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Research Article

On Properties of Meromorphic Solutions of Certain Difference Painlevé III Equations

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We mainly study the exponents of convergence of zeros and poles of difference and divided difference of transcendental meromorphic solutions for certain difference Painlevé III equations.

1. Introduction and Main Results

In this paper, we use the basic notions of Nevanlinna's theory (see [1, 2]). In addition, we use the notations $\sigma(w)$ to denote the order of growth of the meromorphic function $w(z)$, $\lambda(w)$ and $\lambda(1/w)$, respectively, to denote the exponents of convergence of zeros and poles of $w(z)$. The quantity $\delta(a,w)$ is called the deficiency of the value a to $w(z)$. Furthermore, we denote by $S(r,w)$ any quantity satisfying $S(r,w)=o(T(r,w))$ for all r outside of a set with finite logarithmic measure, and by

$$\mathcal{S}(w)=\{\alpha \text{ meromorphic} : T(r,\alpha)=S(r,w)\} \tag{1}$$

the field of small functions with respect to w. A meromorphic solution w of a difference (or differential) equation is called *admissible* if all coefficients of the equation are in $\mathcal{S}(w)$.

At the beginning of the last century, Painlevé, Gambier, and Fuchs classified a large number of second order differential equations in terms of a characteristic which is now known as the Painlevé property [3–6]. They are proven to be integrable by using inverse scattering transform technique, for instance [7].

Recently, a number of papers (such as [8–12]) focus on complex difference equations and difference analogues of Nevanlinna's theory. Ablowitz et al. [13] considered discrete equations as delay equations in the complex plane which enabled them to utilize complex analytic methods. They looked at difference equations of the type

$$w(z+1)+w(z-1)=R(z,w), \tag{2}$$

where R is rational in both of its arguments. It is shown that if (2) has at least one nonrational finite order meromorphic solution, then $\deg_w R\le 2$.

Recently, Halburd and Korhonen [14] considered (2), where the coefficients of $R(z,w)$ are in $\mathcal{S}(w)$ and got Theorem A.

Theorem A. *If* (2) *has an admissible meromorphic solution of finite order, where $R(z,w)$ is rational and irreducible in w and meromorphic in z, then either w satisfies a difference Riccati equation*

$$w(z+1)=\frac{p(z+1)w(z)+q(z)}{w(z)+p(z)}, \tag{3}$$

where $p(z),q(z)\in\mathcal{S}(w)$, or (2) *can be transformed to one of the following equations:*

$$w(z+1)+w(z)+w(z-1)=\frac{\pi_1 z+\pi_2}{w(z)}+\kappa_1, \tag{4a}$$

$$w(z+1)-w(z)+w(z-1)=\frac{\pi_1 z+(-1)^z\pi_2}{w(z)}+\kappa_1, \tag{4b}$$

$$w(z+1)+w(z-1)=\frac{\pi_1 z+\pi_3}{w(z)}+\pi_2, \tag{4c}$$

$$w(z+1)+w(z-1)=\frac{\pi_1 z+\kappa_1}{w(z)}+\frac{\pi_2}{w^2(z)}, \quad (4d)$$

$$w(z+1)+w(z-1)=\frac{\pi_1 z+\kappa_1+\pi_2}{(-1)^z-w^2(z)}, \quad (4e)$$

$$w(z+1)+w(z-1)=\frac{\pi_1 z+\kappa_1+\pi_2}{1-w^2(z)}, \quad (4f)$$

$$w(z+1)+w(z-1)=\frac{p(z)}{w(z)}, \quad (4g)$$

$$w(z+1)+w(z-1)=p(z)w(z)+q(z), \quad (4h)$$

where $\pi_k, \kappa_k \in \mathcal{S}(w)$ *are arbitrary finite order periodic functions with period k.*

Equations (4a), (4c), *and* (4d) *are known as difference Painlevé I equations, while* (4f) *is often viewed as difference Painlevé II equation. Equations* (4b) *and* (4e) *are slight variations of* (4a) *and* (4f), *respectively.*

In 2010, Chen and Shon [15] researched the properties of finite order meromorphic solutions of difference Painlevé I and II equations. They mainly discussed the existence and the forms of rational solutions and value distribution of transcendental meromorphic solutions.

For difference Painlevé III equations, we recall the following.

Theorem B (see [16]). *Assume that equation*

$$w(z+1)w(z-1)=R(z,w) \quad (5)$$

has an admissible meromorphic solution w of hyperorder less than one, where $R(z,w)$ *is rational and irreducible in w and meromorphic in z; then either w satisfies a difference Riccati equation*

$$w(z+1)=\frac{\alpha(z)w(z)+\beta(z)}{w(z)+\gamma(z)}, \quad (6)$$

where $\alpha(z), \beta(z), \gamma(z) \in \mathcal{S}(w)$ *are algebroid functions, or* (5) *can be transformed to one of the following equations:*

$$w(z+1)w(z-1)=\frac{\eta(z)w^2(z)-\lambda(z)w(z)+\mu(z)}{(w(z)-1)(w(z)-\nu(z))}, \quad (7a)$$

$$w(z+1)w(z-1)=\frac{\eta(z)w^2(z)-\lambda(z)w(z)}{w(z)-1}, \quad (7b)$$

$$w(z+1)w(z-1)=\frac{\eta(z)(w(z)-\lambda(z))}{w(z)-1}, \quad (7c)$$

$$w(z+1)w(z-1)=h(z)w^m(z). \quad (7d)$$

In (7a), *the coefficients satisfy* $\kappa^2(z)\mu(z+1)\mu(z-1)=\mu^2(z)$, $\lambda(z+1)\mu(z)=\kappa(z)\lambda(z-1)\mu(z+1)$, $\kappa(z)\lambda(z+2)\lambda(z-1)=\kappa(z-1)\lambda(z)\lambda(z+1)$, *and one of the following:*

