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Research Article

Solving Fractional Difference Equations Using the Laplace Transform Method

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We discuss the Laplace transform of the Caputo fractional difference and the fractional discrete Mittag-Leffer functions. On these bases, linear and nonlinear fractional initial value problems are solved by the Laplace transform method.

1. Introduction

The study of continuous fractional calculus and equations has seen tremendous growth over the past few decades involving many aspects [1–4], such as initial value problem (IVP), boundary value problems (BVP), and stability of fractional equations. Compared with the continuous fractional calculus and fractional order differential equations, we can see that the research about the discrete fractional calculus and fractional difference equations has seen slower progress, but in recent years, a number of papers have appeared, and the study of the discrete fractional calculus and fractional difference equations has been arising. For example, Podlubny et al. [5], Holm [6], and Abdeljawad [7] have explored the definitions of fractional sum and difference operators and obtained many of their properties. Also, Atici and Eloe considered discrete fractional IVPs in paper [8]; moreover, discrete fractional BVPs were discussed in papers by Goodrich [9–11].

We know that the Laplace transform method has played an important role in solving basic problems of differential equations. Holm [6] developed properties of the Laplace transform in a discrete and applied the Laplace transform to solve a fractional initial value problem, which can be described as

$$\begin{gathered}\Delta^{\nu}_{a+\nu-N} y(t) = f(t), \quad t \in N_a, \\ \Delta^{i} y(a+\nu-N) = A_i, \quad i = \{0,1,2\ldots,N-1\};\ A_i \in R.\end{gathered} \tag{1}$$

In this paper, we will discuss the Laplace transform of the Caputo fractional difference and the fractional discrete Mittag-Leffler functions and use the Laplace transform method to solve another kind of discrete fractional IVPs.

2. Preliminaries

Let us start with some definitions and preliminaries.

Definition 1 (see [8]). The generalized falling function is defined by

$$t^{(\mu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}, \quad \text{for } t, \mu \in R, \tag{2}$$

where Γ denotes the special gamma function and $t^{(\mu)} = 0$ whenever $t+1-\mu \in (-N_0)$.

Here are some of the properties of the above fractional function:

(i) $\Delta t^{(\mu)} = \mu t^{(\mu-1)}$;

(ii) $(t-\mu)t^{(\mu)} = \mu t^{(\mu+1)}$;

(iii) $t^{(\mu+\alpha)} = (t-\alpha)^{(\mu)} t^{(\alpha)}$;

(iv) $\mu^{(\mu)} = \Gamma(\mu+1)$.

Definition 2 (see [4]). The αth fractional sum of a function f, for $\alpha > 0$, is defined by

$$\Delta^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-s-1)^{(\alpha-1)} f(s), \tag{3}$$

for $t \in a+\alpha, a+\alpha+1, \ldots =: N_{a+\alpha}$.

Definition 3 (see [7]). The α-order Caputo left fractional difference is defined by

$$\begin{aligned} \Delta_C^{\alpha} f(t) &= \Delta^{-(n-\alpha)} \Delta^n f(s) \\ &= \frac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} (t-s-1)^{(n-\alpha-1)} \Delta^n f(s), \end{aligned} \tag{4}$$

where $n = [\alpha] + 1$, $t \in N_{a+n-\alpha}$. If $\alpha = n \in N$, then $\Delta_C^{\alpha} f(t) = \Delta_C^n f(t)$.

In this paper, we will mainly discuss the problems involving the Caputo left fractional difference.

Definition 4 (see [6]). The Laplace transform of the function f on the time scale

$$N_a := N_0 + a = \{a, a+1, a+2, \ldots\} \quad (a \in R \text{ fixed}), \tag{5}$$

is represented by

$$L_a\{f\}(s) = \sum_{k=0}^{\infty} \frac{f(k+a)}{(s+1)^{k+1}}. \tag{6}$$

Definition 5 (see [6]). One says that a function $f : N_a \to R$ is of exponential order $r > 0$, if there exists a constant $A > 0$ such that

$$|f(t)| \le A r^t, \quad \text{for sufficiently large } t \in N_a. \tag{7}$$

Via a geometric series, it is straightforward to show that if $f : N_a \to R$ is of exponential order $r > 0$, then

$$L\{f\}(s) \quad \text{exist for all } s \in C \setminus \overline{B_{-1}(r)}. \tag{8}$$

