cases} \quad (2)$$

$$M_1^*(P_n) = \begin{cases} \frac{1}{12}n(n-1)(7n-2), & 2 \mid n, \\ \frac{1}{12}(n-1)\left(7n^2-2n-3\right), & 2 \nmid n, \end{cases} \quad (3)$$

$$\xi^c(P_n) = \begin{cases} \frac{1}{2}\left(3n^2-6n+4\right), & 2 \mid n, \\ \frac{3}{2}(n-1)^2, & 2 \nmid n, \end{cases} \quad (4)$$

$$M_2^*(P_n) = \begin{cases} \frac{1}{12}n\left(7n^2-21n+20\right), & 2 \mid n, \\ \frac{1}{12}(n-1)\left(7n^2-14n+3\right), & 2 \nmid n. \end{cases} \quad (5)$$

3. Zagreb Eccentricity Indices of Generalized Hierarchical Product Graphs

In this section, we calculate the Zagreb eccentricity indices of the generalized hierarchical product graphs.

Definition 4 (see [9]). Let G and H be two connected graphs; $\emptyset \neq U \subseteq V(G)$. Then the generalized hierarchical product $G(U)\sqcap H$ is the graph with vertex set $V(G)\times V(H)$ and vertices (u_r, v_i) and (u_s, v_k) are adjacent if and only if $[u_r = u_s \in U$ and $v_iv_k \in E(H)]$ or $[v_i = v_k \in V(H)$ and $u_ru_s \in E(G)]$.

Given a connected graph G and $\emptyset \neq U \subseteq V(G)$, a path connecting vertices x and y through U is a xUy-path of G containing some vertex $z \in U$ (vertex z could be the vertex x or vertex y). Then the distance through U between x and y is denoted by $d_{G(U)}(x, y)$, which is the length of the shortest path xUy in G. Note that if one of the vertex x and y belongs to U, then $d_{G(U)}(x, y) = d_G(x, y)$; see [13]. Similarly, we define some invariants related to U in G as follows:

$\varepsilon_{G(U)}(v) = \max_{u\in V(G)} d_{G(U)}(u, v)$,
$\zeta(G(U)) = \sum_{v\in V(G)} \varepsilon_{G(U)}(v)$,
$\epsilon(G(U)) = \sum_{v\in U} \varepsilon_{G(U)}(v)$; see [15],
$\xi^c(G(U)) = \sum_{uv\in E(G)}(\varepsilon_{G(U)}(u) + \varepsilon_{G(U)}(v))$; see [15],
$M_1^*(G(U)) = \sum_{v\in V(G)} \varepsilon_{G(U)}^2(v)$,
$m_1^*(G(U)) = \sum_{v\in U} \varepsilon_{G(U)}^2(v)$,
$M_2^*(G(U)) = \sum_{uv\in E(G)} \varepsilon_{G(U)}(u)\varepsilon_{G(U)}(v)$.

Theorem 5 (see [15]). *Let graphs G and H be connected; $\emptyset \neq U \subseteq V(G)$. Then*

$$\xi^c(G(U)\sqcap H) = |U|\,\xi^c(H) + |H|\,\xi^c(G(U)) + 2|E(G)|\,\zeta(H) + 2|E(H)|\,\epsilon(G(U)). \quad (6)$$

Lemma 6 (see [10]). *Let G and H be two connected graphs and $\emptyset \neq U \subseteq V(G)$. Then*

$$\varepsilon_{G(U)\sqcap H}(u, v) = \varepsilon_{G(U)}(u) + \varepsilon_H(v). \quad (7)$$

Theorem 7. *Let graphs G and H be connected; $\emptyset \neq U \subseteq V(G)$. Then*

(i)

$$M_1^*(G(U)\sqcap H) = |H|\,M_1^*(G(U)) + |G|\,M_1^*(H) + 2\zeta(G(U))\,\zeta(H), \quad (8)$$

(ii)

$$M_2^*(G(U)\sqcap H) = |E(H)|\, m_1^*(G(U)) + |E(G)|\, M_1^*(H) + |U|\, M_2^*(H) + |H|\, M_2^*(G(U)) + \xi^c(G(U))\,\zeta(H) + \xi^c(H)\,\epsilon(G(U)). \tag{9}$$

Proof. Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_{n'}\}$.

