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Research Article

An Adaptive Nonconforming Finite Element Algorithm for Laplace Eigenvalue Problem

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We establish Crouzeix-Raviart element adaptive algorithm based on Rayleigh quotient iteration and give its a priori/a posteriori error estimates. Our algorithm is performed under the package of Chen, and satisfactory numerical results are obtained.

1. Introduction

A posteriori error estimates and adaptive methods of finite element approximation for eigenvalue problems are topics attracting more attention from mathematical and physical fields; see, for example, [1–8]. Basically, there are the following three ways of combining adaptivity and eigenvalue problems in which the a posteriori error estimators are more or less the same but different in the problem solved in each iteration: (1) solving the original eigenvalue problem $a(u,v) = \lambda b(u,v)$ (see Algorithm 10). The convergence and optimality of this adaptive procedure were proved in [2]; (2) inverse iteration type (with or without correction). The convergence has been studied in [1, 6, 7]; (3) Shifted-inverse iteration type (see [8–11]).

The triangular Crouzeix-Raviart element (C-R element) was first introduced by Crouzeix and Raviart [12] in 1973 to solve the stationary Stokes equation. After that, many scholars developed and applied it to eigenvalue problems, for instance, [13–16] discussed a posteriori error estimates and the adaptive methods of the C-R element. C-R element has important properties; for example, Armentano and Durán [17] discovered and proved that the C-R element eigenvalues approximate the exact ones of the Laplace operator from below, which is a very important property in engineering and mechanics computing.

Based on the above work, this paper further discusses the third kind of adaptive methods of the C-R finite element method for eigenvalue problems and obtains the following new results:

(1) we establish a multiscale discretization scheme of the C-R element based on Rayleigh quotient iteration and prove its convergence and a priori error estimates;

(2) we give residual type a posteriori error estimator for our adaptive algorithm, as well as its reliability and efficiency;

(3) we establish an adaptive algorithm (Algorithm 11), which is performed under the package of Chen (see [18]), and satisfactory numerical results are obtained.

As for the fundamental theory of finite elements and spectral approximation, we refer to [19–22].

Throughout this paper, C denotes a positive constant independent of mesh parameter, which may not be the same constant in different places. For simplicity, we use the notation $a \lesssim b$ to mean that $a \leq Cb$, $a = O(b)$ and to mean that $a \lessapprox b$ and $b \lesssim a$.

2. Preliminaries

Consider Laplace eigenvalue problem

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset R^2$ is a polygonal domain with the maximum interior angle ω.

We denote the real order Sobolev spaces with norm $\|\cdot\|_t$ by $H^t(\Omega)$, $H_0^1(\Omega) = \{v \in H^1(\Omega), v|_{\partial\Omega} = 0\}$. Let $b(\cdot,\cdot)$ and $\|\cdot\|_{0,\Omega}$ be the inner product and the norm in the space $L^2(\Omega)$, respectively.

The weak form of (1) is as follows: find $\lambda \in R$, $u \in H_0^1(\Omega)$, $u \neq 0$ such that

$$a(u,v) = \lambda b(u,v), \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v, \qquad b(u,v) = \int_\Omega uv. \tag{3}$$

As we know, $a(\cdot,\cdot)$ is a symmetric, continuous, and $H_0^1(\Omega)$-elliptic bilinear form on $H_0^1(\Omega)\times H_0^1(\Omega)$, and $b(\cdot,\cdot)$ is a symmetric, continuous, and positive definite bilinear form on $L^2(\Omega)\times L^2(\Omega)$.

Define the operator $T : L^2(\Omega) \to H_0^1(\Omega)$, satisfying

$$a(Tg,v) = b(g,v), \quad \forall v \in H_0^1(\Omega). \tag{4}$$

Then, (2) has the equivalent operator form $Tu = \lambda^{-1}u$, where the operators $T : H_0^1(\Omega) \to H_0^1(\Omega)$ and $T : L^2(\Omega) \to L^2(\Omega)$ are self-adjoint and completely continuous.

Let $\pi_h = \{\kappa\}$ be a regular triangulation of the domain Ω, ε denote the set of all element edges in π_h, $\varepsilon(\Omega)$ denote the set of interior edges, $\varepsilon(\partial\Omega)$ denote the set of all boundary edges, and N_m denote the set of the midside nodes m_e of the edges $e \in \varepsilon$. For the set of midpoints of the edges $e \in \varepsilon(\partial\Omega)$, we use the notation $N_m(\partial\Omega)$; for any element κ, we let $\partial\kappa$ be the union set of edges of κ, and let h_κ be the diameter of κ. The domain ω_κ consists of all elements sharing at least a side with κ. For any edge $e \in \varepsilon$, $h_e = |e|$ is the length of e, and $\vartheta_e = (\vartheta_1, \vartheta_2)$ and $\tau_e = (-\vartheta_2, \vartheta_1)$ are unit outward normal vector and unit tangential vector, respectively.

Given a nonnegative integer k, the space $P_k(\kappa)$ consists of polynomials of total degree at most k defined over κ. The C-R element space is given by $V_h = \{v \in L^2(\Omega), v|_\kappa \in P_1(\kappa), \kappa \in \pi_h$, v is continuous at each $m_e \in N_m \setminus N_m(\partial\Omega)$, and $v(m_e) = 0$ for $m_e \in N_m(\partial\Omega)\}$.

The C-R element approximation of (2) is given as follows: find $\lambda_h \in R$, $u_h \in V_h$, $u_h \neq 0$ such that

$$a_h(u_h,v) = \lambda_h b(u_h,v), \quad \forall v \in V_h, \tag{5}$$

where

$$a_h(u_h,v) = \sum_{\kappa\in\pi_h}\int_\kappa \nabla u_h \cdot \nabla v. \tag{6}$$

$a_h(\cdot,\cdot)$ is a symmetric, continuous, and V_h-elliptic bilinear form on $V_h \times V_h$. $\|v\|_h = \sqrt{a_h(v,v)}$ is well known as the norm of the space V_h; let $D \subset \Omega$, $\overline{D}$ which consists of $\kappa \in \pi_h$,

$$a_{h,\kappa}(u,v) = \int_\kappa uv, \qquad a_{h,D}(u,v) = \sum_{\kappa\in\overline{D}} a_{h,\kappa}(u,v),$$
$$\|v\|_{h,D} = \sqrt{a_{h,D}(v,v)}. \tag{7}$$

Define the operator $T_h : L^2(\Omega) \to V_h$, satisfying

$$a_h(T_h g, v) = b(g,v), \quad \forall v \in V_h. \tag{8}$$

Then, (5) has the equivalent operator form $T_h u_h = \lambda_h^{-1} u_h$, where the operators $T_h : V_h \to V_h$ and $T_h : L^2(\Omega) \to L^2(\Omega)$ are self-adjoint and completely continuous.

Suppose that λ and λ_h are the kth eigenvalue of (2) and (5), respectively, and the algebraic multiplicity of λ is equal to q, $\lambda = \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+q-1}$. Let $M(\lambda_k)$ be the space spanned by all eigenfunctions corresponding to λ_k, and let $M_h(\lambda_k)$ be the direct sum of eigenspaces corresponding to all eigenvalues of (5) that converge to λ_k. Let $\widehat{M}(\lambda_k) = \{v : v \in M(\lambda_k), \|v\|_h = 1\}$, $\widehat{M}_h(\lambda_k) = \{v : v \in M_h(\lambda_k), \|v\|_h = 1\}$.

Define

$$\delta_h^2(\lambda_k) = \left\| (T-T_h)|_{M(\lambda_k)} \right\|_h^2 + \left\| (T-T_h)|_{M(\lambda_k)} \right\|_0. \tag{9}$$

Define $E_h(Tf,v) = a_h(Tf,v) - b(f,v)$, which is the consistent item of nonconforming finite element. Let (λ,u) be an eigenpair of (2), and then

$$E_h(u,v) = a_h(u,v) - b(\lambda u, v). \tag{10}$$

We need the lemmas as follows (see [11, 23]).

Lemma 1 (see [11, Lemma 2.5]). *Let (λ,u) be an eigenpair of (2) and then, for any $w \in V_h$ with $\|w\|_0 \neq 0$, the Rayleigh quotient $a_h(w,w)/\|w\|_0^2$ satisfies*

$$\frac{a_h(w,w)}{\|w\|_0^2} - \lambda = \frac{\|w-u\|_h^2}{\|w\|_0^2} - \lambda\frac{\|w-u\|_0^2}{\|w\|_0^2} + 2\frac{E_h(u,w)}{b(w,w)}. \tag{11}$$

Lemma 2 (see [11, Lemma 2.4] and [23, Lemma 2.3]). *Let $\lambda_{k,h}$ and λ_k be the kth eigenvalue of* (5) *and* (2), *respectively. Then,*

$$\left|\lambda_{k,h} - \lambda_k\right| \lesssim \left\| (T-T_h)|_{M(\lambda_k)} \right\|_0. \tag{12}$$

For any eigenfunction $u_{k,h}$ corresponding to $\lambda_{k,h}$, satisfying $\|u_{k,h}\|_h = 1$, there exists $u_k \in M(\lambda_k)$ such that

$$\|u_{k,h} - u_k\|_0 \lesssim \left\| (T-T_h)|_{M(\lambda_k)} \right\|_0, \tag{13}$$

$$\|u_{k,h} - u_k\|_h \leq \lambda_k \|Tu_k - T_h u_k\|_h + C\left\| (T-T_h)|_{M(\lambda_k)} \right\|_0. \tag{14}$$

For any $u_k \in \widehat{M}(\lambda_k)$, there exists $u_h \in M_h(\lambda_k)$ such that

$$\|u_h - u_k\|_h \lesssim \left\| (T-T_h)|_{M(\lambda_k)} \right\|_h + \left\| (T-T_h)|_{M(\lambda_k)} \right\|_0. \tag{15}$$

3. A Priori Error Estimates for Multiscale Discretization Scheme

In this section, we will discuss a priori error estimates of the C-R finite element multiscale discretization scheme based on the shift-inverse power method. Let $\{\pi_{h_i}\}_0^l$ be a family of shape-regular meshes and let $\{V_{h_i}\}_0^l$ be the C-R finite element spaces defined on $\{\pi_{h_i}\}_0^l$. Besides, let $\pi_H = \pi_{h_0}$, $V_H = V_{h_0}$.

The following condition results from [10, 24].