(1) $\eta\equiv 1, \nu(z+1)\nu(z-1)=1, \kappa(z)=\nu(z)$;

(2) $\eta(z+1)=\eta(z-1)=\nu(z), \kappa\equiv 1$.

In (7b), $\eta(z)\eta(z+1)=1$ *and* $\lambda(z+2)\lambda(z-1)=\lambda(z)\lambda(z+1)$. *In* (7c), *the coefficients satisfy one of the following:*

(1) $\eta\equiv 1$ *and either* $\lambda(z)=\lambda(z+1)\lambda(z-1)$ *or* $\lambda(z+3)\lambda(z-3)=\lambda(z+2)\lambda(z-2)$;

(2) $\lambda(z+1)\lambda(z-1)=\lambda(z+2)\lambda(z-2), \eta(z+1)\lambda(z+1)=\lambda(z+2)\eta(z-1), \eta(z)\eta(z-1)=\eta(z+2)\eta(z-3)$;

(3) $\eta(z+2)\eta(z-2)=\eta(z)\eta(z-1), \lambda(z)=\eta(z-1)$;

(4) $\lambda(z+3)\lambda(z-3)=\lambda(z+2)\lambda(z-2)\lambda(z), \eta(z)\lambda(z)=\eta(z+2)\eta(z-2)$.

In (7d), $h(z)\in\mathcal{S}(w)$ *and* $m\in\mathbb{Z}, |m|\le 2$.

Zhang and Yang [17] investigated difference Painlevé III equations (7a)–(7d) with constant coefficients and obtained the following results.

Theorem C. *If w is a nonconstant meromorphic solution of difference equation* (7d), *where* $m=-2,-1,0,1$ *and h is a nonzero constant, then*

(i) w cannot be a rational function;

(ii) $\lambda(w)=\tau(w)=\sigma(w)$, *where* $\tau(w)$ *denotes the exponent of convergence of fixed points of w.*

Theorem D. *If w is a nonconstant meromorphic solution of difference equation* (7d), *where* $m=2$ *and h is a nonzero constant, then*

(i) w has no nonzero Nevanlinna exceptional value;

(ii) w cannot be a rational function;

(iii) $\tau(w)=\sigma(w)$.

In Theorems C and D, *h* is defined as a nonzero constant. A natural question to ask is what can we say on meromorphic solutions of (7a)–(7d) if *h* is a nonconstant meromorphic function? In this paper, we answer this question. In the following theorems, we study the properties of difference and divide difference of transcendental meromorphic solutions of (7a)–(7d).

Theorem 1. *Suppose that* $h(z)$ *is a nonconstant rational function. If* $w(z)$ *is a transcendental meromorphic solution with finite order of equation*

$$w(z+1)w(z-1)=h(z), \quad (8)$$

set $\Delta w(z)=w(z+1)-w(z)$. *Then*

(i) w has no Nevanlinna exceptional value;

(ii) $\lambda(\Delta w)=\lambda(1/\Delta w)=\sigma(w), \lambda(\Delta w/w)=\lambda(1/(\Delta w/w))=\sigma(w)$.

Example 2. The function $w(z)=z(e^{i(\pi/2)z}-1)/(e^{i(\pi/2)z}+1)$ is a meromorphic solution of difference equation

$$w(z+1)w(z-1)=(z+1)(z-1), \quad (9)$$

where $h(z) = (z+1)(z-1)$. By calculation, this solution satisfies

$$\Delta w(z) = \frac{ie^{i\pi z} + (i-1)(2z+1)e^{i(\pi/2)z} - 1}{(ie^{i(\pi/2)z}+1)(e^{i(\pi/2)z}+1)},$$
$$\frac{\Delta w(z)}{w(z)} = \frac{ie^{i\pi z} + (i-1)(2z+1)e^{i(\pi/2)z} - 1}{z(ie^{i(\pi/2)z}+1)(e^{i(\pi/2)z}-1)}. \tag{10}$$

Thus,

$$\lambda(\Delta w) = \lambda\left(\frac{1}{\Delta w}\right) = \sigma(w) = 1,$$
$$\lambda\left(\frac{\Delta w}{w}\right) = \lambda\left(\frac{1}{\Delta w/w}\right) = \sigma(w) = 1. \tag{11}$$

Theorem 3. *Suppose that $h(z)$ is a nonconstant rational function. If $w(z)$ is a transcendental meromorphic solution with finite order of equation*

$$w(z+1)w(z-1) = h(z)w(z), \tag{12}$$

then

(i) w has no Nevanlinna exceptional value;

(ii) $\lambda(\Delta w) = \lambda(1/\Delta w) = \sigma(w), \lambda(\Delta w/w) = \lambda(1/(\Delta w/w)) = \sigma(w)$.