(i) By the definition of Zagreb eccentricity index M_1^* and Lemma 6, we have

$$\begin{aligned} M_1^*(G(U)\sqcap H) &= \sum_{(u_r,v_i)\in V(G(U)\sqcap H)} \varepsilon^2_{G(U)\sqcap H}((u_r,v_i)) \\ &= \sum_{u_r\in V(G)}\sum_{v_i\in V(H)} (\varepsilon_{G(U)}(u_r) + \varepsilon_H(v_i))^2 \\ &= |H| \sum_{u_r\in V(G)} \varepsilon^2_{G(U)}(u_r) + |G| \sum_{v_i\in V(H)} \varepsilon^2_H(v_i) \\ &\quad + 2\sum_{u_r\in V(G)} \varepsilon_{G(U)}(u_r) \sum_{v_i\in V(H)} \varepsilon_H(v_i) \\ &= |H|\, M_1^*(G(U)) + |G|\, M_1^*(H) \\ &\quad + 2\zeta(G(U))\,\zeta(H). \end{aligned} \tag{10}$$

(ii) We partition the edges of $G(U)\sqcap H$ into two subsets E_1 and E_2, as follows:

$$\begin{aligned} E_1 &= \{(u_r,v_i)(u_r,v_k) \in E(G(U)\sqcap H) \mid v_iv_k \in E(H), u_r \in U\}, \\ E_2 &= \{(u_r,v_i)(u_s,v_i) \in E(G(U)\sqcap H) \mid u_ru_s \in E(G), v_i \in V(H)\}. \end{aligned} \tag{11}$$

From the definition of Zagreb eccentricity index M_2^* and Lemma 6, we get

$$\begin{aligned} &M_2^*(G(U)\sqcap H) \\ &= \sum_{(u_r,v_i)(u_s,v_k)\in E(G(U)\sqcap H)} \varepsilon_{G(U)\sqcap H}((u_r,v_i))\,\varepsilon_{G(U)\sqcap H}((u_s,v_k)) \\ &= \sum_{(u_r,v_i)(u_s,v_k)\in E_1\cup E_2} \varepsilon_{G(U)\sqcap H}((u_r,v_i))\,\varepsilon_{G(U)\sqcap H}((u_s,v_k)) \\ &= \sum_{u_r\in U}\sum_{v_iv_k\in E(H)} [\varepsilon_{G(U)}(u_r) + \varepsilon_H(v_i)]\,[\varepsilon_{G(U)}(u_r) + \varepsilon_H(v_k)] \\ &\quad + \sum_{v_i\in V(H)}\sum_{u_ru_s\in E(G)} [\varepsilon_{G(U)}(u_r) + \varepsilon_H(v_i)] \\ &\qquad \times [\varepsilon_{G(U)}(u_s) + \varepsilon_H(v_i)] \\ &= |E(H)| \sum_{u_r\in U} \varepsilon^2_{G(U)}(u_r) + \sum_{u_r\in U} \varepsilon_{G(U)}(u_r) \\ &\quad \times \sum_{v_iv_k\in E(H)} (\varepsilon_H(v_i) + \varepsilon_H(v_k)) \\ &\quad + |U| \sum_{v_iv_k\in E(H)} \varepsilon_H(v_i)\,\varepsilon_H(v_k) + |E(G)| \sum_{v_i\in V(H)} \varepsilon^2_H(v_i) \\ &\quad + \sum_{v_i\in V(H)} \varepsilon_H(v_i) \sum_{u_ru_s\in E(G)} [\varepsilon_{G(U)}(u_r) + \varepsilon_{G(U)}(u_s)] \\ &\quad + |H| \sum_{u_ru_s\in E(G)} \varepsilon_{G(U)}(u_r)\,\varepsilon_{G(U)}(u_s) \\ &= |E(H)|\, m_1^*(G(U)) + \epsilon(G(U))\,\xi^c(H) + |U|\, M_2^*(H) \\ &\quad + |E(G)|\, M_1^*(H) + \zeta(H)\,\xi^c(G(U)) + |H|\, M_2^*(G(U)). \end{aligned} \tag{12}$$

This completes the proofs. □

Definition 8 (see [13]). Let G and H be two connected graphs. Then the Cartesian product $G\square H$ has the vertex set $V(G) \times V(H)$ and vertices (u_r, v_i) and (u_s, v_k) are adjacent if and only if $[u_r = u_s \in V(G)$ and $v_iv_k \in E(H)]$ or $[v_i = v_k \in V(H)$ and $u_ru_s \in E(G)]$, where $r, s = 1, 2, \ldots, |G|$ and $i, k = 1, 2, \ldots, |H|$.

Note that if $U = V(G)$, then $G(U) \sqcap H \cong G\square H$. So by Theorem 7, the following corollary is obvious.

Corollary 9. *Let G and H be two connected graphs. Then*

$$M_1^*(G\square H) = |G|\, M_1^*(H) + |H|\, M_1^*(G) + 2\zeta(G)\,\zeta(H), \tag{13}$$

$$M_2^*(G\square H) = |E(G)|\, M_1^*(H) + |E(H)|\, M_1^*(G) + |G|\, M_2^*(H) + |H|\, M_2^*(G) + \xi^c(G)\,\zeta(H) + \xi^c(H)\,\zeta(G). \tag{14}$$

Remark 10. Equation (14) corrects the corresponding Corollary 9 in [4]. According to (14), we recompute the second Zagreb eccentricity indices of the C_4 nanotubes as below (see Example 11).