Let $m \in N_0$ be given and suppose $f : N_{a-m} \to R$ and $g : N_a \to R$ are of exponential order $r > 0$. Then for $s \in C \setminus \overline{B_{-1}(r)}$,

$$L_{a-m}\{f\}(s) = \frac{1}{(s+1)^m} L_a\{f\}(s) + \sum_{k=0}^{m-1} \frac{f(k+a-m)}{(s+1)^{k+1}}, \tag{9}$$

$$\begin{aligned} L_{a+m}\{f\}(s) &= (s+1)^m L_a\{f\}(s) \\ &\quad - \sum_{k=0}^{m-1} (s+1)^{m-1-k} g(k+a). \end{aligned} \tag{10}$$

Definition 6 (see [6]). For $f, g : N_a \to R$, define the convolution of f and g by

$$(f * g)(t) = \sum_{r=0}^{t-1} f(r) g(t-1-r+a), \quad \text{for } t \in N_a. \tag{11}$$

By a standard convolution on sums, it is understood that $(f * g)(a) = 0$.

3. The Laplace Transform of Caputo Fractional Difference

Lemma 7 (see [6]). *Suppose $f : N_a \to R$ is of exponential order $r \ge 1$ and let $\alpha > 0$ be given with $N-1 < \alpha \le N$. Then both $L_{a+\alpha-N}\{\Delta^{-\alpha} f\}(s)$ and $L_{a-\alpha+N}\{\Delta^{\alpha} f\}(s)$ converge for all $s \in C \setminus \overline{B_{-1}}(r)$, and*

$$\begin{aligned} L_{a+\alpha-N}\{\Delta^{-\alpha} f\}(s) &= \frac{(s+1)^{\alpha-N}}{s^{\alpha}} L_a\{f\}(s), \\ L_{a+N-\alpha}\{\Delta^{\alpha} f\}(s) &= (s+1)^{N-\alpha} s^{\alpha} \\ &\quad - \sum_{j=0}^{N-1} s^j \Delta_a^{\alpha-1-j} f(a+N-\alpha). \end{aligned} \tag{12}$$

Theorem 8. *Suppose $f : N_a \to R$ is of exponential order $r \ge 1$ and let $\alpha > 0$ be given. Then for each fixed $\epsilon > 0$, $\Delta_C^{\alpha} f$ is of exponential order $r + \epsilon$ and $L_{a+N-\alpha}\{\Delta_C^{\alpha} f\}(s)$ converge for all $s \in C \setminus \overline{B_{-1}}(r)$.*

Proof. Consider the relationship between Caputo fractional difference and Riemann-Liouville difference

$$\Delta_C^{\alpha} f(t) = \Delta^{\alpha} f(t) - \sum_{k=0}^{N-1} \frac{(t-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} \Delta^k f(a), \tag{13}$$

and in Lemma 7 we have

$$\begin{aligned} &\left|\Delta_C^{\alpha} f(t)\right| \\ &= \left| \Delta^{\alpha} f(t) - \sum_{k=0}^{N-1} \frac{(t-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} \Delta^k f(a) \right| \\ &\le \left|\Delta^{\alpha} f(t)\right| + \left| \sum_{k=0}^{N-1} \frac{(t-a)^{(k-\alpha)}}{\Gamma(k-\alpha+1)} \Delta^k f(a) \right| \\ &\le \left|\Delta^{\alpha} f(t)\right| + \left| \sum_{k=0}^{N-1} \frac{\Delta^k f(a)}{\Gamma(k-\alpha+1)(t-a)^{(k-\alpha)}} \right| \\ &\le \sum_{k=0}^{N} (-1)^k \binom{N}{k} (r+\epsilon)^{N-k} (r+\epsilon)^t \\ &\quad + \left| \sum_{k=0}^{N-1} \frac{\Delta^k f(a)}{\Gamma(k-\alpha+1)} \right| (t-a)^{N-1-\alpha}, \end{aligned} \tag{14}$$

since for sufficiently large T_2 when $t > T_2$, $(r+\epsilon)^t$ will eventually grow larger than the function $(t-a)^{N-1-\alpha}$. So we get

$$\begin{aligned} &\left|\Delta_C^{\alpha} f(t)\right| \\ &< \left| \sum_{k=0}^{N} (-1)^k \binom{N}{k} (r+\epsilon)^{N-k} + \sum_{k=0}^{N-1} \frac{\Delta^k f(a)}{\Gamma(k-\alpha+1)} \right| (r+\epsilon)^t \\ &= A(r+\epsilon)^t. \end{aligned} \tag{15}$$