From the following proof of Theorem 3, we have the following.

Remark 4. If $w(z)$ is an admissible meromorphic solution with finite order of (12), then $T(r, \Delta w/w) = T(r,w) + S(r,w)$.

Example 5. The function $w(z) = z\tan(\pi z/6)\tan((\pi z/6) - (\pi/6))$ is a meromorphic solution of difference equation

$$w(z+1)w(z-1) = -\frac{z^2-1}{z}w(z), \tag{13}$$

where $h(z) = -(z^2-1)/z$. By calculation, this solution satisfies

$$\Delta w(z) = \tan\left(\frac{\pi z}{6}\right)\frac{2\sin(\pi z/3) + \sqrt{3}(2z+1)}{2\cos(\pi z/3) - 1},$$
$$\frac{\Delta w(z)}{w(z)} = \cot\left(\frac{\pi z}{6} - \frac{\pi}{6}\right)\frac{2\sin(\pi z/3) + \sqrt{3}(2z+1)}{z(2\cos(\pi z/3) - 1)}. \tag{14}$$

Thus,

$$\lambda(\Delta w) = \lambda\left(\frac{1}{\Delta w}\right) = \sigma(w) = 1,$$
$$\lambda\left(\frac{\Delta w}{w}\right) = \lambda\left(\frac{1}{\Delta w/w}\right) = \sigma(w) = 1. \tag{15}$$

Theorem 6. *Suppose that $h(z)$ is a nonconstant rational function. If $w(z)$ is a transcendental meromorphic solution with finite order of equation*

$$w(z+1)w(z-1)w(z) = h(z), \tag{16}$$

then

(i) w has no Nevanlinna exceptional value;

(ii) $\lambda(\Delta w) = \lambda(1/\Delta w) = \sigma(w), \lambda(\Delta w/w) = \lambda(1/(\Delta w/w)) = \sigma(w)$.

Theorem 7. *Suppose that $h(z)$ is a nonconstant rational function. If $w(z)$ is a transcendental meromorphic solution with finite order of equation*

$$w(z+1)w(z-1)w^2(z) = h(z), \tag{17}$$

then

(i) w has no Nevanlinna exceptional value;

(ii) $\lambda(\Delta w) = \lambda(1/\Delta w) = \sigma(w), \lambda(\Delta w/w) = \lambda(1/(\Delta w/w)) = \sigma(w)$.

From the following proof of Theorem 7, we see the following.

Remark 8. If $w(z)$ is an admissible meromorphic solution with finite order of (17), then $T(r, \Delta w/w) = 2T(r,w) + S(r,w)$.

Example 9. The function $w(z) = z(e^{i\pi z} - 1)/(e^{i\pi z} + 1)$ is a meromorphic solution of difference equation

$$w(z+1)w(z-1)w^2(z) = z^4 - z^2, \tag{18}$$

where $h(z) = z^4 - z^2$. By calculation, this solution satisfies

$$\Delta w(z) = \frac{e^{i2\pi z} + (4z+2)e^{i\pi z} + 1}{e^{i2\pi z} - 1},$$
$$\frac{\Delta w(z)}{w(z)} = \frac{e^{i2\pi z} + (4z+2)e^{i\pi z} + 1}{z(e^{i\pi z} - 1)^2}. \tag{19}$$

Thus,

$$\lambda(\Delta w) = \lambda\left(\frac{1}{\Delta w}\right) = \sigma(w) = 1,$$
$$\lambda\left(\frac{\Delta w}{w}\right) = \lambda\left(\frac{1}{\Delta w/w}\right) = \sigma(w) = 1. \tag{20}$$

From the following proofs of Theorems 1–7, we point out the following.

Remark 10. Suppose that $h(z)$ is a meromorphic function satisfying $h(z+1) \not\equiv h(z)$. If $w(z)$ is an admissible meromorphic solution with finite order of (7d), where $m = -2, -1, 0, 1$, then Theorems 1–7 still hold.

Equations (7a)–(7c) and $w(z+1)w(z-1) = h(z)w^2(z)$ can be discussed similarly; we omit it in the present paper.

2. Lemmas for the Proofs of Theorems

Lemma 11 (see [9]). *Let $f(z)$ be a meromorphic function of finite order and let c be a nonzero complex constant. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f). \tag{21}$$

Lemma 12 (see [9]). *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f), \sigma < \infty$, and let η be a fixed nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T(r, f(z+\eta)) = T(r, f(z)) + O\left(r^{\sigma-1+\varepsilon}\right) + O(\log r). \tag{22}$$

Lemma 13 (see [9]). *Let $f(z)$ be a meromorphic function with exponent of convergence of poles $\lambda(1/f) = \lambda < \infty$, and let $\eta \neq 0$ be fixed. Then for each ε $(0 < \varepsilon < 1)$,*

$$N(r, f(z+\eta)) = N(r, f(z)) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r). \tag{23}$$

Lemmas 11 and 12 show the following.