Example 11. Using the results as above, it is easy to obtain the first and second Zagreb eccentricity indices of the C_4 nanotorus $C_m\square C_n$ $(m \geq n \geq 3)$ and the C_4 nanotubes $P_{2m}\square C_n$ $(n \geq 3)$ and $P_{2m+1}\square C_n$ $(n \geq 3)$. By Corollary 9 and Lemmas 2 and 3, we have

$$\begin{aligned} M_1^*(C_m\square C_n) &= \frac{1}{2} M_2^*(C_m\square C_n) = mn\left(\left\lfloor\frac{m}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor\right)^2, \\ M_1^*(P_{2m}\square C_n) &= mn\left[\frac{14}{3}m^2 - 3m + \frac{1}{3} + 2\left\lfloor\frac{n}{2}\right\rfloor^2 + 2(3m-1)\left\lfloor\frac{n}{2}\right\rfloor\right], \end{aligned}$$

$$M_1^*(P_{2m+1}\square C_n) = mn\left(\frac{14}{3}m^2 + 4m + \frac{1}{3}\right) + n(2m+1)\left\lfloor\frac{n}{2}\right\rfloor^2 + 2mn(3m+2)\left\lfloor\frac{n}{2}\right\rfloor,$$

$$M_2^*(P_{2m}\square C_n) = mn\left(\frac{28}{3}m^2 - 10m + \frac{11}{3}\right) + (4m-1)n\left\lfloor\frac{n}{2}\right\rfloor^2 + 2\left(6m^2 - 4m + 1\right)n\left\lfloor\frac{n}{2}\right\rfloor,$$

$$M_2^*(P_{2m+1}\square C_n) = mn\left(\frac{28}{3}m^2 + 4m - \frac{1}{3}\right) + (4m+1)n\left\lfloor\frac{n}{2}\right\rfloor^2 + 4m(3m+1)n\left\lfloor\frac{n}{2}\right\rfloor. \quad (15)$$

Lemma 12 (see [19]). *Let $\square_{i=1}^n G_i$ be Cartesian product of $n \geq 2$ connected graphs G_i. Then*

(a) $|\square_{i=1}^n G_i| = |V(\square_{i=1}^n G_i)| = \prod_{i=1}^n |G_i|$.

(b) $|E(\square_{i=1}^n G_i)| = \sum_{i=1}^n [\prod_{j=1,j\neq i}^n |G_j||E(G_i)|] = \sum_{i=1}^n |E(G_i)|\prod_{j=1,j\neq i}^n |G_j|$.

(c) $\zeta(\square_{i=1}^n G_i) = \sum_{i=1}^n [\prod_{j=1,j\neq i}^n |G_j|\zeta(G_i)] = \sum_{i=1}^n \zeta(G_i)\prod_{j=1,j\neq i}^n |G_j|$.

Corollary 13. *Let $\square_{i=1}^n G_i$ be Cartesian product of $n \geq 2$ simple connected graphs G_i. Then*

$$M_1^*(\square_{i=1}^n G_i) = \sum_{i=1}^n \left[\prod_{j=1,j\neq i}^n |G_j| M_1^*(G_i)\right] + 2\sum_{1\leq s<t\leq n}\left[\prod_{k=1,k\neq s,t}^n |G_k|\zeta(G_s)\zeta(G_t)\right]. \quad (16)$$

Proof. The case $n = 2$ is proved in Corollary 9. We prove the assertion by induction. Suppose the result is valid for n graphs. Then by Lemma 12, we have

$$\begin{aligned} M_1^*\left(\square_{i=1}^{n+1} G_i\right) &= M_1^*(G_{n+1}\square(\square_{i=1}^n G_i)) \\ &= |G_{n+1}| M_1^*(\square_{i=1}^n G_i) + |\square_{i=1}^n G_i| M_1^*(G_{n+1}) \\ &\quad + 2\zeta(G_{n+1})\zeta(\square_{i=1}^n G_i) \\ &= |G_{n+1}|\sum_{i=1}^n\left[\prod_{j=1,j\neq i}^n |G_j| M_1^*(G_i)\right] + 2|G_{n+1}| \\ &\quad \times \sum_{1\leq s<t\leq n}\left[\prod_{k=1,k\neq s,t}^n |G_k|\zeta(G_s)\zeta(G_t)\right] \\ &\quad + \prod_{i=1}^n |G_i| M_1^*(G_{n+1}) \\ &\quad + 2\zeta(G_{n+1})\sum_{i=1}^n\left[\prod_{j=1,j\neq i}^n |G_j|\zeta(G_i)\right] \\ &= \sum_{i=1}^{n+1}\left[\prod_{j=1,j\neq i}^{n+1} |G_j| M_1^*(G_i)\right] \\ &\quad + 2\sum_{1\leq s<t\leq n+1}\left[\prod_{k=1,k\neq s,t}^{n+1} |G_k|\zeta(G_s)\zeta(G_t)\right]. \end{aligned} \quad (17)$$