Choose $s_0 \in C \setminus \overline{B_{-1}(r)}$; there exists an $\epsilon_0 > 0$ small enough so that $s_0 \in C \setminus \overline{B_{-1}(r+\epsilon_0)}$, and Lemma 7 tells us that $\Delta_C^\alpha f(t)$ is of exponential order $r + \epsilon_0$, so it follows from (7) that $L_{a+N-\alpha}\{\Delta_C^\alpha f\}(s)$ is well defined. □

Theorem 9. *Suppose $f : N_a \to R$ is of exponential order $r \geq 1$ and let $\alpha > 0$ be given with $N-1 < \alpha \leq N$. Then for $s \in C \setminus \overline{B_{-1}(r)}$*

$$L_{a+N-\alpha}\{\Delta_C^\alpha f\}(s) = (s+1)^{N-\alpha} s^\alpha L_a\{f\}(s) - \sum_{j=0}^{N-1} \frac{s^{j+\alpha-N}}{(s+1)^{\alpha-N}} \Delta^{N-1-j} f(a). \tag{16}$$

Proof. Since $\Delta_C^\alpha f(t) = \Delta^{-(N-\alpha)}\Delta^N f(t)$, then

$$L_{a+N-\alpha}\{\Delta_C^\alpha f\}(s) = L_{a+N-\alpha}\left\{\Delta^{-(N-\alpha)}\left(\Delta^N f\right)\right\}(s). \tag{17}$$

Because $L_{a+N-\alpha}\Delta^{-(N-\alpha)}\{g\}(s) = ((s+1)^{N-\alpha}/s^{N-\alpha})L_a\{g\}(s)$ and

$$L_a\left\{\Delta^N f\right\}(s) = s^N L_a\{f\}(s) - \sum_{j=0}^{N-1} s^j \Delta^{N-1-j} f(a), \tag{18}$$

we get

$$\begin{aligned} L_{a+N-\alpha}&\{\Delta_C^\alpha f\}(s) \\ &= \frac{(s+1)^{N-\alpha}}{s^{N-\alpha}} L_a\left\{\Delta^N f\right\}(s) \\ &= \frac{(s+1)^{N-\alpha}}{s^{N-\alpha}}\left(s^N L_a\left\{\Delta^N f\right\}(s) - \sum_{j=0}^{N-1} s^j \Delta^{N-1-j} f(a)\right) \\ &= (s+1)^{N-\alpha} s^\alpha L_a\{f\}(s) - \sum_{j=0}^{N-1}\frac{s^{j+\alpha-N}}{(s+1)^{\alpha-N}}\Delta^{N-1-j} f(a). \end{aligned} \tag{19}$$

□

4. The Laplace Transform of Discrete Mittag-Leffler Function

In paper [8], the discrete Mittag-Leffler function is introduced as the following form.

Definition 10. For any constant $\lambda \in R$ and $\alpha, \beta, t \in C$ with $\mathrm{Re}(\alpha) > 0$, the discrete Mittag-Leffler functions are defined by

$$E_{\alpha,\beta}(\lambda, t) = \sum_{k=0}^{\infty} \lambda^k \frac{(t+(k-1)(\alpha-1))^{(k\alpha)}(t+k(\alpha-1))^{(\beta-1)}}{\Gamma(k\alpha+\beta)}. \tag{20}$$

For $\beta = 1$, it is written that

$$E_{\alpha,1}(\lambda, t) = \sum_{k=0}^{\infty} \lambda^k \frac{(t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}. \tag{21}$$

Theorem 11. *We assume $\alpha > 0$ and $N-1 < \alpha \leq N$ in (20); then for any fixed $r > 1$*

$$\left|E_{\alpha,1}(\lambda, t)\right| \leq Ar^t \quad \textit{for sufficient large } t \in N_a, \ A > 0. \tag{22}$$

And so $L_a\{E_{\alpha,1}(\lambda,t)\}s$ exists for $s \in C \setminus \overline{B_{-1}(r)}$.

