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Research Article

On New p-Valent Meromorphic Function Involving Certain Differential and Integral Operators

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We define new subclasses of meromorphic p-valent functions by using certain differential operator. Combining the differential operator and certain integral operator, we introduce a general p-valent meromorphic function. Then we prove the sufficient conditions for the function in order to be in the new subclasses.

1. Introduction

Let Σ_p denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1}$$

which are analytic and p-valent in the punctured unit disc:

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} - \{0\}. \tag{2}$$

A function $f \in \Sigma_p$ is said to be in the class $\Sigma_p^*(\delta)$ of meromorphic p-valent starlike of order δ $(0 \le \delta < p)$ if it satisfies the following inequality:

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \delta. \tag{3}$$

For $f \in \Sigma_p$, Saif and Kılıçman [1] introduced the linear operator $\mathscr{D}_\lambda^k$, as follows:

$$\begin{gathered} \mathscr{D}_\lambda f(z) = (1 + p\lambda) f(z) + \lambda z f'(z), \quad \lambda \ge 0, \\ \mathscr{D}_\lambda^0 f(z) = f(z), \\ \mathscr{D}_\lambda^1 f(z) = \mathscr{D}_\lambda f(z), \\ \mathscr{D}_\lambda^2 f(z) = \mathscr{D}_\lambda\left(\mathscr{D}_\lambda^1 f(z)\right), \end{gathered} \tag{4}$$

and in general, for $k = 0, 1, 2, \ldots$, we can write

$$\begin{gathered} \mathscr{D}_\lambda^k f(z) = \frac{1}{z^p} + \sum_{n=p+1}^{\infty} (1 + p\lambda + n\lambda)^k a_n z^n, \\ (k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ p \in \mathbb{N}). \end{gathered} \tag{5}$$

It is easy to see that, for $f \in \Sigma_p$, we have

$$\begin{gathered} \lambda z\left(\mathscr{D}_\lambda^k f(z)\right)' = \mathscr{D}_\lambda^{k+1} f(z) - (1 + p\lambda)\mathscr{D}_\lambda^k f(z), \\ (k \in \mathbb{N}_0,\ p \in \mathbb{N}). \end{gathered} \tag{6}$$

Meromorphically multivalent functions have been extensively studied by several authors; see, for example, Uralegaddi and Somanatha [2, 3], Liu and Srivastava [4, 5], Mogra [6, 7], Srivastava et al. [8], Aouf et al. [9, 10], Joshi and Srivastava [11], Owa et al. [12], and Kulkarni et al. [13].

Now, for $f \in \Sigma_p$, we define the following new subclasses.

Definition 1. Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p S_k(\delta, b, \lambda)$ if, and only if, f satisfies

$$\Re\left\{p - \frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{k+1} f(z)}{\mathscr{D}_\lambda^k f(z)} - 1\right)\right\} > \delta, \tag{7}$$

where $\delta \in [0, p)$, $b \in \mathbb{C} \setminus \{0\}$, $\lambda \ge 0$, $k \in \mathbb{N}_0$.

From (6), one can see that (7) is equivalent to

$$\Re\left\{p-\frac{\lambda}{b}\left(\frac{z\left(\mathscr{D}_{\lambda}^{k}f(z)\right)'}{\mathscr{D}_{\lambda}^{k}f(z)}+p\right)\right\}>\delta. \tag{8}$$

Remark 2. In Definition 1, if we set

(i) $k=0$ and $p=\lambda=1$, then we have [14, Definition 1.1];

(ii) $k=0$ and $p=\lambda=b=1$, then we have $\Sigma_p^{\star}(\delta)$, the class of meromorphic p-valent starlike of order δ;

(iii) $k=1$ and $p=\lambda=1$, then we have [14, Definition 1.7].

Definition 3. Let a function $f\in\Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p US_k(\alpha,\delta,b,\lambda)$ if, and only if, f satisfies

$$\begin{aligned}&\Re\left\{p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{k+1}f(z)}{\mathscr{D}_{\lambda}^{k}f(z)}-1\right)\right\}\\&\quad>\alpha\left|\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{k+1}f(z)}{\mathscr{D}_{\lambda}^{k}f(z)}-1\right)\right|+\delta,\end{aligned} \tag{9}$$

where $\alpha\geq 0$, $\delta\in[-1,p)$, $b\in\mathbb{C}\setminus\{0\}$, $\lambda\geq 0$, $k\in\mathbb{N}_0$.