This completes the proof. □

Example 14. The Hamming graph $H_{n_1,n_2,\ldots,n_s} = \square_{i=1}^n K_{n_i}$. Thus, by Corollary 13 and Lemma 1, we have $M_1^*(H_{n_1,n_2,\ldots,n_s}) = s^2\prod_{i=1}^s n_i$. For $n_1 = n_2 = \cdots = n_s = 2$, we attain the s-dimensional hypercubes Q_s. Therefore, $M_1^*(Q_s) = s^2 2^s$.

Remark 15. Ghorbani and Hosseinzadeh computed the second Zagreb eccentricity index of $\square_{i=1}^n G_i$ in [4]. Here, we can also obtain the explicit formula of $M_2^*(\square_{i=1}^n G_i)$ by induction.

For two connected graphs G_i $(i = 1, 2)$, we note that $G_1(U) \sqcap G_2 \cong G_1\square G_2$ if $U = V(G_1)$; then by Theorem 5, we have

$$\xi^c(G_1\square G_2) = |G_1|\xi^c(G_2) + |G_2|\xi^c(G_1) + 2|E(G_1)|\zeta(G_2) + 2|E(G_2)|\zeta(G_1). \quad (18)$$

By induction, we can easily prove that

$$\xi^c(\square_{i=1}^n G_i) = \sum_{i=1}^n\left[\prod_{j=1,j\neq i}^n |G_j|\xi^c(G_i)\right] + 2\sum_{\substack{1\leq s,t\leq n\\ s\neq t}}\left[\prod_{\substack{k=1\\ k\neq s,t}}^n |G_k||E(G_s)|\zeta(G_t)\right]. \quad (19)$$

Therefore, by Corollary 9, Lemma 12 and the formula as above, using a similar method of proof in Corollary 13, we can obtain Corollary 16.

Corollary 16 (see [4]). *Let $\square_{i=1}^n G_i$ be Cartesian product of $n \geq 2$ graphs G_i. Then*

$$M_2^*(\square_{i=1}^n G_i) = \sum_{\substack{1\leq s,t\leq n\\ s\neq t}}\left[\prod_{\substack{k=1\\ k\neq s,t}}^n |G_k|\left(|E(G_s)| M_1^*(G_t) + \xi^c(G_s)\zeta(G_t)\right)\right]$$

$$+\sum_{i=1}^{n}\left[\prod_{\substack{j=1\\ j\neq i}}^{n}\left|G_j\right| M_2^*\left(G_i\right)\right.$$
$$\left.+2\sum_{\substack{1\le s<t\le n\\ s,t\neq i}}\left(\prod_{\substack{k=1\\ k\neq i,s,t}}^{n}\left|G_j\right|\zeta\left(G_s\right)\zeta\left(G_t\right)\right)\left|E\left(G_i\right)\right|\right]. \tag{20}$$

From Corollaries 13 and 16, the following corollary is obvious.

Corollary 17. *Let $G^n=\square_{i=1}^{n}G=\underbrace{G\square\cdots\square G}_{n\ times}$ be Cartesian product of $n\ge 2$ connected graphs G. Then*

$$M_1^*\left(G^n\right)=n|G|^{n-2}\left[|G|\,M_1^*\left(G\right)+(n-1)\,\zeta^2\left(G\right)\right],$$
$$M_2^*\left(G^n\right)=2\binom{n}{2}|G|^{n-2}\left[|E\left(G\right)|\,M_1^*\left(G\right)+\xi^c\left(G\right)\zeta\left(G\right)\right]$$
$$+n|G|^{n-1}M_2^*\left(G\right)+2n\binom{n-1}{2}|G|^{n-3}\zeta^2\left(G\right). \tag{21}$$

4. Zagreb Eccentricity Indices of S-Sum Graphs

Let G be a connected graph. The vertices of a *Line graph* $L(G)$ are the edges of G. Two edges of G that share a vertex are considered to be adjacent in $L(G)$. *A Subdivision graph* $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G. That is, each edge of G is replaced by a path of length two.

Definition 18 (see [15]). For two connected graphs G and H, the S-sum $G+_S H$ of G and H is a graph with vertex set $(V(G)\cup E(G))\times V(H)$ and vertices (u_r,v_i) and (u_s,v_k) are adjacent if and only if $[u_r=u_s\in V(G)$ and $v_iv_k\in E(H)]$ or $[v_i=v_k\in V(H)$ and $u_ru_s\in E(S(G))]$.