Condition 1. There exists a properly small positive number ϵ, $t_i \in (1, 3-\epsilon]$, $i = 1, 2, \ldots$, such that $\delta_{h_i}(\lambda) = O(\delta_{h_{i-1}}(\lambda)^{t_i})$, $\delta_{h_i}(\lambda) \to 0$ $(i \to \infty)$.

The following scheme is proposed by Yang and Bi (see [11]).

Scheme 3 (multiscale discretization scheme). Consider the following steps.
Step 1. Solve (5) on V_H: find $(\lambda_H, u_H) \in R \times V_H$ such that $\|u_H\|_H = 1$ and

$$a_H(u_H, \psi) = \lambda_H b(u_H, \psi), \quad \forall \psi \in V_H. \tag{16}$$

Step 2. Execute $u^{h_0} \Leftarrow u_H$, $\lambda^{h_0} \Leftarrow \lambda_H$, $i \Leftarrow 1$.

Step 3. Solve a linear system on V_{h_i}: find $u' \in V_{h_i}$ such that

$$a_{h_i}(u', \psi) - \lambda^{h_{i-1}} b(u', \psi) = b(u^{h_{i-1}}, \psi), \quad \forall \psi \in V_{h_i}. \tag{17}$$

Set $u^{h_i} = u'/\|u'\|_{h_i}$.

Step 4. Compute the Rayleigh quotient

$$\lambda^{h_i} = \frac{a_{h_i}(u^{h_i}, u^{h_i})}{b(u^{h_i}, u^{h_i})}. \tag{18}$$

Step 5. If $i = l$, then output (λ^{h_l}, u^{h_l}), stop. Else, $i \Leftarrow i + 1$, and return to Step 3.

Let (λ_H, u_H) be the kth eigenpair of (16), and then (λ^{h_l}, u^{h_l}) derived from Scheme 3 is the kth eigenpair approximation of (5).

In the sequel, we also denote $(\lambda_H, u_H) = (\lambda_{k,H}, u_{k,H})$, $(\lambda^{h_l}, u^{h_l}) = (\lambda_k^{h_l}, u_k^{h_l})$.

Lemma 4 (see [11, Lemma 3.1]). *For any nonzero $u, v \in V_h + H_0^1(\Omega)$,*

$$\begin{aligned} \left\| \frac{u}{\|u\|_h} - \frac{v}{\|v\|_h} \right\|_h &\le 2\frac{\|u - v\|_h}{\|u\|_h}, \\ \left\| \frac{u}{\|u\|_h} - \frac{v}{\|v\|_h} \right\|_h &\le 2\frac{\|u - v\|_h}{\|v\|_h}. \end{aligned} \tag{19}$$

Denote $\operatorname{dist}(u, S) = \inf_{v \in S}\|u - v\|_h$.

Our analysis is based on the following crucial property of the shifted-inverse iteration in finite element method (see Lemma 4.2 of [24]), which is a development of Theorem 3.2 in [11]. Let $M = (1/\lambda_k) = M(\lambda_k)$, $M_h = (1/\lambda_k) = M_h(\lambda_k)$.

Lemma 5 (see [24, Lemma 4.2]). *Let $\mu_k = 1/\lambda_k$ and $\mu_{k,h} = 1/\lambda_{k,h}$ be the kth eigenvalue of T and T_h, respectively, (μ_0, u_0) be an approximation for the eigenpair (μ_k, u_k), where μ_0 is not an eigenvalue of T_h, and $u_0 \in V_h$ with $\|u_0\|_h = 1$. Suppose that*

(C1) $\operatorname{dist}(u_0, M_h(\mu_k)) \le 1/2$*;*

(C2) $|\mu_0 - \mu_{j,h}| \ge \rho/2$ *for* $j \ne k, k+1, \ldots, k+q-1$*, where* $\rho = \min_{\mu_j \ne \mu_k}|\mu_j - \mu_k|$ *is the separation constant of the eigenvalue* μ_k*;*

(C3) $u' \in V_h$, $u_k^h \in V_h$ *satisfy*

$$(\mu_0 - T_h) u' = u_0, \qquad u_k^h = \frac{u'}{\|u'\|_h}. \tag{20}$$

Then,

$$\begin{aligned} &\operatorname{dist}(u_k^h, M_h(\mu_k)) \\ &\quad \le \frac{4}{\rho} \max_{k \le j \le k+q-1} |\mu_0 - \mu_{j,h}| \operatorname{dist}(u_0, M_h(\mu_k)). \end{aligned} \tag{21}$$

Let us construct the interpolation postprocessing operator $I_h^c : V_h \to V_h \cap H_0^1(\Omega)$ (see [25]): on the vertex z of elements,

$$(I_h^c u_h)(z) = \begin{cases} 0, & z \in \partial\Omega, \\ \dfrac{1}{J_z} \displaystyle\sum_{\kappa \in \omega_z} u_h|_\kappa(z), & z \notin \partial\Omega, \end{cases} \tag{22}$$

where J_z is the number of elements containing the vertex z and ω_z is the union of elements containing the vertex z.

Lemma 6. *Suppose that Condition 1 holds and H is properly small. Let $(\lambda_k^{h_1}, u_k^{h_1})$ be obtained by Scheme 3 for $l = 1$, and then there exists $u_k \in M(\lambda_k)$ such that*

$$\left\| u_k^{h_1} - u_k \right\|_{h_1} \le C(\delta_H^3(\lambda_k) + \delta_{h_1}(\lambda_k)), \tag{23}$$

$$\left| \lambda_k^{h_1} - \lambda_k \right| \le C(\delta_H^6(\lambda_k) + \delta_{h_1}^2(\lambda_k)). \tag{24}$$

Proof. Based on the proof of Theorem 5.1 in [11] and Lemma 5, we deduce that

$$\begin{aligned} &\left\| u_k^{h_1} - u_k \right\|_{h_1} \\ &\quad \le C\left(\left\| (T - T_H)|_{M(\lambda_k)} \right\|_0^2 + \left\| (T - T_{h_1})|_{M(\lambda_k)} \right\|_{h_1} \right) \\ &\quad \lesssim \delta_H^4(\lambda_k) + \delta_{h_1}(\lambda_k), \end{aligned} \tag{25}$$

and thus (23) holds. Using Strang Lemma and Lemma 3.1 of [25], we deduce that

$$\begin{aligned} &E_{h_1}(u_k, u_k^{h_1}) \\ &\quad = E_{h_1}(u_k, u_k^{h_1} - I_{h_1}^c u_k^{h_1}) \\ &\quad \lesssim \left\| u_k - T_{h_1}(\lambda_k u_k) \right\|_{h_1} \left\| u_k^{h_1} - I_{h_1}^c u_k^{h_1} \right\|_{h_1} \\ &\quad \lesssim \left\| u_k - T_{h_1}(\lambda_k u_k) \right\|_{h_1} \left\| u_k^{h_1} - u_k \right\|_{h_1} \lesssim \delta_{h_1}^2(\lambda_k). \end{aligned} \tag{26}$$

From the above formula and (5.2) in [11], we get

$$\begin{aligned}
&\left|\lambda_k^{h_1} - \lambda_k\right| \\
&\lesssim C\left(\left\| (T - T_H)|_{M(\lambda_k)} \right\|_0^4 + \left\| \left(T - T_{h_1}\right)|_{M(\lambda_k)} \right\|_{h_1}^2 \right. \\
&\qquad \left. +2\frac{E_{h_1}\left(u_k, u_k^{h_1}\right)}{b\left(u_k^{h_1}, u_k^{h_1}\right)} \right) \\
&\lesssim \delta_H^6\left(\lambda_k\right) + \delta_{h_1}^2\left(\lambda_k\right),
\end{aligned} \tag{27}$$

and thus (24) holds. □

Based on [10, 11, 24], we will prove the following Theorems 7 and 8 for Scheme 3.

Theorem 7. *Let (λ^{h_l}, u^{h_l}) be an approximate eigenpair obtained by Scheme 3, and $u^{h_{l-1}}$ and $\lambda^{h_{l-1}}$ approximate $\overline{u} \in \widehat{M}(\lambda)$ and λ, respectively, and $\|u^{h_{l-1}} - \overline{u}\|_{h_{l-1}} \lesssim \delta_{h_{l-1}}(\lambda)$, $|\lambda^{h_{l-1}} - \lambda| \lesssim \delta_{h_{l-1}}^2(\lambda)$. Suppose that H is properly small and Condition 1 holds. Then, there exists $u \in M(\lambda)$ such that*

$$\left\|u - u^{h_l}\right\|_{h_l,D} = \left\|\left(T - T_{h_l}\right)\left(\lambda^{h_l}u^{h_l}\right)\right\|_{h_l,D} + \|R\|_{h_l,D}, \tag{28}$$

$$\left|\lambda^{h_l} - \lambda\right| \lesssim \delta_{h_l}^2(\lambda), \tag{29}$$

$$\left\|u^{h_l} - T_{h_l}\left(\lambda^{h_l}u^{h_l}\right)\right\|_{h_l,D} \lesssim \delta_{h_{l-1}}^3(\lambda) + \delta_{h_l}^2(\lambda), \tag{30}$$

where $\|R\|_{h_l,D} \lesssim \delta_{h_{l-1}}^3(\lambda) + \delta_{h_l}^2(\lambda)$.