Lemma 14. *Let c be a nonzero constant and let $f(z)$ be a finite order meromorphic function. Then*

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f(z)}\right) + S(r, f). \tag{24}$$

Lemma 15 (Valiron-Mohon'ko [18]). *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z, f(z)) = \frac{a_n(z) f(z)^n + \cdots + a_0(z)}{b_m(z) f(z)^m + \cdots + b_0(z)} \tag{25}$$

with meromorphic coefficients $a_i(z), b_j(z)$ being small with respect to f, the characteristic function of $R(z, f(z))$ satisfies

$$T(r, R(z, f(z))) = \max\{m, n\} T(r, f) + S(r, f). \tag{26}$$

Lemma 16 (see [10, 11]). *Let w be a transcendental meromorphic solution with finite order of difference equation*

$$P(z, w) = 0, \tag{27}$$

where $P(z, w)$ is a difference polynomial in $w(z)$. If $P(z, a) \not\equiv 0$ for a meromorphic function $a \in \mathcal{S}(w)$, then

$$m\left(r, \frac{1}{w-a}\right) = S(r, w). \tag{28}$$

Lemma 17 (see [11]). *Let f be a transcendental meromorphic solution with finite order σ of a difference equation of the form*

$$U(z, f) P(z, f) = Q(z, f), \tag{29}$$

where $U(z, f)$, $P(z, f)$, and $Q(z, f)$ are difference polynomials such that the total degree $\deg_f U(z, f) = n$ in $f(z)$ and its shifts and $\deg_f Q(z, f) \leq n$. If $U(z, f)$ contains just one term of maximal total degree in $f(z)$ and its shifts, then for each $\varepsilon > 0$,

$$m(r, P(z, f)) = O\left(r^{\sigma-1+\varepsilon}\right) + S(r, f). \tag{30}$$

3. Proofs of Theorems

Proof of Theorem 1. (i) Set $P(z, w) := w(z+1)w(z-1) - h(z) = 0$. Since $h(z)$ is a nonconstant rational function, for any $a \in \mathbb{C}$, we know $P(z, a) = a^2 - h(z) \not\equiv 0$. Lemma 16 gives $m(r, 1/(w-a)) = S(r, w)$, which follows $N(r, 1/(w-a)) = T(r, w) + S(r, w)$. Thus, $\delta(a, w) = 0$.

From (8), we have that

$$w(z+2) w(z) = h(z+1). \tag{31}$$

Applying Lemma 17 to (31), we know

$$m(r, w) = S(r, w), \tag{32}$$

which implies $N(r, w) = T(r, w) + S(r, w)$. Thus, $\delta(\infty, w) = 0$.

Therefore, for any $a \in \mathbb{C} \cup \{\infty\}$, $\delta(a, w) = 0$. So, w has no Nevanlinna exceptional value.

(ii) First, we prove that $\lambda(\Delta w/w) = \lambda(1/(\Delta w/w)) = \sigma(w)$. By (8) and Lemma 12, we obtain

$$\begin{aligned} 2T(r, w(z)) &= T\left(r, \frac{h(z)}{w^2(z)}\right) + O(\log r) \\ &= T\left(r, \frac{w(z+1)}{w(z)} \frac{w(z-1)}{w(z)}\right) + O(\log r) \\ &\leq T\left(r, \frac{w(z+1)}{w(z)}\right) + T\left(r, \frac{w(z)}{w(z-1)}\right) \\ &\quad + O(\log r) \\ &= 2T\left(r, \frac{w(z+1)}{w(z)}\right) + S\left(r, \frac{w(z+1)}{w(z)}\right) \\ &\quad + O(\log r) \\ &\leq 2T\left(r, \frac{w(z+1)}{w(z)}\right) + S(r, w(z)). \end{aligned} \tag{33}$$

Hence,

$$\begin{aligned} T(r, w(z)) &\leq T\left(r, \frac{w(z+1)}{w(z)}\right) + S(r, w(z)) \\ &= T\left(r, \frac{\Delta w(z)}{w(z)}\right) + S(r, w(z)). \end{aligned} \tag{34}$$

From (34) and Lemmas 11 and 12, we deduce that

$$\begin{aligned} N\left(r, \frac{\Delta w}{w}\right) &= T\left(r, \frac{\Delta w}{w}\right) - m\left(r, \frac{\Delta w}{w}\right) \\ &= T\left(r, \frac{\Delta w}{w}\right) + S(r, w) \geq T(r, w) + S(r, w). \end{aligned} \tag{35}$$

Thus, $\lambda(1/(\Delta w/w)) \geq \sigma(w)$, that is, $\lambda(1/(\Delta w/w)) = \sigma(w)$.

By (8) and (31), we know

$$\frac{w(z+2)}{w(z+1)} \frac{w(z)}{w(z-1)} = \frac{h(z+1)}{h(z)}. \tag{36}$$

Set

$$g(z) = \frac{w(z+1)}{w(z)}. \tag{37}$$

Thus, (36) can be written as $g(z+1)g(z-1) = h(z+1)/h(z)$. Set $P_1(z, g) := g(z+1)g(z-1) - (h(z+1)/h(z)) = 0$. Since $h(z)$

is a nonconstant rational function, $h(z)$ cannot be a periodic function. Then $P_1(z,1) = 1 - (h(z+1)/h(z)) \not\equiv 0$. Since $P_1(z,1) \not\equiv 0$, by (37) and Lemmas 12 and 16, we have

$$m\left(r, \frac{1}{g(z)-1}\right) = S(r, g(z)) = S\left(r, \frac{w(z+1)}{w(z)}\right) \leq S(r, w(z)). \quad (38)$$