Proof. For $N-1 < \alpha < N$, we have

$$\frac{(t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+\beta)} < \frac{(t+(k-1)(N-1))^{(KN)}}{\Gamma(kN+1)}. \tag{23}$$

Moreover, for sufficiently large $t \in N_a$ we get

$$\begin{aligned} &\frac{(t+(k-1)(N-1))^{(kN)}}{\Gamma(kN+1)} \\ &\quad= \frac{(t-k-N+1)\cdots((t+(k-1)(N-1)))}{\Gamma(kN+1)} \\ &\quad< \frac{(t-N+1)\cdots(t-N+1+KN)}{\Gamma(kN+1)} \\ &\quad< \frac{(t-N+1)\cdots(t-N+1+KN)}{\Gamma(kN+1)}; \end{aligned} \tag{24}$$

then,

$$\begin{aligned} &\left|E_{\alpha,1}(\lambda,t)\right| \\ &\quad= \left|\sum_{k=0}^{\infty}\lambda^k \frac{(t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}\right| \\ &\quad< \left|\sum_{k=0}^{\infty}\lambda^k \frac{(t-N+1)\cdots(t-N+1+KN-1)}{\Gamma(kN+1)}\right|. \end{aligned} \tag{25}$$

We know that

$$\begin{aligned} &\left(1-\lambda^{-1}\right)^{-(t-N+1)} \\ &= 1 + \sum_{k=1}^{\infty}\frac{(t-N+1)(t-N+2)\cdots(t-N+k)}{k!}\lambda^k \\ &= 1 + \sum_{k=1}^{\infty}\frac{N(t-N+1)N(t-N+1)\cdots N(t-N+k)}{(kN)!}\lambda^k \\ &> 1 \\ &+ \sum_{k=1}^{\infty}\frac{(t-N+1)(t-N+1+N)\cdots N(t-N+1+kN)}{(kN)!} \\ &\times \lambda^k. \end{aligned} \tag{26}$$

Now, it is easy to see that $|E_{\alpha,1}(\lambda,t)| \leq Ar^t$ for sufficiently large $t \in N_a$, where $A > 1$ is a constant and $r > |\lambda/(\lambda-1)|$, so the function $E_\alpha(\lambda,t)$ converges and $L_a\{E_{\alpha,1}\}(\lambda,s)$ exists for $s \in C \setminus \overline{B_{-1}(r)}$.

We will discuss the Laplace transform of the discrete Mittag-Leffler function $E_{\alpha,1}(\lambda,t)$. □

Theorem 12. *Let $\alpha > 0$, $a = \alpha - 1$; then one gets*

$$L_a\{E_\alpha(\lambda,t)\}(s) = \frac{s^{\alpha-1}}{s^\alpha - \lambda(s+1)^{\alpha-1}}, \qquad \left|\lambda(s+1)^{\alpha-1}\right| < \left|s^\alpha\right|. \tag{27}$$

Proof. When $a = \alpha - 1$, we have

$$\begin{aligned} L_a\{E_\alpha(\lambda,t)\}(s) &= L_{\alpha-1}\left\{\sum_{k=0}^{\infty}\lambda^k\frac{(t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}\right\}(s) \\ &= \sum_{k=0}^{\infty}L_{\alpha-1}\left\{\lambda^k\frac{(t+(k-1))(\alpha-1)^{(k\alpha)}}{\Gamma(k\alpha+1)}\right\}(s). \end{aligned} \tag{28}$$

From Lemma 7 we get

$$L_{k+\alpha-1}\left\{\lambda^k\frac{(t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}\right\}(s) = \frac{(s+1)^{k\alpha}}{s^{k\alpha+1}}; \tag{29}$$

by (10), we conclude that

$$\begin{aligned} &L_{k+\alpha-1}\left\{\frac{(t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}\right\}(s) \\ &\quad= (s+1)^k L_{\alpha-1}\left\{\frac{(t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}\right\}(s) \\ &\qquad - \sum_{l=0}^{k-1}(s+1)^{k-1-l}\frac{(l+k(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}. \end{aligned} \tag{30}$$