Proof. Let $\mu_0 = 1/\lambda^{h_{l-1}}$, $u_0 = \lambda^{h_{l-1}}T_{h_l}u^{h_{l-1}}/\|\lambda^{h_{l-1}}T_{h_l}u^{h_{l-1}}\|_{h_l}$. Since $\overline{u} \in \widehat{M}(\lambda)$, by calculation, we get

$$\begin{aligned}
&\left\|\lambda^{h_{l-1}}T_{h_l}u^{h_{l-1}} - \overline{u}\right\|_{h_l} \\
&\quad = \left\|\lambda^{h_{l-1}}T_{h_l}u^{h_{l-1}} - \lambda T\overline{u}\right\|_{h_l} \\
&\quad \leq \left|\lambda^{h_{l-1}} - \lambda\right| \left\|T_{h_l}u^{h_{l-1}}\right\|_{h_l} + \lambda\left\|T_{h_l}\left(u^{h_{l-1}} - \overline{u}\right)\right\|_{h_l} \\
&\qquad + \lambda\left\|\left(T_{h_l} - T\right)\overline{u}\right\|_{h_l}.
\end{aligned} \tag{31}$$

From the definition of T_{h_l}, it is easy to know that

$$\left\|T_{h_l}v\right\|_{h_l} \leq C\|v\|_0, \quad \forall v \in L^2(\Omega). \tag{32}$$

From $a_{h_l}(T_{h_l}v_{h_l}, v_{h_l}) = b(v_{h_l}, v_{h_l})$, we get

$$\left\|v_{h_l}\right\|_0^2 \lesssim \left\|T_{h_l}v_{h_l}\right\|_{h_l}\left\|v_{h_l}\right\|_{h_l} \lesssim \left\|v_{h_l}\right\|_0\left\|v_{h_l}\right\|_{h_l}, \tag{33}$$

and thus

$$\left\|v_{h_l}\right\|_0 \lesssim \left\|v_{h_l}\right\|_{h_l}, \quad \forall v_{h_l} \in V_{h_l}. \tag{34}$$

By Lemma 3.1 in [25], we get that

$$\begin{aligned}
\left\|u^{h_{l-1}} - I_{h_{l-1}}^c u^{h_{l-1}}\right\|_0 &\lesssim \left\|u^{h_{l-1}} - I_{h_{l-1}}^c u^{h_{l-1}}\right\|_{h_{l-1}}, \\
\left\|u^{h_{l-1}} - I_{h_{l-1}}^c u^{h_{l-1}}\right\|_{h_{l-1}} &\lesssim \left\|u^{h_{l-1}} - \overline{u}\right\|_{h_{l-1}}.
\end{aligned} \tag{35}$$

Thus,

$$\begin{aligned}
&\left\|u^{h_{l-1}} - \overline{u}\right\|_0 \\
&\quad \lesssim \left\|u^{h_{l-1}} - I_{h_{l-1}}^c u^{h_{l-1}}\right\|_0 + \left\|I_{h_{l-1}}^c u^{h_{l-1}} - \overline{u}\right\|_0 \\
&\quad \lesssim \left\|u^{h_{l-1}} - I_{h_{l-1}}^c u^{h_{l-1}}\right\|_{h_{l-1}} + \left\|I_{h_{l-1}}^c u^{h_{l-1}} - \overline{u}\right\|_{h_{l-1}} \\
&\quad \lesssim \left\|u^{h_{l-1}} - \overline{u}\right\|_{h_{l-1}}.
\end{aligned} \tag{36}$$

Using the above formula and (31), we can deduce that

$$\begin{aligned}
\left\|\lambda^{h_{l-1}}T_{h_l}u^{h_{l-1}} - \overline{u}\right\|_{h_l} &\lesssim \left|\lambda^{h_{l-1}} - \lambda\right| + \lambda\left\|u^{h_{l-1}} - \overline{u}\right\|_0 \\
&\quad + \lambda\left\|\left(T_{h_l} - T\right)|_{M(\lambda)}\right\|_{h_l} \\
&\lesssim \delta_{h_{l-1}}(\lambda).
\end{aligned} \tag{37}$$

Using Lemma 4, we get

$$\begin{aligned}
\operatorname{dist}\left(u_0, \widehat{M}(\lambda)\right) \leq \left\|u_0 - \overline{u}\right\|_{h_l} &\leq 2\left\|\lambda^{h_{l-1}}T_{h_l}u^{h_{l-1}} - \overline{u}\right\|_{h_l} \\
&\lesssim \delta_{h_{l-1}}(\lambda).
\end{aligned} \tag{38}$$

Using triangle inequality and (15), we have

$$\operatorname{dist}\left(u_0, M_{h_l}(\lambda)\right) \leq \operatorname{dist}\left(u_0, \widehat{M}(\lambda)\right) + \delta_{h_l}(\lambda). \tag{39}$$

From (12), for $j = k, k+1, \ldots, k+q-1$, we have

$$\left|\mu_0 - \mu_{j,h_l}\right| = \left|\frac{\lambda^{h_{l-1}} - \lambda + \lambda - \lambda_{j,h_l}}{\lambda_{j,h_l}\lambda^{h_{l-1}}}\right| \lesssim \left|\lambda^{h_{l-1}} - \lambda\right| + \delta_{h_l}^2(\lambda). \tag{40}$$

Noticing that H is small enough and Condition 1 holds, then by (38) and (39), we can obtain

$$\operatorname{dist}\left(u_0, M_{h_l}(\lambda)\right) \leq \frac{1}{2}. \tag{41}$$

Since ρ is the separation constant, H is small enough, and Condition 1 holds, we have

$$\left|\mu_0 - \mu_{j,h_l}\right| \geq \frac{\rho}{2}, \quad j \neq k, k+1, \ldots, k+q-1. \tag{42}$$

From the definition of T_{h_l}, we can see that Step 3 in Scheme 3 $(i = l)$ is equivalent to

$$\begin{aligned}
&a_{h_l}\left(u', \psi\right) - \lambda^{h_{l-1}}a_{h_l}\left(T_{h_l}u', \psi\right) \\
&\quad = a_{h_l}\left(T_{h_l}u^{h_{l-1}}, \psi\right), \quad \forall \psi \in V_{h_l},
\end{aligned} \tag{43}$$

where $u^{h_l} = u'/\|u'\|_{h_l}$; that is,

$$\left(\frac{1}{\lambda^{h_{l-1}}} - T_{h_l}\right) u' = \frac{1}{\lambda^{h_{l-1}}} T_{h_l} u^{h_{l-1}}, \qquad u^{h_l} = \frac{u'}{\|u'\|_{h_l}}. \tag{44}$$

Noticing that $(1/\lambda^{h_{l-1}})T_{h_l}u^{h_{l-1}} = \|(1/\lambda^{h_{l-1}})T_{h_l}u^{h_{l-1}}\|_{h_l} u_0$ differs from u_0 by only a constant, then Step 3 is equivalent to

$$\left(\frac{1}{\lambda^{h_{l-1}}} - T_{h_l}\right) u = u_0, \qquad u^{h_l} = \frac{u}{\|u\|_{h_l}}. \tag{45}$$

From the above formulae, (41), (42), and (45), we can see that the conditions in Lemma 5 hold; therefore, substituting (39) and (40) into (21), we derive

$$\begin{aligned} &\operatorname{dist}\left(u^{h_l}, \widehat{M}_{h_l}(\lambda)\right) \\ &\quad \lesssim \left(\left|\lambda^{h_{l-1}} - \lambda\right| + \delta_{h_l}^2(\lambda)\right)\left(\operatorname{dist}\left(u_0, \widehat{M}(\lambda)\right) + \delta_{h_l}(\lambda)\right) \\ &\quad \lesssim \delta_{h_{l-1}}^3 + \delta_{h_{l-1}}^2 \delta_{h_l}(\lambda) \lesssim \delta_{h_{l-1}}^3(\lambda). \end{aligned} \tag{46}$$

Let eigenfunctions $\{u_{j,h_l}\}_k^{k+q-1}$ be an orthonormal basis of $M_{h_l}(\lambda)$ in the sense of inner product $a_{h_l}(\cdot,\cdot)$ and then

$$\operatorname{dist}\left(u^{h_l}, M_{h_l}(\lambda)\right) = \left\| u^{h_l} - \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right) u_{j,h_l} \right\|_{h_l}. \tag{47}$$

Let

$$u^* = \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right) u_{j,h_l}, \tag{48}$$

and then it follows directly from (46) that

$$\left\|u^{h_l} - u^*\right\|_{h_l} \le \operatorname{dist}\left(u^{h_l}, \widehat{M}_{h_l}(\lambda)\right) \lesssim \delta_{h_{l-1}}^3(\lambda). \tag{49}$$

By Lemma 2, there exists $\{u_j^0\}_k^{k+q-1} \subset \widehat{M}(\lambda)$ so that $u_{j,h_l} - u_j^0$ satisfies (14).

Let

$$u = \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right) u_j^0. \tag{50}$$

Then, $u \in M(\lambda)$, and

$$u - u^* = \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right)\left(u_j^0 - u_{j,h_l}\right). \tag{51}$$

By calculation,

$$\begin{aligned} u_j^0 - u_{j,h_l} &= \lambda_j T u_j^0 - \lambda_{j,h_l} T_{h_l} u_{j,h_l} \\ &= \lambda_j T u_j^0 - \lambda_{j,h_l} T u_j^0 + \lambda_{j,h_l} T u_j^0 - \lambda_{j,h_l} T u_{j,h_l} \\ &\quad + \lambda_{j,h_l} T u_{j,h_l} - \lambda_{j,h_l} T_{h_l} u_{j,h_l} \\ &= \left(\lambda_j - \lambda_{j,h_l}\right) T u_j^0 + \lambda_{j,h_l} T\left(u_j^0 - u_{j,h_l}\right) \\ &\quad + \lambda_{j,h_l}\left(T - T_{h_l}\right) u_{j,h_l} \\ &= \lambda_{j,h_l}\left(T - T_{h_l}\right) u_{j,h_l} + R_j', \end{aligned} \tag{52}$$

where $R_j' = (\lambda_j - \lambda_{j,h_l})Tu_j^0 + \lambda_{j,h_l}T(u_j^0 - u_{j,h_l})$.