Thus,

$$m\left(r, \frac{1}{\Delta w(z)/w(z)}\right) = m\left(r, \frac{1}{(w(z+1)/w(z))-1}\right) = m\left(r, \frac{1}{g(z)-1}\right) = S(r, w(z)). \quad (39)$$

By (34) and (39), we have

$$N\left(r, \frac{1}{\Delta w/w}\right) = T\left(r, \frac{\Delta w}{w}\right) - m\left(r, \frac{1}{\Delta w/w}\right) = T\left(r, \frac{\Delta w}{w}\right) + S(r,w) \geq T(r,w) + S(r,w). \quad (40)$$

Then, $\lambda(\Delta w/w) \geq \sigma(w)$, that is, $\lambda(\Delta w/w) = \sigma(w)$.

Next, we prove $\lambda(\Delta w) = \lambda(1/\Delta w) = \sigma(w)$. By (8),

$$\begin{aligned} \Delta w(z) + \Delta w(z-1) &= (w(z+1) - w(z)) + (w(z) - w(z-1)) \\ &= w(z+1) - w(z-1) \\ &= w(z+1) - \frac{h(z)}{w(z+1)} \\ &= \frac{w^2(z+1) - h(z)}{w(z+1)}. \end{aligned} \quad (41)$$

Applying Lemmas 12 and 15 to (41), we have

$$\begin{aligned} 2T(r, w(z)) &= 2T(r, w(z+1)) + S(r, w(z)) \\ &= T\left(r, \frac{w^2(z+1) - h(z)}{w(z+1)}\right) + S(r, w(z)) \\ &= T(r, \Delta w(z) + \Delta w(z-1)) + S(r, w(z)) \\ &\leq T(r, \Delta w(z)) + T(r, \Delta w(z-1)) + S(r, w(z)) \\ &= 2T(r, \Delta w(z)) + S(r, \Delta w(z)) + S(r, w(z)) \\ &\leq 2T(r, \Delta w(z)) + S(r, w(z)). \end{aligned} \quad (42)$$

Hence,

$$T(r, w(z)) \leq T(r, \Delta w(z)) + S(r, w(z)). \quad (43)$$

Obviously, it follows from (32) and Lemma 11 that

$$m(r, \Delta w) \leq m\left(r, \frac{\Delta w}{w}\right) + m(r,w) = S(r,w). \quad (44)$$

Together with (43), we have

$$N(r, \Delta w) = T(r, \Delta w) + S(r,w) \geq T(r,w) + S(r,w), \quad (45)$$

which yields $\lambda(1/\Delta w) \geq \sigma(w)$. That is, $\lambda(1/\Delta w) = \sigma(w)$.

Set $a = 0$ in (i). By (39), we obtain

$$m\left(r, \frac{1}{\Delta w}\right) = m\left(r, \frac{1}{\Delta w/w}\frac{1}{w}\right) \leq m\left(r, \frac{1}{\Delta w/w}\right) + m\left(r, \frac{1}{w}\right) = S(r,w). \quad (46)$$

Combining this with (43), we have

$$N\left(r, \frac{1}{\Delta w}\right) = T(r, \Delta w) - m\left(r, \frac{1}{\Delta w}\right) = T(r, \Delta w) + S(r,w) \geq T(r,w) + S(r,w). \quad (47)$$

Then $\lambda(\Delta w) \geq \sigma(w)$, that is, $\lambda(\Delta w) = \sigma(w)$. □

Proof of Theorem 3. (i) By (12) and Lemma 11, we see that

$$\begin{aligned} m(r, w(z)) &= m\left(r, w(z)\frac{h(z)w(z)}{w(z+1)w(z-1)}\right) \\ &= m\left(r, \frac{h(z)w^2(z)}{w(z+1)w(z-1)}\right) = S(r,w), \\ m\left(r, \frac{1}{w(z)}\right) &= m\left(r, \frac{1}{w(z)}\frac{w(z+1)w(z-1)}{h(z)w(z)}\right) \\ &= m\left(r, \frac{w(z+1)w(z-1)}{h(z)w^2(z)}\right) = S(r,w). \end{aligned} \quad (48)$$

Hence,

$$N(r, w(z)) = T(r, w(z)) + S(r,w), \quad (49)$$

$$N\left(r, \frac{1}{w(z)}\right) = T(r, w(z)) + S(r,w). \quad (50)$$

So, $\delta(0,w) = \delta(\infty, w) = 0$.

Set

$$P(z,w) := w(z+1)w(z-1) - h(z)w(z) = 0. \quad (51)$$

Since $h(z)$ is a nonconstant rational function, for any $a \in \mathbb{C} \setminus \{0\}$, we have $P(z,a) = a^2 - ah(z) \not\equiv 0$. Lemma 16 gives $m(r, 1/(w-a)) = S(r,w)$, which follows $N(r, 1/(w-a)) = T(r,w) + S(r,w)$. Thus, $\delta(a,w) = 0$. Combining with $\delta(0,w) = \delta(\infty,w) = 0$, we know w has no Nevanlinna exceptional value.