For $0 \le l \le k-1$ Definition 1 implies that

$$\frac{(l+k(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)} = \frac{\Gamma(l+k(\alpha-1)+1)}{\Gamma(l+1-k\alpha)\Gamma(k\alpha+1)} = 0. \tag{31}$$

So, we get

$$\begin{aligned} &L_{k+\alpha-1}\left\{\frac{(t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}\right\}(s) \\ &\quad= (s+1)^k L_{\alpha-1}\left\{\frac{(t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}\right\}(s). \end{aligned} \tag{32}$$

Recalling (28), we have

$$L_{\alpha-1}\left\{\frac{(t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}\right\}(s) = \frac{(s+1)^{k(\alpha-1)}}{s^{k\alpha+1}}. \tag{33}$$

Using (33), (27) can be rewritten as

$$\begin{aligned} L_a\{E_{\alpha,1}(\lambda,t)\}(s) &= \sum_{k=0}^{\infty}\lambda^k L_{\alpha-1}\left\{\frac{(t+(k-1)(\alpha-1))^{(k\alpha)}}{\Gamma(k\alpha+1)}\right\}(s) \\ &= \sum_{k=0}^{\infty}\lambda^k\frac{(s+1)^{k(\alpha-1)}}{s^{k\alpha+1}} \\ &= \frac{s^{\alpha-1}}{s^\alpha-\lambda(s+1)^{\alpha-1}}. \end{aligned} \tag{34}$$

When $\alpha = 1$, the result is $L_0\{E_{1,1}(\lambda,t)\}(s) = 1/(s-\lambda)$, which coincided with integer order.

With this in mind, let us discuss the Laplace transform of the Mittag-Leffler function $E_{\alpha,\alpha}(\lambda,t)$; we will use this result in the following section. □

Theorem 13. *Letting $\alpha > 0$, $a = \alpha - 1$, then one has*

$$L_a\{E_{\alpha,\alpha}(\lambda,t)\}(s) = \frac{1}{(s+1)^{1-\alpha}s^\alpha - \lambda}. \tag{35}$$

Proof. We recall that

$$E_{\alpha,\alpha}(\lambda,t) = \sum_{k=0}^{\infty}\lambda^k\frac{(t+(k-1)(\alpha-1))^{(k\alpha)}(t+k(\alpha-1))^{(\alpha-1)}}{\Gamma(k\alpha+\alpha)}. \tag{36}$$

By property (iii) of the generalized falling function, we get

$$E_{\alpha,\alpha}(\lambda,t) = \sum_{k=0}^{\infty}\lambda^k\frac{(t+k(\alpha-1))^{(k\alpha+\alpha-1)}}{\Gamma((k+1)\alpha)}. \tag{37}$$

Then

$$\begin{aligned} L_a\{E_{\alpha,\alpha}(\lambda,t)\}(s) &= \sum_{k=0}^{\infty}\lambda^k L_{\alpha-1}\left\{\frac{(t+k(\alpha-1))^{(k\alpha+\alpha-1)}}{\Gamma((k+1)\alpha)}\right\}(s) \\ &= \sum_{k=0}^{\infty}\lambda^k\frac{(s+1)^{k(\alpha-1)+\alpha-1}}{s^{k\alpha+\alpha}} \\ &= \frac{(s+1)^{\alpha-1}}{s^\alpha}\sum_{k=0}^{\infty}\lambda^k\frac{(s+1)^{k(\alpha-1)}}{s^{k\alpha}} \\ &= \frac{1}{(s+1)^{1-\alpha}s^\alpha-\lambda}. \end{aligned} \tag{38}$$

□

5. Laplace Transform Method for Solving Fractional Difference Equation with Caputo Fractional Difference

In this section, we first consider the following Caputo fractional difference equations:

$$\Delta_C^\alpha\, y(t) = \lambda y(t+\alpha-1), \quad y(a) = a_0,\ t \in N_0, \tag{39}$$

where $\lambda \in R$, $0 < \alpha \le 1$, and $a = \alpha - 1$.

The solution of (39) is given by Atici and Eloe in [8] using the method of successive approximation; we will give the solution of (39) by the method of Laplace transform.