From (12) and (13), we deduce that

$$\left\|R_j'\right\|_{h_l} = \left\|\left(\lambda_j - \lambda_{j,h_l}\right) T u_j^0 + \lambda_{j,h_l} T\left(u_j^0 - u_{j,h_l}\right)\right\|_{h_l} \lesssim \delta_{h_l}^2(\lambda). \tag{53}$$

Substituting (52) into (51), we have

$$\begin{aligned} u - u^* &= \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right)\left(\lambda_{j,h_l}\left(T - T_{h_l}\right) u_{j,h_l} + R_j'\right) \\ &= \lambda_{j,h_l}\left(T - T_{h_l}\right) \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right) u_{j,h_l} \\ &\quad + \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right) R_j' \\ &= \left(T - T_{h_l}\right)\left(\lambda_{j,h_l} u^*\right) + \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right) R_j' \\ &= \left(T - T_{h_l}\right)\left(\lambda_{j,h_l} u^{h_l}\right) + \lambda_{j,h_l}\left(T - T_{h_l}\right)\left(u^* - u^{h_l}\right) \\ &\quad + \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right) R_j'. \end{aligned} \tag{54}$$

Let

$$\begin{aligned} R' = \lambda_{j,h_l}\left(T - T_{h_l}\right)\left(u^* - u^{h_l}\right) + \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right) R_j' \\ + u^* - u^{h_l}. \end{aligned} \tag{55}$$

By the above two equalities, we obtain

$$u - u^{h_l} = u - u^* + u^* - u^{h_l} = \left(T - T_{h_l}\right)\left(\lambda_{j,h_l} u^{h_l}\right) + R'. \tag{56}$$

From (49) and (53), we have

$$\begin{aligned} \left\|R'\right\|_{h_l,D} &= \Bigg\| \lambda_{j,h_l}\left(T - T_{h_l}\right)\left(u^* - u^{h_l}\right) \\ &\qquad + \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right) R_j' + u^* - u^{h_l} \Bigg\|_{h_l,D} \\ &\le \left|\lambda_{j,h_l}\right| \left\|T - T_{h_l}\right\|_{h_l,D} \left\|u^* - u^{h_l}\right\|_{h_l,D} \\ &\quad + \left\| \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right) R_j' \right\|_{h_l,D} + \left\|u^* - u^{h_l}\right\|_{h_l,D} \\ &\lesssim \left\|u^* - u^{h_l}\right\|_{h_l,D} + \left\| \sum_{j=k}^{k+q-1} a_{h_l}\left(u^{h_l}, u_{j,h_l}\right) R_j' \right\|_{h_l,D} \\ &\lesssim \delta_{h_{l-1}}^3(\lambda) + \delta_{h_l}^2(\lambda). \end{aligned} \tag{57}$$

Therefore,

$$\left\|u-u^{h_l}\right\|_{h_l,D}=\left\|\left(T-T_{h_l}\right)\left(\lambda_{j,h_l}u^{h_l}\right)\right\|_{h_l,D}+\left\|R'\right\|_{h_l,D} \tag{58}$$

$$\lesssim \delta_{h_l}(\lambda). \tag{59}$$

By Lemma 1, we have

$$\frac{a_h\left(u^{h_l},u^{h_l}\right)}{\left\|u^{h_l}\right\|_0^2}-\lambda=\frac{\left\|u^{h_l}-u\right\|_{h_l}^2}{\left\|u^{h_l}\right\|_0^2}-\lambda\frac{\left\|u^{h_l}-u\right\|_0^2}{\left\|u^{h_l}\right\|_0^2}+2\frac{E_h\left(u,u^{h_l}\right)}{b\left(u^{h_l},u^{h_l}\right)}. \tag{60}$$

Since $I_{h_l}^c u^{h_l}\in C^0(\overline{\Omega})$, using Strang Lemma and Lemma 3.1 of [25], we deduce that

$$\begin{aligned}E_{h_l}\left(u,u^{h_l}\right)&=E_{h_l}\left(u,u^{h_l}-I_{h_l}^c u^{h_l}\right)\\&\lesssim\left\|u-T_{h_l}(\lambda u)\right\|_{h_l}\left\|u^{h_l}-I_{h_l}^c u^{h_l}\right\|_{h_l}\\&\lesssim\left\|u-T_{h_l}(\lambda u)\right\|_{h_l}\left\|u^{h_l}-u\right\|_{h_l}\\&\lesssim\delta_{h_l}^2(\lambda).\end{aligned} \tag{61}$$

Substituting (59) and (61) into (60), (29) holds.

By (56) and triangle inequality, we have

$$\begin{aligned}\left\|u-u^{h_l}\right\|_{h_l,D}&=\left\|\left(T-T_{h_l}\right)\left(\lambda_{j,h_l}u^{h_l}\right)+R'\right\|_{h_l,D}\\&\le\left\|\left(T-T_{h_l}\right)\left(\lambda^{h_l}u^{h_l}\right)\right\|_{h_l,D}\\&\quad+\left\|\left(T-T_{h_l}\right)\left(\lambda_{j,h_l}-\lambda^{h_l}\right)u^{h_l}+R'\right\|_{h_l,D}\\&\equiv\left\|\left(T-T_{h_l}\right)\left(\lambda^{h_l}u^{h_l}\right)\right\|_{h_l,D}+\|R\|_{h_l,D},\end{aligned} \tag{62}$$

where $R=(T-T_{h_l})(\lambda_{j,h_l}-\lambda^{h_l})u^{h_l}+R'$.

By (57) and (29), we know that $\|R\|_{h_l,D}\lesssim\delta_{h_{l-1}}^3(\lambda)+\delta_{h_l}^2(\lambda)$; thus, (28) holds.

By calculation,

$$\begin{aligned}&\sum_{j=k}^{k+q-1}a_{h_l}\left(u^{h_l},u_{j,h_l}\right)\lambda^{h_l}T_{h_l}u_{j,h_l}\\&=\lambda^{h_l}T_{h_l}\left(\sum_{j=k}^{k+q-1}a_{h_l}\left(u^{h_l},u_{j,h_l}\right)u_{j,h_l}\right)\\&=\lambda^{h_l}T_{h_l}u^*.\end{aligned} \tag{63}$$

By the above formulae and (12), we deduce that

$$\begin{aligned}&\left\|u^{h_l}-T_{h_l}(\lambda^{h_l}u^{h_l})\right\|_{h_l,D}\\&=\left\|u^{h_l}-u^*+u^*-T_{h_l}\left(\lambda^{h_l}u^*\right)\right.\\&\qquad\left.+T_{h_l}(\lambda^{h_l}u^*)-T_{h_l}(\lambda^{h_l}u^{h_l})\right\|_{h_l,D}\\&\lesssim\left\|u^{h_l}-u^*\right\|_{h_l,D}\\&\quad+\left\|\sum_{j=k}^{k+q-1}\left(\lambda_{j,h_l}-\lambda^{h_l}\right)a_{h_l}\left(u^{h_l},u_{j,h_l}\right)T_{h_l}u_{j,h_l}\right\|_{h_l,D}\\&\quad+\left\|\lambda^{h_l}T_{h_l}\left(u^*-u^{h_l}\right)\right\|_{h_l,D}\\&\lesssim\left\|u^{h_l}-u^*\right\|_{h_l,D}+\sum_{j=k}^{k+q-1}\left|\lambda_{j,h_l}-\lambda^{h_l}\right|\\&\lesssim\left\|u^{h_l}-u^*\right\|_{h_l}+\delta_{h_l}^2(\lambda),\end{aligned} \tag{64}$$

which together with (49) leads to (30). This completes the proof. □

Theorem 8. *Let (λ^{h_l},u^{h_l}) be the kth approximate eigenpair of* (1) *obtained by Scheme 3, let λ be the kth eigenvalue of* (1), *and let H be properly small. Suppose that Condition 1 holds, then there exists $u\in M(\lambda)$ such that*

$$\begin{aligned}\left\|u^{h_l}-u\right\|_{h_l}&\le C\delta_{h_l}(\lambda),\\\left|\lambda^{h_l}-\lambda\right|&\le C\delta_{h_l}^2(\lambda),\quad l\ge1.\end{aligned} \tag{65}$$

Proof. The proof of (65) is completed by using induction. When $l=1$, by Lemma 6, we know that Theorem 8 holds.

Suppose that Theorem 8 holds for $l-1$; that is,

$$\begin{aligned}\left\|u^{h_{l-1}}-u\right\|_{h_{l-1}}&\le C\delta_{h_{l-1}}(\lambda),\\\left|\lambda^{h_{l-1}}-\lambda\right|&\le C\delta_{h_{l-1}}^2(\lambda),\end{aligned} \tag{66}$$

which together with the assumptions in Theorem 8, we know that Theorem 7 holds. For l, by (29) and (59), we get (65). The proof is completed. □

4. A Posteriori Error Estimates for Multiscale Discretization Scheme

Based on the work of [14, 26], in this section, we will discuss a posteriori error estimates of the C-R element approximation for Laplace eigenvalue problem.

Consider the boundary value problem corresponding to (2): find $w\in H_0^1(\Omega)$ such that

$$a(w,v)=b(f,v),\quad\forall v\in H_0^1(\Omega), \tag{67}$$

and its C-R element approximation: find $w_h\in V_h$ such that

$$a_h(w_h,v)=b(f,v),\quad\forall v\in V_h. \tag{68}$$

Let $\kappa^+ \in \pi_h$, $\kappa^- \in \pi_h$ be two elements sharing one edge e. For any piecewise continuous function φ, we denote by $[[\varphi]]_e = (\varphi|_{\kappa^+})|_e - (\varphi|_{\kappa^-})|_e$ the jump of φ across e.

Let w_h be the solution of (68), $\widetilde{J}_{e,\vartheta}$ be the jump of ∇w_h across e along ϑ_e, and $\widetilde{J}_{e,\tau}$ be the jump of ∇w_h across e along τ_e; let $\widetilde{R}_\kappa(w_h)$ be element residual; that is,

$$\widetilde{R}_\kappa(w_h) = f + \Delta w_h \quad \kappa \in \pi_h,$$
$$\widetilde{J}_{e,\vartheta}(w_h) = \begin{cases} [[\nabla w_h]]_e \cdot \vartheta_e, & e \in \varepsilon(\Omega), \\ 0, & e \in \varepsilon(\partial\Omega), \end{cases} \tag{69}$$
$$\widetilde{J}_{e,\tau}(w_h) = \begin{cases} [[\nabla w_h]]_e \cdot \tau_e, & e \in \varepsilon(\Omega), \\ -\sqrt{2}\nabla w_h \cdot \tau_e, & e \in \varepsilon(\partial\Omega). \end{cases}$$

For $\kappa \in \pi_h$, define the residual on the element κ as

$$\widetilde{\eta}_h(w_h,\kappa) = \left(h_\kappa^2 \left\| \widetilde{R}_\kappa(w_h) \right\|_{0,\kappa}^2 + \frac{1}{2}\sum_{e\in\partial\kappa} h_e \left(\left\| \widetilde{J}_{e,\tau}(w_h) \right\|_{0,e}^2 + \left\| \widetilde{J}_{e,\vartheta}(w_h) \right\|_{0,e}^2 \right) \right)^{1/2}, \tag{70}$$

and thus, for $G \subset \Omega$, the residual sum on G is given by

$$\widetilde{\eta}_h(w_h,G) = \left(\sum_{\kappa\in\pi_h,\kappa\subset G} \widetilde{\eta}_h^2(w_h,\kappa) \right)^{1/2}. \tag{71}$$

For $f \in L^2(\Omega)$, define the date oscillation by

$$\operatorname{osc}(f,\pi_h) = \left(\sum_{\kappa\in\pi_h} h_\kappa^2 \| f - f_h \|_{0,\kappa}^2 \right)^{1/2}, \tag{72}$$

where f_h stands for a piecewise polynomial approximation of f over π_h.