Note that if $\emptyset\neq U=V(G)\subset V(S(G))$, then $G+_SH\cong S(G)(U)\sqcap H\cong S(G)(V(G))\sqcap H$. So by Theorem 7, we can compute the Zagreb eccentricity indices of $G+_SH$ easily.

Lemma 19 (see [15]). *Let G and H be two connected graphs. If $U=V(G)$, then we have*

(a) $|V(S(G))|=|G|+|E(G)|$, $|E(S(G))|=2|E(G)|$,

(b) *for each vertex $v\in U$, we have $d_{S(G)}(v)=d_G(v)$,*

(c) *for each vertex $v\in V(S(G))\setminus U$, we have $d_{S(G)}(v)=2$.*

Lemma 20 (see [15]). *Let $G=T_n$ $(n\ge 2)$ be a tree with n vertices. If $U=V(G)$, then*

(a) *for each vertex $v\in U$, we have $\varepsilon_{S(G)(U)}(v)=2\varepsilon_G(v)$,*

(b) *for each vertex $v\in V(S(G))\setminus U$, we have $\varepsilon_{S(G)(U)}(v)=2\varepsilon_{L(G)}(v)+1$.*

Theorem 21. *Let T_n be a tree with order n $(n\ge 2)$ and let H be a connected graph; $\emptyset\neq U=V(T_n)\subset V(S(T_n))$. Then*

(i)

$$M_1^*\left(T_n+_SH\right)=|H|\left[4M_1^*\left(T_n\right)+4M_1^*\left(L\left(T_n\right)\right)+4\zeta\left(L\left(T_n\right)\right)+n-1\right]$$
$$+(2n-1)\,M_1^*\left(H\right)+2\zeta\left(H\right)\times\left[2\zeta\left(T_n\right)+2\zeta\left(L\left(T_n\right)\right)+n-1\right], \tag{22}$$

(ii)

$$M_2^*\left(T_n+_SH\right)=4\left|E\left(H\right)\right|M_1^*\left(T_n\right)+2\left(n-1\right)M_1^*\left(H\right)$$
$$+nM_2^*\left(H\right)+2\xi^c\left(H\right)\zeta\left(T_n\right)$$
$$+2\left|H\right|\left[2\rho\left(T_n\right)+\xi^c\left(T_n\right)\right]$$
$$+2\zeta\left(H\right)\left[\xi^c\left(T_n\right)+2\zeta\left(L\left(T_n\right)\right)+n-1\right], \tag{23}$$

where $\rho(T_n)=\sum_{uv\in E(T_n)}\varepsilon_{L(T_n)}(uv)[\varepsilon_{T_n}(u)+\varepsilon_{T_n}(v)]$.

Proof. (i) We start to calculate $M_1^*(T_n+_SH)$. By Lemmas 19 and 20, we have

$$M_1^*\left(S\left(T_n\right)\left(U\right)\right)=\sum_{v\in V(S(T_n))}\varepsilon^2_{S(T_n)(U)}\left(v\right)$$
$$=4\sum_{v\in V(T_n)}\varepsilon^2_{T_n}\left(v\right)+\sum_{v\in V(L(T_n))}\left[2\varepsilon_{L(T_n)}\left(v\right)+1\right]^2$$
$$=4M_1^*\left(T_n\right)+4M_1^*\left(L\left(T_n\right)\right)+4\zeta\left(L\left(T_n\right)\right)+n-1,$$
$$\zeta\left(S\left(T_n\right)\left(U\right)\right)=\sum_{v\in V(S(T_n))}\varepsilon_{S(T_n)(U)}\left(v\right)$$
$$=2\sum_{v\in V(T_n)}\varepsilon_{T_n}\left(v\right)+2\sum_{v\in V(L(T_n))}\varepsilon_{L(T_n)}\left(v\right)+\left|E\left(T_n\right)\right|$$
$$=2\zeta\left(T_n\right)+2\zeta\left(L\left(T_n\right)\right)+n-1. \tag{24}$$

Combing these with (8) in Theorem 7, we can obtain the corresponding result.

(ii) Now, let us compute $M_2^*(T_n+_SH)$. By Lemmas 19 and 20, we get

$$m_1^*\left(S\left(T_n\right)\left(U\right)\right)=\sum_{v\in U=V(T_n)}\varepsilon^2_{S(T_n)(U)}\left(v\right)$$
$$=4\sum_{v\in V(T_n)}\varepsilon^2_{T_n}\left(v\right)=4M_1^*\left(T_n\right),$$
$$M_2^*\left(S\left(T_n\right)\left(U\right)\right)=\sum_{uv\in E(T_n)}\varepsilon_{S(T_n)(U)}\left(uv\right)$$