Theorem 14. *Equation* (39) *has its solution given by*

$$y(t) = a_0 E_{\alpha,1}(\lambda,t), \quad t \in N_0. \tag{40}$$

Proof. Both sides of (39) carried out Laplace transform; we get

$$L_{a+N-\alpha}\{\Delta_C^\alpha y\}(s) = \lambda L_{a+N-\alpha}\{y(t+\alpha-1)\}(s). \tag{41}$$

Equations (6) and (9) and Theorem 9 imply that

$$(s+1)^{1-\alpha}s^{\alpha}L_a\{y\}(s)-\frac{s^{\alpha-1}}{(s+1)^{\alpha-1}}a_0=\lambda L_a\{y\}(s). \quad (42)$$

Then

$$L_a\{y\}(s)=\frac{s^{\alpha-1}}{s^{\alpha}-\lambda(s+1)^{\alpha-1}}a_0. \quad (43)$$

Since Theorem 11, we have the solution of (39):

$$y(t)=a_0E_{\alpha,1}(\lambda,t),\quad t\in N_0. \quad (44)$$

This result coincides with the result of paper [7], which obtains the solution of (40) using the method of successive approximation.

Next, we consider the solution of Caputo nonhomogeneous difference equation

$$\Delta_C^{\alpha}y(t)=\lambda y(t+\alpha-1)+f(t),\quad y(a)=a_0,\ t\in N_0, \quad (45)$$

where $f:N_0\rightarrow R$, $0<\alpha\le 1$, $a=\alpha-1$.

The following standard rule for composing the Laplace transform with the convolution is necessary for solving the fractional initial value problems (45). □

Lemma 15 (see [6]). *Let $f,g:N_a\rightarrow R$ be of exponential order $r>0$. Then*

$$L_a\{(f*g)\}(s)=L_a\{f\}(s)L_a\{g\}(s),\quad \text{for } s\in C\setminus(B_{-1}(r)). \quad (46)$$

Theorem 16. *Let $f(t):N_a\rightarrow R$ of be exponential order $r>0$; then (45) has its solution given by*

$$y(t)=a_0E_{\alpha,1}(\lambda,t)+\sum_{s=0}^{t-\alpha}E_{\alpha,\alpha}(\lambda,t-1-s)f(s). \quad (47)$$

Proof. Using Laplace transform on both sides of (45), we obtain

$$\begin{aligned}&L_{a+N-\alpha}\{\Delta_C^{\alpha}y\}(s)\\&\quad=\lambda L_{a+N-\alpha}\{y(t+\alpha-1)\}(s)+L_{a+N-\alpha}\{f\}(s),\end{aligned} \quad (48)$$

because $0<\alpha\le 1$; that is, $N=1$; similar to the above discussion, it is easy to obtain the following:

$$\begin{aligned}(s+1)^{1-\alpha}s^{\alpha}L_a\{y\}(s)-\frac{s^{\alpha-1}}{(s+1)^{\alpha-1}}a_0\\=\lambda L_a\{y\}(s)+L_a\{f(t+1-\alpha)\}(s).\end{aligned} \quad (49)$$

Then we obtain

$$\begin{aligned}L_a\{y\}(s)=&\frac{s^{\alpha-1}}{s^{\alpha}-\lambda(s+1)^{\alpha-1}}a_0\\&+\frac{1}{(s+1)^{1-\alpha}s^{\alpha}-\lambda}L_a\{f(t+1-\alpha)\}(s).\end{aligned} \quad (50)$$

Carrying out Laplace inverse transform of both sides of (45), according to (10), (28), (33), and (35), we have

$$\begin{aligned}y(t)&=a_0E_{\alpha,1}(\lambda,t)\\&\quad+\sum_{r=a}^{t-1}E_{\alpha,\alpha}(\lambda,t-1-r+a)f(r+1-\alpha).\end{aligned} \quad (51)$$

Letting $s=r-a=r-\alpha+1$, formula (45) yields

$$y(t)=a_0E_{\alpha,1}(\lambda,t)+\sum_{s=0}^{t-\alpha}E_{\alpha,\alpha}(\lambda,t-1-s)f(s), \quad (52)$$

which is the expression of the Caputo nonhomogeneous difference equation (45).

In our future research work, we will consider the solution of fractional difference equations (39) and (45) in general situation: $N-1<\alpha\le N$, $N>1$. □

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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