(ii) First, we prove $\lambda(\Delta w) = \lambda(1/\Delta w) = \sigma(w)$. Since $w(z+1) = w(z) + \Delta w(z)$, $w(z-1) = w(z) - \Delta w(z-1)$, by (12), we have

$$(w(z) + \Delta w(z))(w(z) - \Delta w(z-1)) = h(z)w(z); \quad (52)$$

that is,

$$(\Delta w(z)-\Delta w(z-1))\,w(z)-\Delta w(z)\,\Delta w(z-1) = -w^2(z)+h(z)\,w(z). \tag{53}$$

Let z_0 be a zero of $w(z)$, not pole of $h(z)$. From (52), z_0 is a zero of $w(z)+\Delta w(z)$ or $w(z)-\Delta w(z-1)$. Since $w(z_0)=0$, then z_0 must be a zero of $\Delta w(z)$ or $\Delta w(z-1)$. Thus, by (50) and Lemma 14, we obtain

$$\begin{aligned}
T(r,w(z)) &= N\left(r,\frac{1}{w(z)}\right)+S(r,w)\\
&\le N\left(r,\frac{1}{\Delta w(z)}\right)+N\left(r,\frac{1}{\Delta w(z-1)}\right)\\
&\quad + N(r,h(z))+S(r,w)\\
&= 2N\left(r,\frac{1}{\Delta w(z)}\right)+S(r,\Delta w(z))\\
&\quad + O(\log r)+S(r,w)\\
&\le 2N\left(r,\frac{1}{\Delta w(z)}\right)+S(r,w).
\end{aligned} \tag{54}$$

Hence, $\sigma(w)\le\lambda(\Delta w)$, that is, $\lambda(\Delta w)=\sigma(w)$.

If z_1 is a pole of $w(z)$ with multiplicity k, not pole of $h(z)$, then z_1 is a pole of $-w^2(z)+h(z)w(z)$ with multiplicity $2k$. From (53), one of $\Delta w(z)$ and $\Delta w(z-1)$ must have the pole z_1 with multiplicity not less than k. Thus, by (49) and Lemma 13, we get

$$\begin{aligned}
T(r,w(z)) &= N(r,w(z))+S(r,w)\\
&\le N(r,\Delta w(z))+N(r,\Delta w(z-1))\\
&\quad + N(r,h(z))+S(r,w)\\
&= 2N(r,\Delta w(z))+S(r,\Delta w(z))\\
&\quad + O(\log r)+S(r,w)\\
&\le 2N(r,\Delta w(z))+S(r,w).
\end{aligned} \tag{55}$$

Hence, $\sigma(w)\le\lambda(1/\Delta w)$, that is, $\lambda(1/\Delta w)=\sigma(w)$.

Next, we prove that $\lambda(\Delta w/w)=\lambda(1/(\Delta w/w))=\sigma(w)$. By (12), we have

$$\begin{aligned}
\frac{\Delta w(z)}{w(z)} &= \frac{w(z+1)-w(z)}{w(z)}=\frac{w(z+1)}{w(z)}-1\\
&= \frac{h(z)}{w(z-1)}-1=\frac{h(z)-w(z-1)}{w(z-1)}.
\end{aligned} \tag{56}$$

From (56) and Lemmas 11 and 12, we deduce that

$$\begin{aligned}
N\left(r,\frac{\Delta w(z)}{w(z)}\right) &= T\left(r,\frac{\Delta w(z)}{w(z)}\right)-m\left(r,\frac{\Delta w(z)}{w(z)}\right)\\
&= T\left(r,\frac{\Delta w(z)}{w(z)}\right)+S(r,w(z))\\
&= T\left(r,\frac{h(z)}{w(z-1)}-1\right)+S(r,w(z))\\
&= T(r,w(z-1))+S(r,w(z))\\
&= T(r,w(z))+S(r,w(z)).
\end{aligned} \tag{57}$$

Thus, $\lambda(1/(\Delta w/w))=\sigma(w)$.

Since $h(z)$ is a nonconstant rational function, $h(z)$ cannot be a periodic function. Thus, by (51), $P(z,h(z+1))=h(z+2)h(z)-h(z)h(z+1)=h(z)(h(z+2)-h(z+1))\not\equiv 0$. Lemma 16 gives $m(r,1/(w(z)-h(z+1)))=S(r,w(z))$, which follows

$$N\left(r,\frac{1}{w(z)-h(z+1)}\right)=T(r,w(z))+S(r,w(z)). \tag{58}$$

By (56), if z_0 is a common zero of $h(z)-w(z-1)$ and $w(z-1)$, then z_0 must be a zero of $h(z)$. Thus, by (56), (58), and Lemma 14, we have

$$\begin{aligned}
N\left(r,\frac{1}{\Delta w(z)/w(z)}\right) &\ge N\left(r,\frac{1}{w(z-1)-h(z)}\right)\\
&\quad - N\left(r,\frac{1}{h(z)}\right)\\
&= N\left(r,\frac{1}{w(z)-h(z+1)}\right)\\
&\quad + O(\log r)+S(r,w(z))\\
&= T(r,w(z))+S(r,w(z)).
\end{aligned} \tag{59}$$

Hence, $\lambda(\Delta w/w)\ge\sigma(w)$, that is, $\lambda(\Delta w/w)=\sigma(w)$. □

Proof of Theorem 6. (i) Set $P(z,w):=w(z+1)w(z-1)w(z)-h(z)=0$. Since $h(z)$ is a nonconstant rational function, for any $a\in\mathbb{C}$, we have $P(z,a)=a^3-h(z)\not\equiv 0$. Lemma 16 shows $m(r,1/(w-a))=S(r,w)$, which yields $N(r,1/(w-a))=T(r,w)+S(r,w)$. Thus, $\delta(a,w)=0$.