Inequality (9) is equivalent to

$$\begin{aligned}&\Re\left\{p-\frac{\lambda}{b}\left(\frac{z\left(\mathscr{D}_{\lambda}^{k}f(z)\right)'}{\mathscr{D}_{\lambda}^{k}f(z)}+p\right)\right\}\\&\quad>\alpha\left|\frac{\lambda}{b}\left(\frac{z\left(\mathscr{D}_{\lambda}^{k}f(z)\right)'}{\mathscr{D}_{\lambda}^{k}f(z)}+p\right)\right|+\delta.\end{aligned} \tag{10}$$

Remark 4. In Definition 3, if we set

(i) $k=0$ and $p=\lambda=1$, then we have [14, Definition 1.3];

(ii) for $k=1$ and $p=\lambda=1$, then we have [14, Definition 1.8].

Definition 5. Let a function $f\in\Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Sigma_p SH_k(\alpha,b,\lambda)$, if, and only if, f satisfies

$$\begin{aligned}&\left|p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{k+1}f(z)}{\mathscr{D}_{\lambda}^{k}f(z)}-1\right)-2\alpha\left(\sqrt{2}-1\right)\right|\\&\quad<\sqrt{2}\Re\left\{p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{k+1}f(z)}{\mathscr{D}_{\lambda}^{k}f(z)}-1\right)\right\}+2\alpha\left(\sqrt{2}-1\right),\end{aligned} \tag{11}$$

where $\alpha>0$, $b\in\mathbb{C}\setminus\{0\}$, $\lambda\geq 0$, $k\in\mathbb{N}_0$.

Inequality (11) is equivalent to

$$\begin{aligned}&\left|p-\frac{\lambda}{b}\left(\frac{z\left(\mathscr{D}_{\lambda}^{k}f(z)\right)'}{\mathscr{D}_{\lambda}^{k}f(z)}+p\right)-2\alpha\left(\sqrt{2}-1\right)\right|\\&\quad<\sqrt{2}\Re\left\{p-\frac{\lambda}{b}\left(\frac{z\left(\mathscr{D}_{\lambda}^{k}f(z)\right)'}{\mathscr{D}_{\lambda}^{k}f(z)}+p\right)\right\}+2\alpha\left(\sqrt{2}-1\right).\end{aligned} \tag{12}$$

Remark 6. In Definition 5, if we set

(i) $k=0$ and $p=\lambda=1$, then we have [14, Definition 1.5];

(ii) for $k=1$ and $p=\lambda=1$, then we have [14, Definition 1.9].

Recently, Mohammed and Darus [15] introduced the following p-valent meromorphic function:

$$G(z)=z\mathscr{F}'_{p,\gamma_1,\ldots,\gamma_n}(z)+(p+1)\mathscr{F}_{p,\gamma_1,\ldots,\gamma_n}(z), \tag{13}$$

where $\mathscr{F}_{p,\gamma_1,\ldots,\gamma_n}$ is the integral operator introduced and studied by the authors [15, 16] and defined by

$$\mathscr{F}_{p,\gamma_1,\ldots,\gamma_n}(z)=\frac{1}{z^{p+1}}\int_0^z\left(u^p f_1(u)\right)^{\gamma_1}\cdots\left(u^p f_n(u)\right)^{\gamma_n}du, \tag{14}$$

where

$$n,p\in\mathbb{N},\quad j\in\{1,2,3,\ldots,n\},\quad \gamma_j>0. \tag{15}$$

For $p=1$ we obtain [17]. It is clear that

$$G(z)=\frac{1}{z^p}\left(z^p f_1(z)\right)^{\gamma_1}\cdots\left(z^p f_n(z)\right)^{\gamma_n}. \tag{16}$$

By using the differential operator given by (4), we introduce the following p-valent meromorphic function.

Definition 7. Let $k\in\mathbb{N}_0$, $l=(l_1,\ldots,l_n)\in\mathbb{N}_0^n$ and $\gamma_j>0$, $1\leq j\leq n$. One defines the p-valent meromorphic function $I_{k,n,l,\gamma}:\Sigma_p^n\to\Sigma_p$,

$$I_{k,n,l,\gamma}(f_1,\ldots,f_n)=\Phi, \tag{17}$$

$$\mathscr{D}_{\lambda}^{k}\Phi(z)=\frac{1}{z^p}\left[\left(z^p\mathscr{D}_{\lambda}^{l_1}f_1(z)\right)^{\gamma_1}\cdots\left(z^p\mathscr{D}_{\lambda}^{l_n}f_n(z)\right)^{\gamma_n}\right], \tag{18}$$

where $f_1,\ldots,f_n\in\Sigma_p$, and $\mathscr{D}_{\lambda}$ is the differential operator given by (4).