$$\times\left[\varepsilon_{S(T_n)(U)}(u)+\varepsilon_{S(T_n)(U)}(v)\right]$$
$$=2\sum_{uv\in E(T_n)}\left[2\varepsilon_{L(T_n)}(uv)+1\right]$$
$$\times\left[\varepsilon_{T_n}(u)+\varepsilon_{T_n}(v)\right]$$
$$=4\sum_{uv\in E(T_n)}\varepsilon_{L(T_n)}(uv)\left[\varepsilon_{T_n}(u)+\varepsilon_{T_n}(v)\right]$$
$$+2\sum_{uv\in E(T_n)}\left[\varepsilon_{T_n}(u)+\varepsilon_{T_n}(v)\right]$$
$$=4\rho(T_n)+2\xi^c(T_n),$$
$$\xi^c(S(T_n)(U))=\sum_{uv\in E(S(T_n))}\left[\varepsilon_{S(T_n)(U)}(u)+\varepsilon_{S(T_n)(U)}(v)\right]$$
$$=\sum_{v\in V(S(T_n))}d_{S(T_n)(U)}(v)\,\varepsilon_{S(T_n)(U)}(v)$$
$$=2\sum_{v\in U=V(T_n)}d_{T_n}(v)\,\varepsilon_{T_n}(v)$$
$$+2\sum_{v\in V(L(T_n))}\left(2\varepsilon_{L(T_n)}(v)+1\right)$$
$$=2\xi^c(T_n)+4\zeta(L(T_n))+2(n-1),$$
$$\epsilon(S(T_n)(U))=\sum_{v\in U=V(T_n)}\varepsilon_{S(T_n)(U)}(v)$$
$$=2\sum_{v\in V(T_n)}\varepsilon_{T_n}(v)=2\zeta(T_n).\qquad(25)$$

Combing these results with (9) in Theorem 7, we obtain the desired result. □

Clearly, if n is even, then $\rho(P_n)=(1/2)n(n-2)+2\sum_{i=(n/2)}^{n-2}i(2i+1)$. Otherwise, if n is odd, then $\rho(P_n)=2\sum_{i=((n-1)/2)}^{n-2}i(2i+1)$. So, the following lemma holds.

Lemma 22. *Let P_n be a path of order n. Then*

$$\rho(P_n)=\begin{cases}\dfrac{1}{12}(n-2)\left(14n^2-23n+12\right), & 2\mid n,\\[2mm] \dfrac{1}{12}(n-1)\left(14n^2-37n+21\right), & 2\nmid n.\end{cases}\qquad(26)$$

Example 23. Suppose L_n is a linear hexagonal chain with $n\geq 2$ hexagons (see Figure 1); note that $L_n\cong P_{n+1}+_SP_2$. Thus, by Lemmas 3 and 22 and Theorem 21, we have

$$M_1^*(L_n)=\frac{2}{3}\left(14n^3+30n^2+19n+3\right),$$
$$M_2^*(L_n)=\begin{cases}\dfrac{1}{3}\left(35n^3+57n^2+25n+3\right), & 2\mid n,\\[2mm] \dfrac{1}{3}\left(35n^3+57n^2+28n+6\right), & 2\nmid n.\end{cases}\qquad(27)$$

Let n be an integer with $n\geq 3$ and $U=V(C_n)$. Note that $\varepsilon_{S(C_n)(U)}(v)=n$ for any vertex $v\in V(S(C_n))$. Then by Theorem 7, we can obtain the following theorem.

Theorem 24. *Let C_n $(n\geq 3)$ be a cycle and let H be an arbitrary connected graph; $\emptyset\neq U=V(C_n)\subset V(S(C_n))$. Then*

$$M_1^*(C_n+_SH)=2n\left(n^2|H|+M_1^*(H)+2n\zeta(H)\right),$$
$$M_2^*(C_n+_SH)=n\left[(|E(H)|+2|H|)n^2+\left(4\zeta(H)+\xi^c(H)\right)n+2M_1^*(H)+M_2^*(H)\right].\qquad(28)$$

Example 25. Let n be an integer with $n\geq 3$ and let Γ be the zigzag polyhex nanotube $TUHC_6[2n,2]$ (see Figure 2); then $\Gamma\cong C_n+_SP_2$. By Theorem 24 and Lemma 3, we have $M_1^*(\Gamma)=4n(n+1)^2$ and $M_2^*(\Gamma)=5n(n+1)^2$.

5. Zagreb Eccentricity Indices of Cluster and Corona Product Graphs

The cluster product, corona product, and join of two graphs are important graph operations defined as below.

Definition 26 (see [20]). The cluster product graph $G\{H\}$ is obtained by taking one copy of G and $|G|$ copies of a rooted graph H and by identifying the root of the ith copy of H with the ith vertex of G, $i=1,2,\ldots,|G|$.

Definition 27 (see [20]). The corona product graph $G\odot H$ is obtained by taking one copy of G and $|G|$ copies of H and by joining each vertex of the ith copy of H to the ith vertex of G, $i=1,2,\ldots,|G|$.