For the boundary value problem (67), Carstensen and Hu [27] have proved the following a posteriori error estimates :

$$\| w - w_h \|_{h,\Omega} \le \widetilde{C}_1 \left(\widetilde{\eta}_h(w_h,\Omega) + \operatorname{osc}(f,\pi_h) \right), \tag{73}$$

where constant $\widetilde{C}_1$ is only dependent on minimum angle of π_h, and if the right-hand side f of (67) is a piecewise linear polynomial over π_h, then

$$\widetilde{C}_2 \widetilde{\eta}_h(w_h,\Omega) \le \| w - w_h \|_{h,\Omega}. \tag{74}$$

Selecting $f = \lambda^{h_l} u^{h_l}$ in (67) and (68), then the generalized solution and the nonconforming finite element solution are $w = T(\lambda^{h_l} u^{h_l})$ and $w_h = T_{h_l}(\lambda^{h_l} u^{h_l})$, respectively, and the a posteriori error indicator of w_h is $\widetilde{\eta}_h(w_h,\Omega)$, which is defined by (71).

Define the element residual $R_\kappa(u^{h_l})$ and the jump residual $J_{e,\vartheta}(u^{h_l})$ and $J_{e,\tau}(u^{h_l})$ for u^{h_l} as follows:

$$R_\kappa(u^{h_l}) = \lambda^{h_l} u^{h_l} + \Delta u^{h_l} \quad \kappa \in \pi_{h_l},$$
$$J_{e,\vartheta}(u^{h_l}) = \begin{cases} [[\nabla u^{h_l}]]_e \cdot \vartheta_e, & e \in \varepsilon(\Omega), \\ 0, & e \in \varepsilon(\partial\Omega), \end{cases} \tag{75}$$
$$J_{e,\tau}(u^{h_l}) = \begin{cases} [[\nabla u^{h_l}]]_e \cdot \tau_e, & e \in \varepsilon(\Omega), \\ -\sqrt{2}\nabla u^{h_l} \cdot \tau_e, & e \in \varepsilon(\partial\Omega). \end{cases}$$

For $\kappa \in \pi_{h_l}$, define the residual on the element κ as

$$\eta_{h_l}(u^{h_l},\kappa) = \left(h_\kappa^2 \left\| R_\kappa(u^{h_l}) \right\|_{0,\kappa}^2 + \frac{1}{2}\sum_{e\in\partial\kappa} h_e \left(\left\| J_{e,\tau}(u^{h_l}) \right\|_{0,e}^2 + \left\| J_{e,\vartheta}(u^{h_l}) \right\|_{0,e}^2 \right) \right)^{1/2}. \tag{76}$$

For $G \subset \Omega$, define the residual sum on G as

$$\eta_{h_l}(u^{h_l},G) = \left(\sum_{\kappa\in\pi_{h_l},\kappa\subset G} \eta_{h_l}^2(u^{h_l},\kappa) \right)^{1/2}. \tag{77}$$

Theorem 9. *Suppose that the conditions in Theorem 7 hold and V_{h_l} is a finite element space consisting of piecewise linear polynomials, then there exists a positive constant δ which is independent of mesh parameter, such that*

$$\| u - u^{h_l} \|_{h_l,\Omega} \le (\widetilde{C}_1 + \delta)\, \eta_{h_l}(u^{h_l},\Omega), \tag{78}$$
$$(\widetilde{C}_2 + \delta)\, \eta_{h_l}(u^{h_l},\Omega) \le \| u - u^{h_l} \|_{h_l,\Omega}, \tag{79}$$
$$|\lambda - \lambda^{h_l}| \lesssim \eta_{h_l}^2(u^{h_l},\Omega). \tag{80}$$

Proof. Let $w_{h_l} = T_{h_l}\lambda^{h_l}u^{h_l}$, and by calculation

$$\begin{aligned} \widetilde{\eta}_{h_l}(w_{h_l},\Omega) &= \left(\sum_{\kappa\in\pi_{h_l},\kappa\subset\Omega} \widetilde{\eta}_{h_l}^2(w_{h_l},\kappa) \right)^{1/2} \\ &= \left(\sum_{\kappa\in\pi_{h_l},\kappa\subset\Omega} \eta_{h_l}^2(u^{h_l},\kappa) \right)^{1/2} \\ &\quad + \left(\sum_{\kappa\in\pi_{h_l},\kappa\subset\Omega} \widetilde{\eta}_{h_l}^2(w_{h_l},\kappa) \right)^{1/2} \\ &\quad - \left(\sum_{\kappa\in\pi_{h_l},\kappa\subset\Omega} \eta_{h_l}^2(u^{h_l},\kappa) \right)^{1/2} \\ &\equiv \eta_{h_l}(u^{h_l},\Omega) + R_2. \end{aligned} \tag{81}$$

By triangle inequality, we have

$$\begin{aligned}\left|R_2\right| &= \left|\left(\sum_{\kappa\in\pi_{h_l},\kappa\subset\Omega}\widetilde{\eta}_{h_l}^2\left(w_{h_l},\kappa\right)\right)^{1/2}\right.\\ &\qquad\left.-\left(\sum_{\kappa\in\pi_{h_l},\kappa\subset\Omega}\eta_{h_l}^2\left(u^{h_l},\kappa\right)\right)^{1/2}\right|\\ &\leq\left(\sum_{\kappa\in\pi_{h_l},\kappa\subset\Omega}\left(\widetilde{\eta}_{h_l}\left(w_{h_l},\kappa\right)-\eta_{h_l}\left(u^{h_l},\kappa\right)\right)^2\right)^{1/2}.\end{aligned}\tag{82}$$

From triangle inequality, (69)-(70), and (75)-(76), we deduce that

$$\begin{aligned}&\left|\widetilde{\eta}_{h_l}\left(w_{h_l},\kappa\right)-\eta_{h_l}\left(u^{h_l},\kappa\right)\right|\\ &\quad=\left|\left(h_\kappa^2\left\|\widetilde{R}_\kappa\left(w_{h_l}\right)\right\|_{0,\kappa}^2\right.\right.\\ &\qquad\left.+\frac{1}{2}\sum_{e\in\partial\kappa}h_e\left(\left\|\widetilde{J}_{e,\tau}\left(w_{h_l}\right)\right\|_{0,e}^2+\left\|\widetilde{J}_{e,\vartheta}\left(w_{h_l}\right)\right\|_{0,e}^2\right)\right)^{1/2}\\ &\qquad-\left(h_\kappa^2\left\|R_\kappa\left(u^{h_l}\right)\right\|_{0,\kappa}^2\right.\\ &\qquad\left.\left.+\frac{1}{2}\sum_{e\in\partial\kappa}h_e\left(\left\|J_{e,\tau}\left(u^{h_l}\right)\right\|_{0,e}^2+\left\|J_{e,\vartheta}\left(u^{h_l}\right)\right\|_{0,e}^2\right)\right)^{1/2}\right|\\ &\quad\leq\left(h_\kappa^2\left\|\widetilde{R}_\kappa\left(w_{h_l}\right)-R_\kappa\left(u^{h_l}\right)\right\|_{0,\kappa}^2\right.\\ &\qquad+\frac{1}{2}\sum_{e\in\partial\kappa}h_e\left(\left\|\widetilde{J}_{e,\tau}\left(w_{h_l}\right)-J_{e,\tau}\left(u^{h_l}\right)\right\|_{0,e}^2\right.\\ &\qquad\left.\left.+\left\|\widetilde{J}_{e,\vartheta}\left(w_{h_l}\right)-J_{l,\vartheta}\left(u^{h_l}\right)\right\|_{0,e}^2\right)\right)^{1/2}\\ &\quad\lesssim\left(h_\kappa^2\left\|\Delta\left(w_{h_l}-u^{h_l}\right)\right\|_{0,\kappa}^2\right.\\ &\qquad+\frac{1}{2}\sum_{e\in\partial\kappa}h_e\left(\left\|\left[\left[\nabla\left(w_{h_l}-u^{h_l}\right)\right]\right]_e\cdot\vartheta_e\right\|_{0,e}^2\right.\\ &\qquad\left.\left.+\left\|\left[\left[\nabla\left(w_{h_l}-u^{h_l}\right)\right]\right]_e\cdot\tau_e\right\|_{0,e}^2\right)\right)^{1/2}.\end{aligned}\tag{83}$$

It is obvious that $\|\Delta(w_{h_l}-u^{h_l})\|_{0,\kappa}^2 = 0$, and, by the trace theorem (see e.g., [28]) and the inverse estimates, we get

$$\begin{aligned}&\frac{1}{2}\sum_{e\in\partial\kappa}h_e\left(\left\|\left[\left[\nabla\left(w_{h_l}-u^{h_l}\right)\right]\right]_e\cdot\vartheta_e\right\|_{0,e}^2\right.\\ &\qquad\left.+\left\|\left[\left[\nabla\left(w_{h_l}-u^{h_l}\right)\right]\right]_e\cdot\tau_e\right\|_{0,e}^2\right)\\ &\quad\lesssim h_\kappa\left(h_\kappa^{-1}\left\|\nabla\left(w_{h_l}-u^{h_l}\right)\right\|_{0,\omega_\kappa}^2\right.\\ &\qquad\left.+h_\kappa\left|\nabla\left(w_{h_l}-u^{h_l}\right)\right|_{1,\omega_\kappa}^2\right)\\ &\quad\lesssim\left\|w_{h_l}-u^{h_l}\right\|_{h_l,\omega_\kappa}^2.\end{aligned}\tag{84}$$

Thus,

$$\begin{aligned}\left|\widetilde{\eta}_{h_l}\left(w_{h_l},\kappa\right)-\eta_{h_l}\left(u^{h_l},\kappa\right)\right| &\lesssim\left\|T_{h_l}\left(\lambda^{h_l}u^{h_l}\right)-u^{h_l}\right\|_{h_l,\kappa}\\ &=\left\|w_{h_l}-u^{h_l}\right\|_{h_l,\omega_\kappa}.\end{aligned}\tag{85}$$

Combining (82), (85), and (30), we get

$$\left|R_2\right|\lesssim\left\|w_{h_l}-u^{h_l}\right\|_{h_l}\lesssim\delta_{h_{l-1}}^3(\lambda)+\delta_{h_l}^2(\lambda).\tag{86}$$

Hence, from Condition 1, we know that R_2 is a small quantity of higher order than $\widetilde{\eta}_{h_l}(w_{h_l},\Omega)$. Using (81), we obtain that R_2 is also a small quantity of higher order than $\eta_{h_l}(u^{h_l},\Omega)$.