Remark 8. If we set $\lambda=1$, $k=0$, and $l_1=\cdots=l_n=0$, then we have the p-valent meromorphic function given by (13).

2. Main Results

To prove our main results, we need the following lemma.

Lemma 9. *For the p-valent meromorphic function $I_{k,n,l,\gamma}(f_1,\ldots,f_n)=\Phi$ given by* (18), *one has*

$$-\frac{\lambda z\left(\mathscr{D}_{\lambda}^{k}\Phi(z)\right)'}{\mathscr{D}_{\lambda}^{k}\Phi(z)}=-\sum_{j=1}^{n}\gamma_j\frac{\mathscr{D}_{\lambda}^{l_j+1}f_j(z)}{\mathscr{D}_{\lambda}^{l_j}f_j(z)}+p\lambda+\sum_{j=1}^{n}\gamma_j. \tag{19}$$

Proof. From (18), we have

$$z^p\mathscr{D}_{\lambda}^{k}\Phi(z)=\left[\left(z^p\mathscr{D}_{\lambda}^{l_1}f_1(z)\right)^{\gamma_1}\cdots\left(z^p\mathscr{D}_{\lambda}^{l_n}f_n(z)\right)^{\gamma_n}\right]. \tag{20}$$

Differentiating (20) logarithmically and then by simple computation, we get

$$\frac{z\left(\mathscr{D}_\lambda^k\Phi(z)\right)'}{\mathscr{D}_\lambda^k\Phi(z)}=\sum_{j=1}^n\gamma_j\left(\frac{z\left(\mathscr{D}_\lambda^{l_j}f_j(z)\right)'+p\mathscr{D}_\lambda^{l_j}f_j(z)}{\mathscr{D}_\lambda^{l_j}f_j(z)}\right)-p. \tag{21}$$

From (6), we obtain

$$\left(\mathscr{D}_\lambda^{l_j}f_j(z)\right)'=\frac{\mathscr{D}_\lambda^{l_j+1}f_j(z)-(1+p\lambda)\mathscr{D}_\lambda^{l_j}f_j(z)}{\lambda z}. \tag{22}$$

Then using (22) on the right-hand side of (21), one gets

$$\frac{z\left(\mathscr{D}_\lambda^k\Phi(z)\right)'}{\mathscr{D}_\lambda^k\Phi(z)}=\sum_{j=1}^n\gamma_j\left(\frac{\mathscr{D}_\lambda^{l_j+1}f_j(z)}{\lambda\mathscr{D}_\lambda^{l_j}f_j(z)}-\frac{1}{\lambda}\right)-p. \tag{23}$$

Multiplying (23) by λ yields that

$$\frac{\lambda z\left(\mathscr{D}_\lambda^k\Phi(z)\right)'}{\mathscr{D}_\lambda^k\Phi(z)}=\sum_{j=1}^n\gamma_j\left(\frac{\mathscr{D}_\lambda^{l_j+1}f_j(z)}{\mathscr{D}_\lambda^{l_j}f_j(z)}-1\right)-p\lambda, \tag{24}$$

or, equivalently, we can write that

$$-\frac{\lambda z\left(\mathscr{D}_\lambda^k\Phi(z)\right)'}{\mathscr{D}_\lambda^k\Phi(z)}=-\sum_{j=1}^n\gamma_j\frac{\mathscr{D}_\lambda^{l_j+1}f_j(z)}{\mathscr{D}_\lambda^{l_j}f_i(z)}+p\lambda+\sum_{j=1}^n\gamma_j, \tag{25}$$

which is the desired result. □

Our first theorem is as follows.