Definition 28 (see [20]). The join graph $G+H$: $V(G+H)=V(G)\cup V(H)$; $E(G+H)=E(G)\cup E(H)\cup\{(u,v)\mid u\in V(G),v\in V(H)\}$.

Let G and H be connected graphs; x is a root-vertex of H. Note that; if $\emptyset\neq U=\{x\}\subset V(H)$, then $G\{H\}\cong H(U)\sqcap G\cong H(\{x\})\sqcap G$. We define $\omega_x(H)=\sum_{uv\in E(H)}d_H(u,x)d_H(v,x)$ and $\varsigma_x(H)=\sum_{uv\in E(H)}[d_H(u,x)+d_H(v,x)]=\sum_{v\in V(H)}d_H(v)d_H(v,x)$.

Theorem 29. *Let G and H be two connected graphs; x is a root-vertex of H. Then*

$$\xi^c(G\{H\})=\xi^c(G)+2|E(H)|\zeta(G)+2[|E(G)|+|G||E(H)|]\varepsilon_H(x)+|G|\varsigma_x(H),\qquad(29)$$

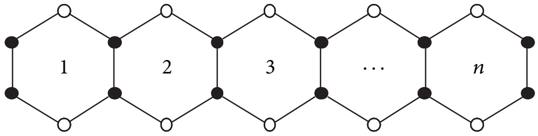

Figure 1: The linear hexagonal chain with n hexagons L_n.

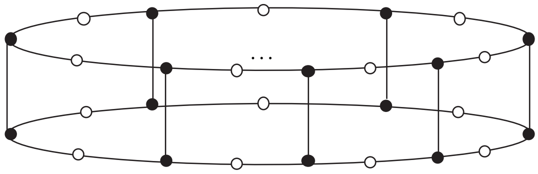

Figure 2: The zig-zay polyhex nanotube $TUHC_6[2n,2]$.

$$\begin{aligned} M_1^*(G\{H\}) = {} & |H|\, M_1^*(G) \\ & + 2\left[|H|\,\varepsilon_H(x) + d(x \mid H)\right]\zeta(G) + |G| \\ & \times \left[|H|\,\varepsilon_H^2(x) + 2\varepsilon_H(x)\, d(x \mid H) \right. \\ & \left. + d^2(x \mid H)\right], \end{aligned} \tag{30}$$

$$\begin{aligned} M_2^*(G\{H\}) = {} & \varepsilon_H(x)\, \xi^c(G) + q' M_1^*(G) \\ & + M_2^*(G) + \left[\varsigma_x(H) + 2\,|E(H)|\,\varepsilon_H(x)\right]\zeta(G) \\ & + \left[|E(G)| + |G|\,|E(H)|\right]\varepsilon_H^2(x) \\ & + |G|\,\varepsilon_H(x)\,\varsigma_x(H) + |G|\,\omega_x(H). \end{aligned} \tag{31}$$

Proof. Let $\emptyset \neq U = \{x\} \subset V(H)$. Then

$$\begin{aligned} \xi^c(H(U)) = \xi^c(H(\{x\})) &= \sum_{uv\in E(H)} \left[\varepsilon_{H(\{x\})}(u) + \varepsilon_{H(\{x\})}(v)\right] \\ &= \sum_{uv\in E(H)} \left[d_H(u,x) + d_H(u,x) + 2\varepsilon_H(x)\right] \\ &= \varsigma_x(H) + 2\,|E(H)|\,\varepsilon_H(x). \end{aligned} \tag{32}$$

On the other hand, $\epsilon(H(U)) = \epsilon(H(\{x\})) = \sum_{v=x}\varepsilon_{H(\{x\})}(v) = \varepsilon_H(x)$. Note that $G\{H\} \cong G \sqcap H(\{x\})$. Thus, combing these results with Theorem 5, we can obtain (29).

Similarly, we can determine (30) and (31) in terms of Theorem 7, respectively. □

Let G and H be two simple graphs. If $|G| = n$ and $|E(G)| = q$, then we say that G is an (n,q)-graph. According to the definitions of the cluster and corona products, if G is an (n,q)-graph and H is a (n',q')-graph, then $G\{H\}$ is an $(nn', q+nq')$-graph and $G\odot H$ is an $(nn'+n, q+nq'+nn')$-graph.