Therefore, by (28), (73), (81), and (86), we have

$$\begin{aligned}&\left\|u-u^{h_l}\right\|_{h_l,\Omega}\\ &\quad=\left\|\left(T-T_{h_l}\right)\left(\lambda^{h_l}u^{h_l}\right)\right\|_{h_l,\Omega}+\|R\|_{h_l,\Omega}\\ &\quad\leq\widetilde{C}_1\widetilde{\eta}_{h_l}\left(T_{h_l}\left(\lambda^{h_l}u^{h_l}\right),\Omega\right)+\|R\|_{h_l,\Omega}\\ &\quad\leq\widetilde{C}_1\eta_{h_l}\left(u^{h_l},\Omega\right)\\ &\qquad+\widetilde{C}_1\left(\widetilde{\eta}_{h_l}\left(T_{h_l}\left(\lambda^{h_l}u^{h_l}\right),\Omega\right)-\eta_{h_l}\left(u^{h_l},\Omega\right)\right)\\ &\qquad+\|R\|_{h_l,\Omega}\\ &\quad\leq\widetilde{C}_1\eta_{h_l}\left(u^{h_l},\Omega\right)+\widetilde{C}_1\left\|R_2\right\|_{h_l,\Omega}+\|R\|_{h_l,\Omega}\\ &\quad\leq\left(\widetilde{C}_1+\delta\right)\eta_{h_l}\left(u^{h_l},\Omega\right),\end{aligned}\tag{87}$$

which is (78).

Similarly, by (28) and (74), we get

$$\begin{aligned}
\|u-u^{h_l}\|_{h_l,\Omega} &= \left\|(T-T_{h_l})(\lambda^{h_l}u^{h_l})\right\|_{h_l,\Omega}+\|R\|_{h_l,\Omega}\\
&\geq \widetilde{C}_2\widetilde{\eta}_{h_l}\left(T_{h_l}(\lambda^{h_l}u^{h_l}),\Omega\right)+\|R\|_{h_l,\Omega}\\
&\geq \widetilde{C}_2\eta_{h_l}\left(u^{h_l},\Omega\right)\\
&\quad+\widetilde{C}_2\left(\widetilde{\eta}_{h_l}\left(T_{h_l}(\lambda^{h_l}u^{h_l}),\Omega\right)-\eta_{h_l}\left(u^{h_l},\Omega\right)\right)\\
&\quad+\|R\|_{h_l,\Omega}\\
&\geq \widetilde{C}_2\eta_{h_l}\left(u^{h_l},\Omega\right)+\widetilde{C}_2\|R_2\|_{h_l,\Omega}+\|R\|_{h_l,\Omega}\\
&\geq \left(\widetilde{C}_2+\delta\right)\eta_{h_l}\left(u^{h_l},\Omega\right),
\end{aligned} \tag{88}$$

and thus (79) holds.

By (61) and (28), we get

$$\begin{aligned}
E_{h_l}\left(u,u^{h_l}\right) &\lesssim \left\|u-T_{h_l}(\lambda u)\right\|_{h_l}\left\|u^{h_l}-u\right\|_{h_l}\\
&\lesssim \left\|u^{h_l}-u\right\|_{h_l}^2,
\end{aligned} \tag{89}$$

and, by substituting the above relation into (60), we obtain

$$\left|\lambda^{h_l}-\lambda\right| \lesssim \left\|u^{h_l}-u\right\|_{h_l}^2, \tag{90}$$

which together with (78) yields (80). This completes the proof. □

5. Adaptive Finite Element Algorithm Based on Multiscale Discretizations

As we know, The following Algorithm 10 is fundamental and important; see [14, 16] for its detailed theoretical results.

Algorithm 10. Choose parameter $0<\theta<1$.

Step 1. Pick any initial mesh π_{h_0} with mesh size h_0.

Step 2. Solve (5) on π_{h_0} for discrete solution (λ_{h_0},u_{h_0}).

Step 3. $l \Leftarrow 0$.

Step 4. Compute the local indicators $\eta_{h_l}(u_{h_l},\kappa)$.

Step 5. Construct $\widehat{\pi}_{h_l} \subset \pi_{h_l}$ by Marking Strategy E and parameter θ.

Step 6. Refine π_{h_l} to get a new mesh $\pi_{h_{l+1}}$.

Step 7. Solve (5) on $\pi_{h_{l+1}}$ for discrete solution $(\lambda_{h_{l+1}},u_{h_{l+1}})$.

Step 8. $l \Leftarrow l+1$, and go to Step 4.

Marking Strategy E. Give parameter $0<\theta<1$.

Step 1. Construct a minimal subset $\widehat{\pi}_{h_l}$ of π_{h_l} by selecting some elements in π_{h_l} such that

$$\sum_{\kappa\in\widehat{\pi}_{h_l}}\eta_{h_l}^2\left(u_{h_l},\kappa\right)\geq\theta\eta_{h_l}^2\left(u_{h_l},\Omega\right). \tag{91}$$

Step 2. Mark all the elements $\widehat{\pi}_{h_l}$.

$\eta_{h_l}(u_{h_l},\kappa)$ and $\eta_{h_l}(u_{h_l},\Omega)$ are defined as (76) and (77), respectively, with u^{h_l} and λ^{h_l} replaced by u_{h_l} and λ_{h_l}.

We have the following adaptive algorithm on the basis of Scheme 3.

Algorithm 11. Choose parameter $0<\theta<1$.

Step 1. Pick any initial mesh π_{h_0} with mesh size h_0.

Step 2. Solve (5) on π_{h_0} for discrete solution (λ^{h_0},u^{h_0}).

Step 3. $l \Leftarrow 0$, $\lambda_0 \Leftarrow \lambda^{h_0}$.

Step 4. Compute the local indicators $\eta_{h_l}(u^{h_l},\kappa)$.

Step 5. Construct $\widehat{\pi}_{h_l} \subset \pi_{h_l}$ by Marking Strategy E and parameter θ.

Step 6. Refine π_{h_l} to get a new mesh $\pi_{h_{l+1}}$

Step 7. Find $u' \in V_{h_{l+1}}$ such that

$$a_{h_{l+1}}\left(u',\psi\right)-\lambda_0 b\left(u',\psi\right)=b\left(u^{h_l},\psi\right),\quad \forall\psi\in V_{h_{l+1}}. \tag{92}$$

Set $u^{h_{l+1}}=u'/\|u'\|_{h_{l+1}}$ and compute the Rayleigh quotient

$$\lambda^{h_{l+1}}=\frac{a_{h_{l+1}}\left(u^{h_{l+1}},u^{h_{l+1}}\right)}{b\left(u^{h_{l+1}},u^{h_{l+1}}\right)}. \tag{93}$$

Step 8. $\lambda_0 \Leftarrow \lambda^{h_{l+1}}$, $l \Leftarrow l+1$ and go to Step 4.

Marking Strategy E in Algorithm 11 will be the same as that in Algorithm 10, except for replacing u_{h_l} with u^{h_l}.

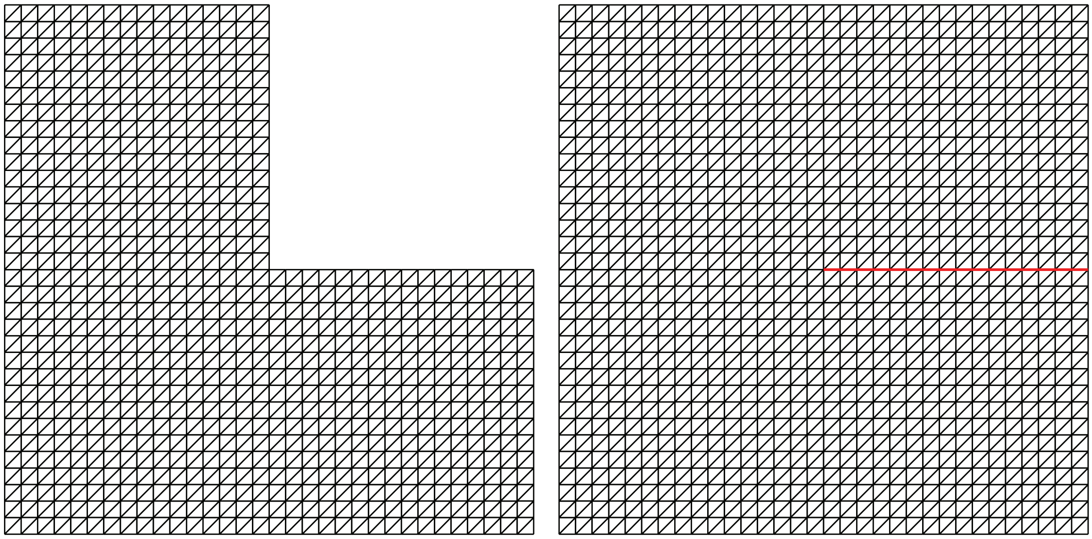
Note that when $|\lambda_0-\lambda|$ is too small, (92) is an almost singular linear equation. Although it has no difficulty in solving (92) numerically (see Lecture 27.4 in [29]), one would like to think of selecting a proper integer $l_0\geq 0$. When $l\geq l_0$, set $\lambda^{h_l}=\lambda^{h_{l_0}}$ in (92). So, we can establish the following algorithm (see e.g., Scheme 3.2 in [24]).

Algorithm 12. Choose parameter $0<\theta<1$.

Step 1–Step 7. Execute Step 1–Step 7 of Algorithm 11.