Theorem 10. *Let* $\alpha_j\geq 0,\ \delta_j\in[-1,p),\ \alpha_j+\delta_j\geq 0,\ (1\leq j\leq n)$ *and* $b\in\mathbb{C}\setminus\{0\},\ \lambda\geq 0$. *Suppose that*

$$\sum_{j=1}^n\gamma_j\left(\frac{p-\delta_j}{\alpha_j+1}\right)\leq p. \tag{26}$$

If $f_j\in\Sigma_pUS_{l_j}(\alpha_j,\delta_j,b,\lambda)$ $(1\leq j\leq n)$, *then the function* $\mathscr{D}_\lambda^k\Phi(z)$ *defined by* (18) *is in the class* $\Sigma_pS_k(\mu,b,\lambda)$, *where*

$$\mu=p-\sum_{j=1}^n\gamma_j\left(\frac{p-\delta_j}{\alpha_j+1}\right). \tag{27}$$

Proof. Since $f_j\in\Sigma_pUS_{l_j}(\alpha_j,\delta_j,b,\lambda)$ $(1\leq j\leq n)$, by (9), we have

$$\Re\left\{p-\frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1}f_j(z)}{\mathscr{D}_\lambda^{l_j}f_j(z)}-1\right)\right\}>\frac{p\alpha_j+\delta_j}{1+\alpha_j}. \tag{28}$$

By (19), we get

$$-\frac{\lambda z\left(\mathscr{D}_\lambda^k\Phi(z)\right)'}{\mathscr{D}_\lambda^k\Phi(z)}-p\lambda=-\sum_{j=1}^n\gamma_j\left(\frac{\mathscr{D}_\lambda^{l_j+1}f_j(z)}{\mathscr{D}_\lambda^{l_j}f_j(z)}-1\right). \tag{29}$$

This is equivalent to

$$\begin{aligned}p-\frac{\lambda}{b}&\left(\frac{z\left(\mathscr{D}_\lambda^k\Phi(z)\right)'}{\mathscr{D}_\lambda^k\Phi(z)}+p\right)\\&=p-\frac{1}{b}\sum_{j=1}^n\gamma_j\left(\frac{\mathscr{D}_\lambda^{l_j+1}f_j(z)}{\mathscr{D}_\lambda^{l_j}f_j(z)}-1\right)\\&=\sum_{j=1}^n\gamma_j\left[p-\frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1}f_j(z)}{\mathscr{D}_\lambda^{l_j}f_j(z)}-1\right)\right]+p-p\sum_{j=1}^n\gamma_j.\end{aligned} \tag{30}$$

From (28) together with (30), we can get

$$\begin{aligned}\Re&\left\{p-\frac{\lambda}{b}\left(\frac{z\left(\mathscr{D}_\lambda^k\Phi(z)\right)'}{\mathscr{D}_\lambda^k\Phi(z)}+p\right)\right\}\\&=\sum_{j=1}^n\gamma_j\Re\left[p-\frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1}f_j(z)}{\mathscr{D}_\lambda^{l_j}f_j(z)}-1\right)\right]\\&\quad+p-p\sum_{j=1}^n\gamma_j\\&>\sum_{j=1}^n\gamma_j\left(\frac{p\alpha_j+\delta_j}{1+\alpha_j}\right)-p\sum_{j=1}^n\gamma_j+p\\&=p-\sum_{j=1}^n\gamma_j\left(\frac{p-\delta_j}{1+\alpha_j}\right).\end{aligned} \tag{31}$$

Hence, we obtain $\mathscr{D}_\lambda^k\Phi(z)\in\Sigma_pS_k(\mu,b,\lambda)$, where $\mu=p-\sum_{j=1}^n\gamma_j((p-\delta_j)/(\alpha_j+1))$. □

Corollary 11. *Let* $\alpha_j\geq 0,\ \delta_j\in[-1,p),\ \alpha_j+\delta_j\geq 0,\ (1\leq j\leq n)$, *and* $b\in\mathbb{C}\setminus\{0\},\ \lambda\geq 0$. *Suppose that*

$$\sum_{j=1}^n\gamma_j\left(\frac{p-\delta_j}{\alpha_j+1}\right)\leq p. \tag{32}$$

If $f_j\in\Sigma_pUS_{l_j}(\alpha_j,\delta_j,b,1)$ $(1\leq j\leq n)$, *then the function* $\mathscr{D}_\lambda^k\Phi(z)$, *defined by* (18), *is in the class* $\Sigma_pS_{k+1}(\mu,b,1)$, *where* μ *is defined as in* (27).

Proof. In Theorem 10, we consider $\lambda=1$. □

By Corollary 11, we easily get the following.

Corollary 12. *Let* $\alpha_j\geq 0,\ \delta_j\in[-1,p),\ \alpha_j+\delta_j\geq 0,\ (1\leq j\leq n)$, *and* $b\in\mathbb{C}\setminus\{0\},\ \lambda\geq 0$. *Suppose that*

$$\sum_{j=1}^n\gamma_j\left(\frac{p-\delta_j}{\alpha_j+1}\right)\leq p. \tag{33}$$

If $f_j\in\Sigma_pUS_{l_j}(\alpha_j,\delta_j,b,1)$ $(1\leq j\leq n)$, *then the function* $\mathscr{D}_\lambda^k\Phi(z)$, *defined by* (18), *is in the class* $\Sigma_pS_{k+1}(0,b,1)$.