We see from (16) and Lemma 17 that

$$m(r,w)=S(r,w), \tag{60}$$

which follows $N(r,w)=T(r,w)+S(r,w)$; thus, $\delta(\infty,w)=0$.

Therefore, for any $a\in\mathbb{C}\cup\{\infty\}$, $\delta(a,w)=0$. So, w has no Nevanlinna exceptional value.

(ii) First, we prove $\lambda(1/(\Delta w/w))=\lambda(1/\Delta w)=\sigma(w)$. By (16) and Lemma 12, we have

$$\begin{aligned}
3T(r,w(z)) &= T\left(r,\frac{h(z)}{w^3(z)}\right)+O(\log r)\\
&= T\left(r,\frac{w(z+1)}{w(z)}\frac{w(z-1)}{w(z)}\right)+O(\log r)\\
&\le T\left(r,\frac{w(z+1)}{w(z)}\right)+T\left(r,\frac{w(z)}{w(z-1)}\right)\\
&\quad + O(\log r)
\end{aligned}$$

$$= 2T\left(r, \frac{w(z+1)}{w(z)}\right) + S\left(r, \frac{w(z+1)}{w(z)}\right) + O(\log r) \le 2T\left(r, \frac{w(z+1)}{w(z)}\right) + S(r, w(z)). \quad (61)$$

Thus,

$$T\left(r, \frac{\Delta w(z)}{w(z)}\right) = T\left(r, \frac{w(z+1)}{w(z)}\right) + O(1) \ge \frac{3}{2}T(r, w(z)) + S(r, w). \quad (62)$$

We deduce from (62) and Lemmas 11 and 12 that

$$N\left(r, \frac{\Delta w}{w}\right) = T\left(r, \frac{\Delta w}{w}\right) - m\left(r, \frac{\Delta w}{w}\right) = T\left(r, \frac{\Delta w}{w}\right) + S(r, w) \ge \frac{3}{2}T(r, w) + S(r, w). \quad (63)$$

Then $\lambda(1/(\Delta w/w)) \ge \sigma(w)$. So, $\lambda(1/(\Delta w/w)) = \sigma(w)$.

By (62), we obtain

$$T(r, \Delta w) = T\left(r, \frac{\Delta w}{w}w\right) \ge T\left(r, \frac{\Delta w}{w}\right) - T(r, w) \ge \frac{1}{2}T(r, w) + S(r, w). \quad (64)$$

By (60), (64), and Lemma 11, we have

$$\begin{aligned} N(r, \Delta w) &= T(r, \Delta w) - m(r, \Delta w) \\ &\ge T(r, \Delta w) - m\left(r, \frac{\Delta w}{w}\right) - m(r, w) \\ &= T(r, \Delta w) + S(r, w) \\ &\ge \frac{1}{2}T(r, w) + S(r, w). \end{aligned} \quad (65)$$

Then $\lambda(1/\Delta w) \ge \sigma(w)$, that is, $\lambda(1/\Delta w) = \sigma(w)$.

Next, we prove that $\lambda(\Delta w) = \lambda(\Delta w/w) = \sigma(w)$. By (16), we know

$$w(z+2)\,w(z)\,w(z+1) = h(z+1). \quad (66)$$

By this and (16), we have

$$\frac{w(z+2)}{w(z+1)}\frac{w(z)}{w(z-1)}\frac{w(z+1)}{w(z)} = \frac{h(z+1)}{h(z)}. \quad (67)$$

Set

$$g(z) = \frac{w(z+1)}{w(z)}. \quad (68)$$

Substituting (68) into (67), we have $g(z+1)g(z-1)g(z) = h(z+1)/h(z)$. Set $P_1(z, g) := g(z+1)g(z-1)g(z) - (h(z+1)/h(z)) = 0$. Since $h(z)$ is a nonconstant rational function, $h(z)$ cannot be a periodic function. Thus, $P_1(z, 1) = 1 - (h(z+1)/h(z)) \not\equiv 0$. By this and by (68) and Lemmas 12 and 16, we obtain

$$m\left(r, \frac{1}{g(z)-1}\right) = S(r, g(z)) = S\left(r, \frac{w(z+1)}{w(z)}\right) \le S(r, w(z)). \quad (69)$$

That is,

$$m\left(r, \frac{1}{\Delta w(z)/w(z)}\right) = m\left(r, \frac{1}{(w(z+1)/w(z)) - 1}\right) = m\left(r, \frac{1}{g(z)-1}\right) = S(r, w(z)). \quad (70)$$

By (62) and (70), we have

$$N\left(r, \frac{1}{\Delta w/w}\right) = T\left(r, \frac{\Delta w}{w}\right) - m\left(r, \frac{1}{\Delta w/w}\right) + O(1) = T\left(r, \frac{\Delta w}{w}\right) + S(r, w) \ge \frac{3}{2}T(r, w) + S(r, w). \quad (71)$$

Thus, $\lambda(\Delta w/w) \ge \sigma(w)$, that is, $\lambda(\Delta w/w) = \sigma(w)$.