Corollary 30. *Let G be a connected (n,q)-graph and H is an arbitrary (n',q')-graph. Then*

$$\xi^c(G\odot H) = \xi^c(G) + 2\left(q'+n'\right)\zeta(G) + 2q + 4nq' + 3nn', \tag{33}$$

$$\begin{aligned} M_1^*(G\odot H) = {} & \left(n'+1\right) M_1^*(G) \\ & + 2\left(2n'+1\right)\zeta(G) + n\left(4n'+1\right), \end{aligned} \tag{34}$$

$$\begin{aligned} M_2^*(G\odot H) = {} & \xi^c(G) + \left(n'+q'\right) M_1^*(G) + M_2^*(G) \\ & + \left(3n'+4q'\right)\zeta(G) + 2n\left(n'+2q'\right) + q. \end{aligned} \tag{35}$$

Proof. For any (n',q')-graph H, let x be the root-vertex of graph $H+x$ (the join of graphs H and K_1, x is the unique vertex in K_1). Then $H+x$ is a $(n'+1, n'+q')$-graph. It is easy to see that $\varepsilon_{H+x}(x) = 1$, $\varsigma_x(H+x) = n'+2q'$. Note that $G\odot H \cong G\{H+x\}$; hence $\xi^c(G\odot H) = \xi^c(G\{H+x\})$. Equation (33) is obtained by (29). Moreover, we note that $d(x \mid H+x) = d^2(x \mid H+x) = n'$ and $\omega_x(H+x) = q'$. Hence, by (30) and (31), using the same method as above, the corresponding equations (34) and (35) are also obtained, respectively. □

As applications, we present some examples as below, these results can be attained by means of Corollary 30, Lemmas 2, and 3.

Example 31. The following equations hold:

$$\begin{aligned} \xi^c(P_m\odot P_n) &= \begin{cases} 3m^2n + 5mn - 4m, & 2 \mid m,\ m \geq 2; \\ 3m^2n + 5mn - 4m - n, & 2 \nmid m,\ m \geq 3, \end{cases} \\ \xi^c(C_m\odot C_n) &= m\left(2(2n+1)\left\lfloor \frac{m}{2} \right\rfloor + 7n + 2\right), \\ & \qquad m \geq 3,\ n \geq 3, \\ \xi^c(P_m\odot C_n) & \\ &= \begin{cases} \frac{1}{2}\left(6m^2n + 3m^2 + 10mn - 2m\right), \\ \qquad 2 \mid m,\ m \geq 2,\ n \geq 3; \\ \frac{1}{2}\left(6m^2n + 3m^2 + 10mn - 2m - 2n - 1\right), \\ \qquad 2 \nmid m,\ m \geq 3,\ n \geq 3, \end{cases} \\ \xi^c(C_m\odot P_n) &= m\left(4n\left\lfloor \frac{m}{2} \right\rfloor + 7n - 2\right), \quad m \geq 3. \end{aligned} \tag{36}$$

Example 32. Let H be an arbitrary graph with n vertices. Then

$$\begin{aligned} & M_1^*(P_m\odot H) \\ & = \begin{cases} \frac{1}{12} m\left(7m^2n + 7m^2 + 27mn + 9m + 26n + 2\right), \\ \qquad 2 \mid m,\ m \geq 2; \\ \frac{1}{12}\left(7m^3n + 7m^3 + 27m^2n + 9m^2 - m - 9n - 3\right), \\ \qquad 2 \nmid m,\ m \geq 3, \end{cases} \end{aligned} \tag{37}$$

$$M_1^*(C_m \odot H) = m\left((n+1)\left\lfloor\frac{m}{2}\right\rfloor^2 + 2(2n+1)\left\lfloor\frac{m}{2}\right\rfloor + 4n + 1\right), \quad m \geq 3. \tag{38}$$

Example 33. The following equations hold:

$$M_2^*(P_m \odot P_n) = \begin{cases} \frac{1}{12}\left(14m^3n + 45m^2n - 30m^2 + 34mn - 30m + 12\right), & 2 \mid m,\ m \geq 2; \\ \frac{1}{12}\left(14m^3n + 45m^2n - 30m^2 + 28mn - 30m - 15n + 12\right), & 2 \nmid m,\ m \geq 3, \end{cases}$$

$$M_2^*(C_m \odot C_n) = m\left((2n+1)\left\lfloor\frac{m}{2}\right\rfloor^2 + (7n+2)\left\lfloor\frac{m}{2}\right\rfloor + 6n + 1\right), \quad m \geq 3,\ n \geq 3, \tag{39}$$

$$M_2^*(P_m \odot C_n) = \begin{cases} \frac{1}{12}\left(14m^3n + 7m^3 + 45m^2n - 3m^2 + 34mn - 4m + 12\right), & 2 \mid m,\ m \geq 2,\ n \geq 3; \\ \frac{1}{12}\left(14m^3n + 7m^3 + 45m^2n - 3m^2 + 28mn - 7m - 15n + 3\right), & 2 \nmid m,\ m \geq 3,\ n \geq 3, \end{cases}$$

$$M_2^*(C_m \odot P_n) = m\left(2n\left\lfloor\frac{m}{2}\right\rfloor^2 + (7n-2)\left\lfloor\frac{m}{2}\right\rfloor + 6n - 3\right), \quad m \geq 3.$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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