Now, we prove a sufficient condition for the function $\mathscr{D}_\lambda^k \Phi(z)$ defined by (18) to belong to the class $\Sigma_p US_k(\alpha, \delta, b, \lambda)$.

Theorem 13. *Let $\alpha \geq 0$, $\delta \in [-1, p)$, $\alpha + \delta \geq 0$ $(1 \leq j \leq n)$, and $b \in \mathbb{C} \setminus \{0\}$, $\lambda \geq 0$. Suppose that*

$$\sum_{j=1}^{n} \gamma_j \leq 1. \tag{34}$$

If $f_j \in \Sigma_p US_{l_j}(\alpha, \delta, b, \lambda)$ $(1 \leq j \leq n)$, then the function $\mathscr{D}_\lambda^k \Phi(z)$ defined by (18) is in the class $\Sigma_p US_k(\alpha, \delta, b, \lambda)$.

Proof. Since $f_j \in \Sigma_p US_{l_j}(\alpha, \delta, b, \lambda)$ $(1 \leq j \leq n)$, by (9), we have

$$\begin{aligned} &\Re\left\{p - \frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1} f(z)}{\mathscr{D}_\lambda^{l_j} f(z)} - 1\right)\right\} \\ &\quad > \alpha\left|\frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1} f_j(z)}{\mathscr{D}_\lambda^{l_j} f_j(z)} - 1\right)\right| + \delta. \end{aligned} \tag{35}$$

On the other hand, from (19), we obtain the following:

$$\begin{aligned} &p - \frac{\lambda}{b}\left(\frac{z\left(\mathscr{D}_\lambda^k \Phi(z)\right)'}{\mathscr{D}_\lambda^k \Phi(z)} + p\right) \\ &\quad = \sum_{j=1}^{n} \gamma_j\left[p - \frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1} f_j(z)}{\mathscr{D}_\lambda^{l_j} f_j(z)} - 1\right)\right] + p - p\sum_{j=1}^{n} \gamma_j. \end{aligned} \tag{36}$$

Considering (10) with the above equality, we find

$$\begin{aligned} &\Re\left\{p - \frac{\lambda}{b}\left(\frac{z\left(\mathscr{D}_\lambda^k f(z)\right)'}{\mathscr{D}_\lambda^k f(z)} + p\right)\right\} \\ &\quad - \alpha\left|\frac{\lambda}{b}\left(\frac{z\left(\mathscr{D}_\lambda^k f(z)\right)'}{\mathscr{D}_\lambda^k f(z)} + p\right)\right| - \delta \\ &= p - p\sum_{j=1}^{n} \gamma_j + \sum_{j=1}^{n} \gamma_j \Re\left[p - \frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1} f_j(z)}{\mathscr{D}_\lambda^{l_j} f_j(z)} - 1\right)\right] \\ &\quad - \alpha\left|\sum_{j=1}^{n} \gamma_j \frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1} f_j(z)}{\mathscr{D}_\lambda^{l_j} f_j(z)} - 1\right)\right| - \delta \\ &\geq p - p\sum_{j=1}^{n} \gamma_j + \sum_{j=1}^{n} \gamma_j \Re\left[p - \frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1} f_j(z)}{\mathscr{D}_\lambda^{l_j} f_j(z)} - 1\right)\right] \\ &\quad - \alpha\sum_{j=1}^{n} \gamma_j\left|\frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1} f_j(z)}{\mathscr{D}_\lambda^{l_j} f_j(z)} - 1\right)\right| - \delta \\ &> p - p\sum_{j=1}^{n} \gamma_j + \sum_{j=1}^{n} \gamma_j\left[\alpha\left|\frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1} f_j(z)}{\mathscr{D}_\lambda^{l_j} f_j(z)} - 1\right)\right| + \delta\right] \\ &\quad - \alpha\sum_{j=1}^{n} \gamma_j\left|\frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1} f_j(z)}{\mathscr{D}_\lambda^{l_j} f_j(z)} - 1\right)\right| - \delta \\ &= (p - \delta)\left(1 - \sum_{j=1}^{n} \gamma_j\right) \geq 0. \end{aligned} \tag{37}$$