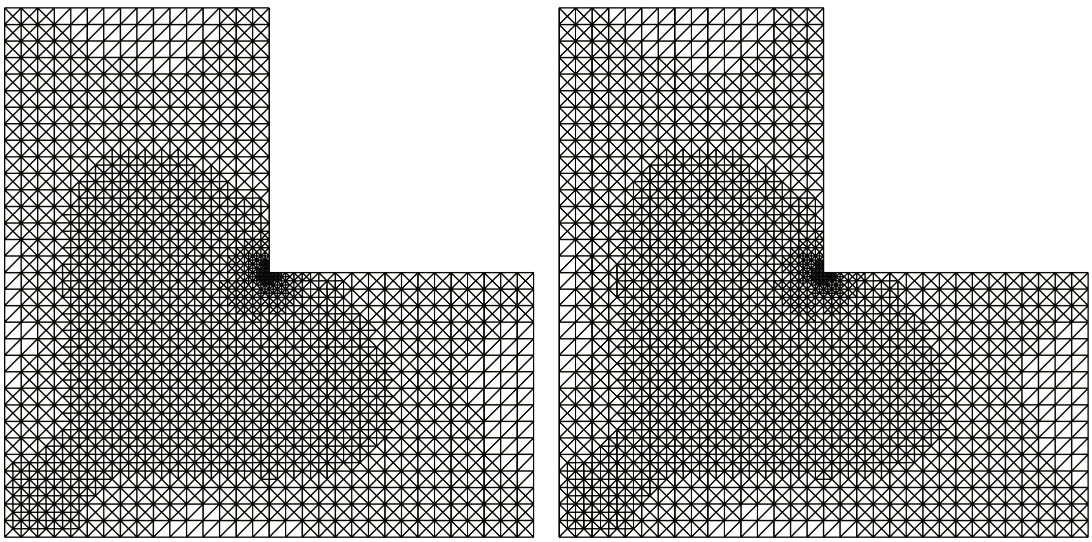
Step 8. If $l<l_0$, $\lambda_0 \Leftarrow \lambda^{h_{l+1}}$, $l \Leftarrow l+1$, go to Step 4; else $l \Leftarrow l+1$, go to Step 4.

Marking Strategy E in Algorithm 12 will be the same as that in Algorithm 11.



(a) (b)

Figure 1: The initial mesh $H = \sqrt{2}/16$.



(a) (b)

Figure 2: The adaptive meshes of 1st eigenvalue after 5th iteration by Algorithm 10 (a) and Algorithm 11 (b).

6. Numerical Experiments

In this section, we will report two numerical examples for Algorithms 10 and 11 to illustrate the theoretical results in this paper. We use MATLAB 2012 to solve Examples 1 and 2. Our program is compiled under the package of Chen. We take $\theta = 0.5$ in two Algorithms.

For reading convenience, we use the following notations in our tables.

l^*: The l^*th iteration of Algorithm 10

$\lambda_{k,h_{l^*}}$:The kth approximate eigenvalue derived from the l^*th iteration obtained by Algorithm 10

dof_{k,l^*}:The degrees of freedom of the l^*th iteration for computing $\lambda_{k,h_{l^*}}$

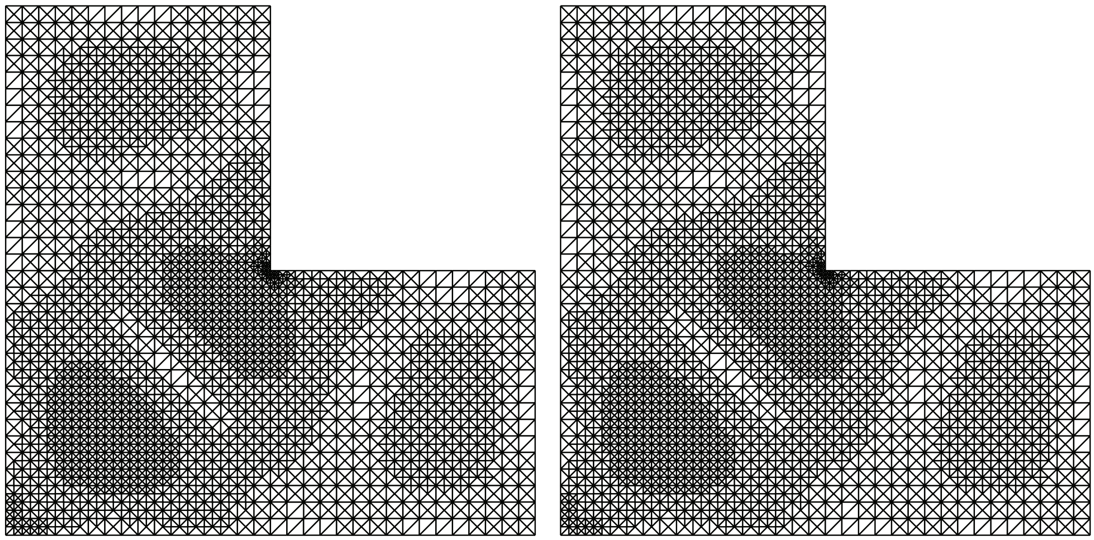
$\mathrm{CPU}_{k,l^*}(s)$: The total CPU time(s) for computing $\lambda_{k,h_{l^*}}$

$|\lambda_{k,h_{l^*}} - \lambda_k|$: the error of kth approximate eigenvalue $\lambda_{k,h_{l^*}}$

l: The lth iteration of Algorithm 11

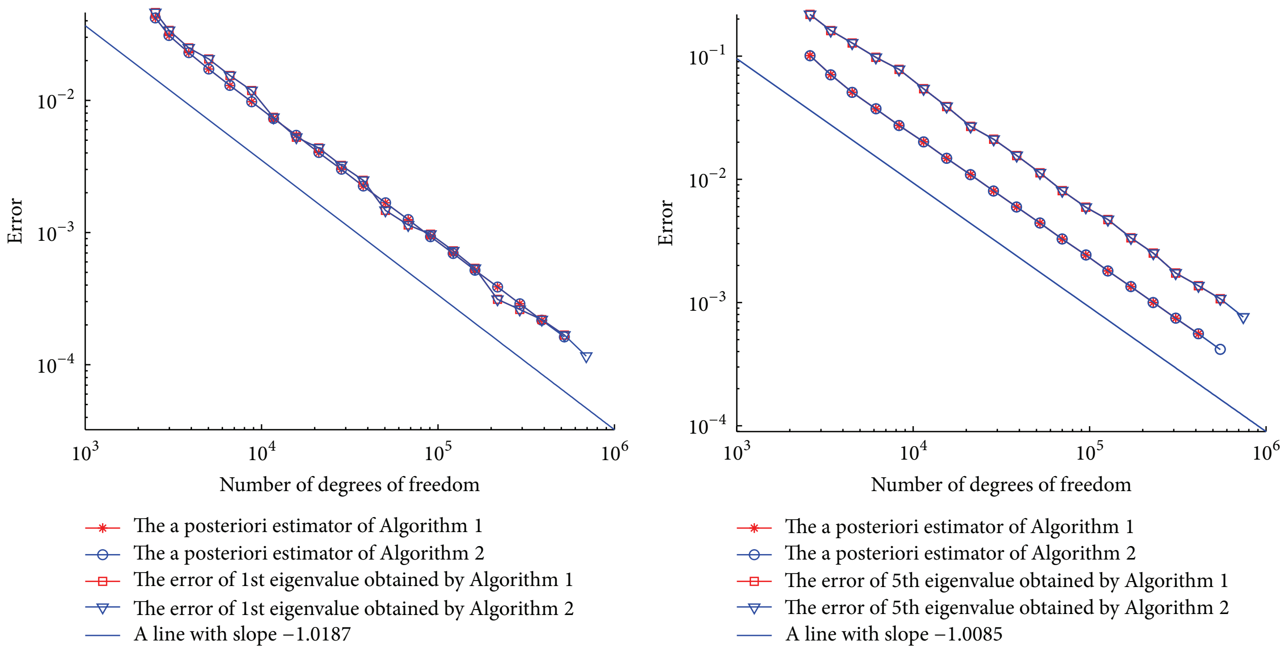
$\lambda_k^{h_l}$:The kth approximate eigenvalue derived from the lth iteration obtained by Algorithm 11

$\mathrm{dof}_{k,l}$: The degrees of freedom of the lth iteration for computing $\lambda_k^{h_l}$



(a) (b)

Figure 3: The adaptive meshes of 5th eigenvalue after 5th iteration by Algorithm 10 (a) and Algorithm 11 (b).



(a) (b)

Figure 4: The error curves of two algorithms on L-shaped domain.

$\mathrm{CPU}_{k,l}(s)$: The total CPU time(s) for computing $\lambda_k^{h_l}$

$|\lambda_k^{h_l} - \lambda_k|$: The error of kth approximate eigenvalue $\lambda_k^{h_l}$.

Example 1. We use Algorithms 10 and 11 to compute the approximate eigenvalues of (1) on the L-shaped domain $\Omega = ((0,2) \times (0,2)) \setminus ([1,2] \times [1,2])$ (see Figure 1(a)).

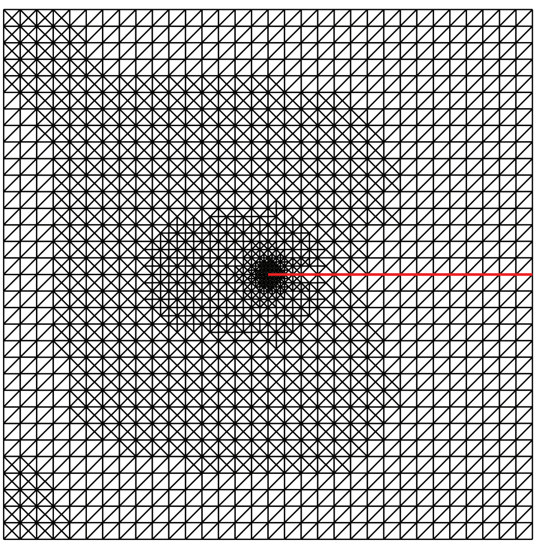
The first and fifth eigenvalues of (1) are $\lambda_1 \approx 9.639723844$ and $\lambda_5 \approx 31.912636$ on this domain, respectively. The associated numerical results are presented in Table 1 and Figures 1(a), 2, 3, and 4. Figure 1(a) gives the uniform initial mesh with $H = \sqrt{2}/16$. Figures 2 and 3 show the adaptive meshes of the first and fifth eigenvalues after the fifth iteration by two algorithms, respectively. It is indicated in Figure 4 that the error curves of the first and fifth approximate eigenvalues and the curves of the associated a posteriori error estimators obtained by Algorithms 10 and 11 are approximately parallel to the line with slope -1, respectively; this coincides with our theory in Section 4.

Table 1: The 1st and 5th eigenvalues obtained by two algorithms on L-shaped domain with $H = \sqrt{2}/16$.