Set $a = 0$ in (i). By (70), we have

$$m\left(r, \frac{1}{\Delta w}\right) = m\left(r, \frac{1}{\Delta w/w}\frac{1}{w}\right) \le m\left(r, \frac{1}{\Delta w/w}\right) + m\left(r, \frac{1}{w}\right) = S(r, w). \quad (72)$$

Thus, by (64),

$$N\left(r, \frac{1}{\Delta w}\right) = T(r, \Delta w) - m\left(r, \frac{1}{\Delta w}\right) = T(r, \Delta w) + S(r, w) \ge \frac{1}{2}T(r, w) + S(r, w). \quad (73)$$

Hence, $\lambda(\Delta w) \ge \sigma(w))$, that is, $\lambda(\Delta w) = \sigma(w)$. □

Proof of Theorem 7. The proof of (i) is similar to the proof of (i) in Theorem 6; we omit it here.

(ii) We conclude from (17) and Lemmas 12 and 15 that

$$\begin{aligned} 4T(r, w(z)) &= T\left(r, \frac{h(z)}{w^4(z)}\right) + O(\log r) \\ &= T\left(r, \frac{w(z+1)}{w(z)}\frac{w(z-1)}{w(z)}\right) + O(\log r) \\ &\le T\left(r, \frac{w(z+1)}{w(z)}\right) + T\left(r, \frac{w(z)}{w(z-1)}\right) + O(\log r) \end{aligned}$$

$$= 2T\left(r, \frac{w(z+1)}{w(z)}\right) + S\left(r, \frac{w(z+1)}{w(z)}\right) + O(\log r) \le 4T(r, w(z)) + S(r, w(z)). \tag{74}$$

Thus,

$$T\left(r, \frac{\Delta w(z)}{w(z)}\right) = T\left(r, \frac{w(z+1)}{w(z)}\right) + O(1) = 2T(r, w(z)) + S(r, w(z)). \tag{75}$$

By (75) and Lemma 11, we know

$$N\left(r, \frac{\Delta w}{w}\right) = T\left(r, \frac{\Delta w}{w}\right) - m\left(r, \frac{\Delta w}{w}\right) = T\left(r, \frac{\Delta w}{w}\right) + S(r, w) = 2T(r, w) + S(r, w). \tag{76}$$

Therefore, $\lambda(1/(\Delta w/w)) = \sigma(w)$.

By (17), we know

$$w(z+2)\, w(z)\, w^2(z+1) = h(z+1). \tag{77}$$

By this and (17), we have

$$\frac{w(z+2)}{w(z+1)} \frac{w(z)}{w(z-1)} \left(\frac{w(z+1)}{w(z)}\right)^2 = \frac{h(z+1)}{h(z)}. \tag{78}$$

Set

$$g(z) = \frac{w(z+1)}{w(z)}. \tag{79}$$

Then (78) can be written as $g(z+1)g(z-1)g^2(z) = h(z+1)/h(z)$. Set $P(z, g) := g(z+1)g(z-1)g^2(z) - (h(z+1)/h(z)) = 0$. Since $h(z)$ is a nonconstant rational function, $h(z)$ cannot be a periodic function. Thus, $P(z, 1) = 1 - (h(z+1)/h(z)) \not\equiv 0$. Since $P(z, 1) \not\equiv 0$, by Lemmas 12 and 16, we have

$$m\left(r, \frac{1}{g(z)-1}\right) = S(r, g(z)) = S\left(r, \frac{w(z+1)}{w(z)}\right) \le S(r, w(z)), \tag{80}$$

thus,

$$m\left(r, \frac{1}{\Delta w(z)/w(z)}\right) = m\left(r, \frac{1}{(w(z+1)/w(z)) - 1}\right) = m\left(r, \frac{1}{g(z)-1}\right) = S(r, w(z)). \tag{81}$$

By this and (75), we have

$$N\left(r, \frac{1}{\Delta w/w}\right) = T\left(r, \frac{\Delta w}{w}\right) - m\left(r, \frac{1}{\Delta w/w}\right) = T\left(r, \frac{\Delta w}{w}\right) + S(r, w) = 2T(r, w) + S(r, w). \tag{82}$$

Then $\lambda(\Delta w/w) = \sigma(w)$. □

We see from (76) that

$$N(r, \Delta w) = N\left(r, \frac{\Delta w}{w} w\right) \ge N\left(r, \frac{\Delta w}{w}\right) - N\left(r, \frac{1}{w}\right) \ge N\left(r, \frac{\Delta w}{w}\right) - T(r, w) = T(r, w) + S(r, w). \tag{83}$$

We deduce from (82) that

$$N\left(r, \frac{1}{\Delta w}\right) = N\left(r, \frac{1}{\Delta w/w} \frac{1}{w}\right) \ge N\left(r, \frac{1}{\Delta w/w}\right) - N(r, w) \ge N\left(r, \frac{1}{\Delta w/w}\right) - T(r, w) = T(r, w) + S(r, w). \tag{84}$$

The last two inequalities show $\lambda(1/\Delta w) \ge \sigma(w)$ and $\lambda(\Delta w) \ge \sigma(w)$, respectively. Thus, $\lambda(\Delta w) = \lambda(1/\Delta w) = \sigma(w)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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