The proof is complete. □

Corollary 14. *Let $\alpha \geq 0$, $\delta \in [-1, p)$, $\alpha + \delta \geq 0$ $(1 \leq j \leq n)$, and $b \in \mathbb{C} \setminus \{0\}$. Suppose that*

$$\sum_{j=1}^{n} \gamma_j \leq 1. \tag{38}$$

If $f_j \in \Sigma_p US_{l_j}(\alpha, \delta, b, 1)$ $(1 \leq j \leq n)$, then the function $\mathscr{D}_\lambda^k \Phi(z)$ defined by (18) is in the class $\Sigma_p US_{k+1}(\alpha, \delta, b, 1)$.

Proof. In Theorem 13, we consider that $\lambda = 1$ □

Next, for the function $\mathscr{D}_\lambda^k \Phi$ defined by (18) to belong to the class $\Sigma_p SH_k(\alpha, b, \lambda)$, we have the following result.

Theorem 15. *Let $\alpha \geq 0$, $\lambda \geq 0$, and $b \in \mathbb{C} \setminus \{0\}$. Suppose that*

$$\sum_{j=1}^{n} \gamma_j \leq 1. \tag{39}$$

If $f_j \in \Sigma_p SH_{l_j}(\alpha, b, \lambda)$, then the function $\mathscr{D}_\lambda^k \Phi(z) \in \Sigma_p SH_k(\alpha, b, \lambda)$.

Proof. Since $f_j \in \Sigma_p SH_{l_j}(\alpha, b, \lambda)$, by (11), we have

$$\begin{aligned} &\sqrt{2}\Re\left\{p - \frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1} f_j(z)}{\mathscr{D}_\lambda^{l_j} f_j(z)} - 1\right)\right\} + 2\alpha\left(\sqrt{2} - 1\right) \\ &\quad - \left|p - \frac{1}{b}\left(\frac{\mathscr{D}_\lambda^{l_j+1} f_j(z)}{\mathscr{D}_\lambda^{l_j} f_j(z)} - 1\right) - 2\alpha\left(\sqrt{2} - 1\right)\right| > 0. \end{aligned} \tag{40}$$

Combining (12), (30), and the above inequality, we obtain

$$
\begin{aligned}
&\sqrt{2}\mathfrak{R}\left\{p-\frac{\lambda}{b}\left(\frac{z\left(\mathscr{D}_{\lambda}^{k}f(z)\right)'}{\mathscr{D}_{\lambda}^{k}f(z)}+p\right)\right\}+2\alpha\left(\sqrt{2}-1\right)\\
&\quad-\left|p-\frac{\lambda}{b}\left(\frac{z\left(\mathscr{D}_{\lambda}^{k}f(z)\right)'}{\mathscr{D}_{\lambda}^{k}f(z)}+p\right)-2\alpha\left(\sqrt{2}-1\right)\right|\\
&=\sqrt{2}\mathfrak{R}\left\{\sum_{j=1}^{n}\gamma_j\left[p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{l_j+1}f_j(z)}{\mathscr{D}_{\lambda}^{l_j}f_j(z)}-1\right)\right]\right.\\
&\qquad\left.+p-p\sum_{j=1}^{n}\gamma_j\right\}+2\alpha\left(\sqrt{2}-1\right)\\
&\quad-\left|\sum_{j=1}^{n}\gamma_j\left[p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{l_j+1}f_j(z)}{\mathscr{D}_{\lambda}^{l_j}f_j(z)}-1\right)\right]\right.\\
&\qquad\left.+p-p\sum_{j=1}^{n}\gamma_j-2\alpha\left(\sqrt{2}-1\right)\right|\\
&=\sum_{j=1}^{n}\gamma_j\left\{\sqrt{2}\mathfrak{R}\left(p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{l_j+1}f_j(z)}{\mathscr{D}_{\lambda}^{l_j}f_j(z)}-1\right)\right)\right.\\
&\qquad\left.+2\alpha\left(\sqrt{2}-1\right)\right\}-2\alpha\left(\sqrt{2}-1\right)\sum_{j=1}^{n}\gamma_j\\
&\quad+\sqrt{2}\left(p-p\sum_{j=1}^{n}\gamma_j\right)+2\alpha\left(\sqrt{2}-1\right)\\
&\quad-\left|\sum_{j=1}^{n}\gamma_j\left\{\left(p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{l_j+1}f_j(z)}{\mathscr{D}_{\lambda}^{l_j}f_j(z)}-1\right)\right)\right.\right.\\
&\qquad\left.-2\alpha\left(\sqrt{2}-1\right)\right\}\\
&\qquad\left.+2\alpha\left(\sqrt{2}-1\right)\sum_{j=1}^{n}\gamma_j-2\alpha\left(\sqrt{2}-1\right)+p-p\sum_{j=1}^{n}\gamma_j\right|\\
&=\sum_{j=1}^{n}\gamma_j\left\{\sqrt{2}\mathfrak{R}\left(p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{l_j+1}f_j(z)}{\mathscr{D}_{\lambda}^{l_j}f_j(z)}-1\right)\right)\right.\\
&\qquad\left.+2\alpha\left(\sqrt{2}-1\right)\right\}\\
&\quad+\left[\sqrt{2}p+2\alpha\left(\sqrt{2}-1\right)\right]\left(1-\sum_{j=1}^{n}\gamma_j\right)\\
&\quad-\left|\sum_{j=1}^{n}\gamma_j\left\{\left(p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{l_j+1}f_j(z)}{\mathscr{D}_{\lambda}^{l_j}f_j(z)}-1\right)\right)\right.\right.\\
&\qquad\left.-2\alpha\left(\sqrt{2}-1\right)\right\}\\
&\qquad\left.+\left[p-2\alpha\left(\sqrt{2}-1\right)\right]\left(1-\sum_{j=1}^{n}\gamma_j\right)\right|,
\end{aligned}
\tag{41}
$$