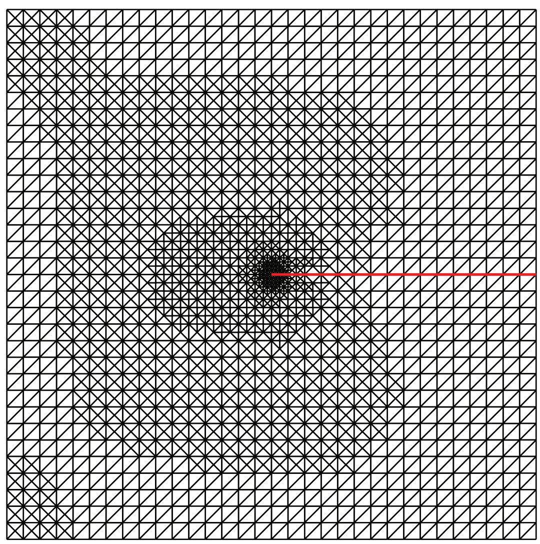
k	l^*	dof_{k,l^*}	$\lambda_{k,h_{l^*}}$	CPU_{k,l^*} (s)	l	$\mathrm{dof}_{k,l}$	$\lambda_k^{h_l}$	$\mathrm{CPU}_{k,l}$ (s)
1	5	6615	9.624308	0.43	5	6615	9.624308	0.26
1	10	28350	9.636503	1.83	10	28350	9.636503	0.99
1	15	122122	9.638997	9.27	15	122122	9.638997	5.17
1	19	387527	9.639505	35.6	19	387527	9.639505	19.3
1	20	520999	9.639556	52.0	20	520999	9.639556	26.6
1	21	—	—	—	21	692323	9.639608	38.2
5	5	8319	31.83478	0.62	5	8319	31.83478	0.35
5	10	38601	31.89705	3.07	10	38601	31.89705	1.75
5	15	171591	31.90928	17.1	15	171591	31.90928	9.79
5	18	414327	31.91127	47.9	18	414327	31.91127	26.7
5	19	550331	31.91156	68.6	19	550331	31.91156	37.0
5	20	—	—	—	20	743563	31.91187	53.0

Table 2: The 1st and 6th eigenvalues obtained by two algorithms on slit domain with $H = \sqrt{2}/16$.

k	l^*	dof_{k,l^*}	$\lambda_{k,h_{l^*}}$	CPU_{k,l^*} (s)	l	$\mathrm{dof}_{k,l}$	$\lambda_k^{h_l}$	$\mathrm{CPU}_{k,l}$ (s)
1	5	5229	8.329697	0.43	5	5229	8.329697	0.25
1	10	17488	8.361012	1.33	10	17491	8.361012	0.79
1	15	64863	8.368891	5.14	15	64863	8.368891	2.90
1	20	249713	8.370756	23.4	20	249713	8.370756	12.9
1	23	569429	8.371082	61.5	23	569429	8.371082	31.2
1	24	—	—	—	24	748279	8.371148	44.0
6	5	8733	30.39428	0.69	5	8721	30.39422	0.41
6	10	35253	30.50367	3.21	10	35212	30.50363	1.68
6	15	141116	30.52867	16.2	15	140930	30.52866	8.20
6	19	426746	30.53371	58.8	19	426194	30.53371	28.9
6	20	566328	30.53432	82.6	20	565526	30.53431	39.1
6	21	—	—	—	21	743261	30.53464	54.9



(a)



(b)

Figure 5: The adaptive meshes of 1st eigenvalue after 5th iteration by **Algorithm 10** (a) and **Algorithm 11** (b).

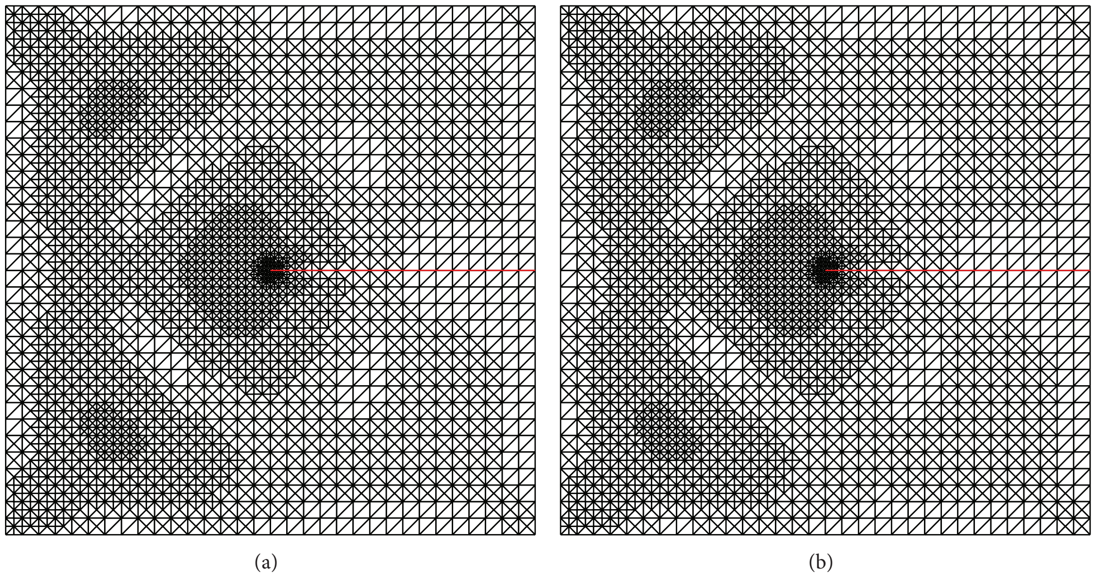


FIGURE 6: The adaptive meshes of 6th eigenvalue after 5th iteration by Algorithm 10 (a) and Algorithm 11 (b).

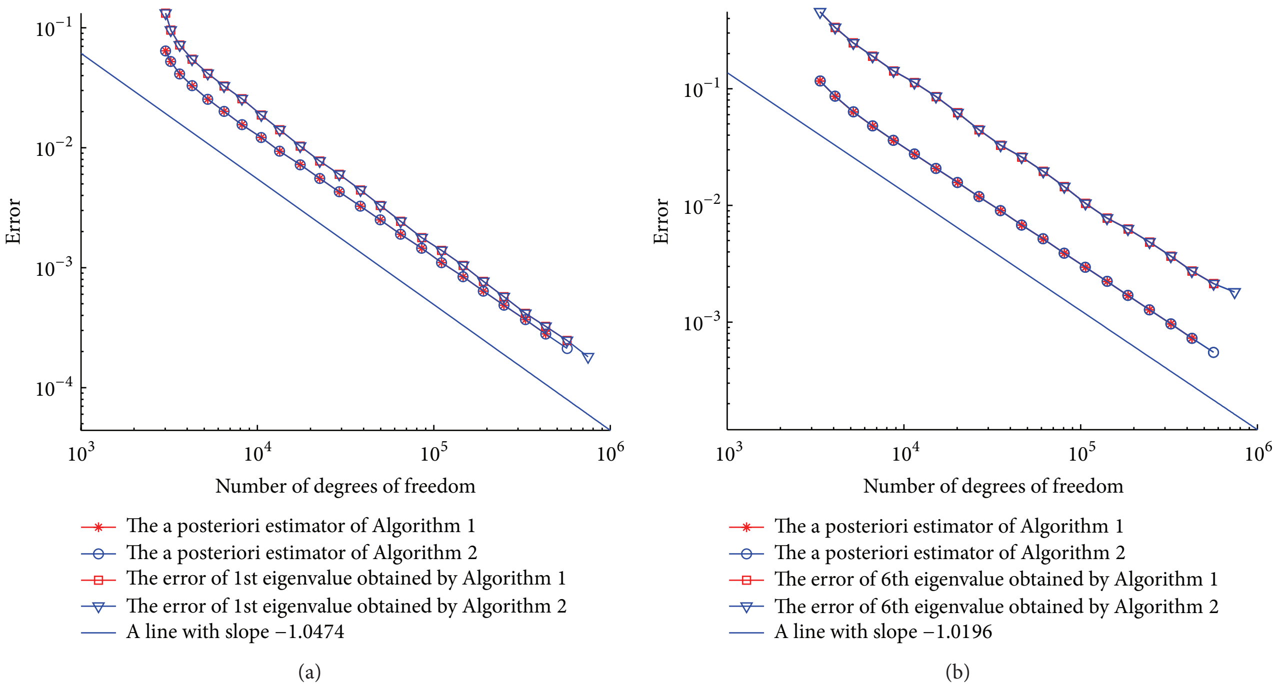


FIGURE 7: The error curves of two algorithms on slit domain.

But from Table 1, using Algorithm 11, we will spend much less time in the case of the same number of degrees of freedom but get the same accuracy to Algorithm 10. In addition, Algorithm 10, due to not having enough memory, can not proceed, while Algorithm 11 can have one more iteration; thus, more accurate numerical results will be obtained.

Example 2. We use Algorithms 10 and 11 to compute the approximate eigenvalues of (1) on $\Omega = ((0,2)\times(0,2))\backslash([1,2]\times\{1\})$ with a slit (see Figure 1(b)).

The first and sixth eigenvalues of (1) are $\lambda_1 \approx 8.3713297112$ and $\lambda_6 \approx 30.536$ on this domain, respectively. The associated numerical results are presented in Table 2 and Figures 5, 6, and 7. Figure 7 show that the error curves of the first and sixth approximate eigenvalues and the curves of the associated a posteriori error estimators obtained by Algorithms 10 and 11 are approximately parallel to the line with slope −1, respectively, which suffices to support our theory.

From Table 2, using Algorithm 11, compared with Algorithm 10, we can get the same accurate results in the case of the almost same degrees of freedom, but the CPU time is significantly decreased.

Remark 13. Based on the work of [30], we would like to believe that λ^{h_l} and Rayleigh quotient $a(I_{h_l}^c u^{h_l}, I_{h_l}^c u^{h_l})/b(I_{h_l}^c u^{h_l}, I_{h_l}^c u^{h_l})$ of $I_{h_l}^c u^{h_l}$ are the lower and upper bounds of the exact eigenvalue λ, respectively. To see this point, the numerical results of Tables 1 and 2 also illustrate that the C-R element eigenvalues approximate the exact ones of the Laplace operator from below. Thus, we can establish iterative control condition by computing λ^{h_l} and $a(I_{h_l}^c u^{h_l}, I_{h_l}^c u^{h_l})/b(I_{h_l}^c u^{h_l}, I_{h_l}^c u^{h_l})$ for the two algorithms.

Remark 14. For Algorithm 12, by calculating, in the case of the almost same number of degrees of freedom, we can get the same accurate results to Algorithm 11, and CPU time is almost the same; thus, we do not list the associated numerical results in this paper.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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