which is

$$
\begin{aligned}
&\geq\sum_{j=1}^{n}\gamma_j\left\{\sqrt{2}\mathfrak{R}\left(p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{l_j+1}f_j(z)}{\mathscr{D}_{\lambda}^{l_j}f_j(z)}-1\right)\right)\right.\\
&\qquad\left.+2\alpha\left(\sqrt{2}-1\right)\right\}\\
&\quad+\left[\sqrt{2}p+2\alpha\left(\sqrt{2}-1\right)\right]\left(1-\sum_{j=1}^{n}\gamma_j\right)\\
&\quad-\sum_{j=1}^{n}\gamma_j\left|\left\{\left(p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{l_j+1}f_j(z)}{\mathscr{D}_{\lambda}^{l_j}f_j(z)}-1\right)\right)\right.\right.\\
&\qquad\left.\left.-2\alpha\left(\sqrt{2}-1\right)\right\}\right|\\
&\quad-\left|p-2\alpha\left(\sqrt{2}-1\right)\right|\left(1-\sum_{j=1}^{n}\gamma_j\right)\\
&=\sum_{j=1}^{n}\gamma_j\left\{\sqrt{2}\mathfrak{R}\left[p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{l_j+1}f_j(z)}{\mathscr{D}_{\lambda}^{l_j}f_j(z)}-1\right)\right]\right.\\
&\qquad+2\alpha\left(\sqrt{2}-1\right)\\
&\qquad\left.-\left|p-\frac{1}{b}\left(\frac{\mathscr{D}_{\lambda}^{l_j+1}f_j(z)}{\mathscr{D}_{\lambda}^{l_j}f_j(z)}-1\right)-2\alpha\left(\sqrt{2}-1\right)\right|\right\}\\
&\quad+\left[\sqrt{2}p+2\alpha\left(\sqrt{2}-1\right)-\left|p-2\alpha\left(\sqrt{2}-1\right)\right|\right]\\
&\quad\times\left(1-\sum_{j=1}^{n}\gamma_j\right)\\
&>\left[\sqrt{2}p+2\alpha\left(\sqrt{2}-1\right)-\left|p-2\alpha\left(\sqrt{2}-1\right)\right|\right]\\
&\quad\times\left(1-\sum_{j=1}^{n}\gamma_j\right),
\end{aligned}
\tag{42}
$$

and finally

$$>\left(1-\sum_{j=1}^{n}\gamma_j\right)\min\left\{\left(\sqrt{2}-1\right)(p+4\alpha),p\left(\sqrt{2}+1\right)\right\}\geq 0. \tag{43}$$

Hence, by (12), we have $\mathcal{D}_\lambda^k\Phi(z)\in\Sigma_pSH_k(\alpha,b,\lambda)$. □

Corollary 16. *Let* $\alpha\geq 0$ *and* $b\in\mathbb{C}\setminus\{0\}$. *Suppose that*

$$\sum_{j=1}^{n}\gamma_j\leq 1. \tag{44}$$

If $f_j\in\Sigma_pSH_{l_j}(\alpha,b,1)$, *then the function* $\mathcal{D}_\lambda^k\Phi(z)$ *defined by* (18) *is in the class* $\Sigma_pSH_{k+1}(\alpha,b,1)$.

Proof. In Theorem 15, we consider $\lambda=1$. □

Finally, we end this paper by the following theorem and its consequence.

Theorem 17. *Let* $\alpha\geq 0$, $\lambda\geq 0$, *and* $b\in\mathbb{C}\setminus\{0\}$. *Suppose that*

$$\left(p+\sqrt{2}\alpha\left(\sqrt{2}-1\right)\right)\sum_{j=1}^{n}\gamma_j<p. \tag{45}$$

If $f_j\in\Sigma_pSH_{l_j}(\alpha,b,\lambda)$, *then the function* $\mathcal{D}_\lambda^k\Phi(z)$ *defined by* (18) *is in the class* $\Sigma_pSH_k(0,b,\lambda)$.

Proof. Since $f_j\in\Sigma_pSH_{l_j}(\alpha,b,\lambda)$, by (11), we have

$$\begin{aligned}&\sqrt{2}\Re\left\{p-\frac{1}{b}\left(\frac{\mathcal{D}_\lambda^{l_j+1}f_j(z)}{\mathcal{D}_\lambda^{l_j}f_j(z)}-1\right)\right\}+2\alpha\left(\sqrt{2}-1\right)\\&\quad>\left|p-\frac{1}{b}\left(\frac{\mathcal{D}_\lambda^{l_j+1}f_j(z)}{\mathcal{D}_\lambda^{l_j}f_j(z)}-1\right)-2\alpha\left(\sqrt{2}-1\right)\right|.\end{aligned} \tag{46}$$

Considering this inequality and (30), we obtain

$$\begin{aligned}&\sqrt{2}\Re\left\{p-\frac{\lambda}{b}\left(\frac{z\left(\mathcal{D}_\lambda^kf(z)\right)'}{\mathcal{D}_\lambda^kf(z)}+p\right)\right\}\\&\quad=\sqrt{2}\Re\left\{\sum_{j=1}^{n}\gamma_j\left[p-\frac{1}{b}\left(\frac{\mathcal{D}_\lambda^{l_j+1}f_j(z)}{\mathcal{D}_\lambda^{l_j}f_j(z)}-1\right)\right]+p-p\sum_{j=1}^{n}\gamma_j\right\}\\&\quad=\sum_{j=1}^{n}\gamma_j\left\{\sqrt{2}\Re\left[p-\frac{1}{b}\left(\frac{\mathcal{D}_\lambda^{l_j+1}f_j(z)}{\mathcal{D}_\lambda^{l_j}f_j(z)}-1\right)\right]+2\alpha\left(\sqrt{2}-1\right)\right\}\\&\qquad+\sqrt{2}p\left(1-\sum_{j=1}^{n}\gamma_j\right)-2\alpha\left(\sqrt{2}-1\right)\sum_{j=1}^{n}\gamma_j\\&\quad>\sqrt{2}p\left(1-\sum_{j=1}^{n}\gamma_j\right)-2\alpha\left(\sqrt{2}-1\right)\sum_{j=1}^{n}\gamma_j\\&\quad=\sqrt{2}\left(p-\left(p+\sqrt{2}\alpha\left(\sqrt{2}-1\right)\right)\sum_{j=1}^{n}\gamma_j\right)>0.\end{aligned} \tag{47}$$

Hence, we have $\mathcal{D}_\lambda^k\Phi(z)\in\Sigma_pSH_k(0,b,\lambda)$. □

Corollary 18. *Let* $\alpha\geq 0$ *and* $b\in\mathbb{C}\setminus\{0\}$. *Suppose that*

$$\left(p+\sqrt{2}\alpha\left(\sqrt{2}-1\right)\right)\sum_{j=1}^{n}\gamma_j<p. \tag{48}$$

If $f_j\in\Sigma_pSH_{l_j}(\alpha,b,1)$, *then the function* $\mathcal{D}_\lambda^k\Phi(z)$ *defined by* (18) *is in the class* $\Sigma_pSH_{k+1}(0,b,1)$.

Proof. In Theorem 17, we consider that $\lambda=1$. □

For other work that we can look at regarding differential and integral operators, see [14, 18–24].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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