Research Article

The Lagrangian, Self-Adjointness, and Conserved Quantities for a Generalized Regularized Long-Wave Equation

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We consider the Lagrangian and the self-adjointness of a generalized regularized long-wave equation and its transformed equation. We show that the third-order equation has a nonlocal Lagrangian with an auxiliary function and is strictly self-adjoint; its transformed equation is nonlinearly self-adjoint and the minimal order of the differential substitution is equal to one. Then by Ibragimov's theorem on conservation laws we obtain some conserved qualities of the generalized regularized long-wave equation.

1. Introduction

The generalized regularized long-wave (GRLW) equation

$$u_t + u_x + au^m u_x + bu_{xxt} = 0 \tag{1}$$

was first put forward as a model for small-amplitude long waves on the surface of water in a channel by Peregrine [1, 2], where m is a positive integer, and a and b are two constants. It is an alternative description of nonlinear dispersive waves to the Korteweg de Vries equation [1] and has been used to describe phenomena with weak nonlinearity and dispersion waves, including nonlinear transverse waves in shallow water, ion-acoustic and magnetohydrodynamic waves in plasma and phonon packets in nonlinear crystals, pressure waves in liquid-gas bubble mixture, rotating flow down a tube, and lossless propagation of shallow water waves; see [3] and the references therein.

Equation (1) has been studied extensively in the past both analytically and numerically by means of, for example, the Adomian decomposition method, the Fourier method, the finite difference method, the finite element method, the variational iteration method, the mesh-free method, the cubic B-spline collocation procedure, the sine-cosine method, and the double reduction method [4–11], and so forth. A few of the conserved quantities by using the 1-soliton solution and solitary-wave solution of the $R(m, n)$ equation were calculated in [12]. In [11], the authors employed the Lie symmetry method and double reduction theory to obtain some exact solutions and derived the conservation laws for (1) via the so-called "partial Noether approach" after increasing its order. Noting that the GRLW equation is a third-order partial differential equation and it is difficult to obtain a variational formulation/Lagrangian for (1) by a classical approach, the authors converted the equation to the following fourth-order equation by assuming new dependent variable v to be the derivative of original dependent variable u by setting $u = v_x$ as follows:

$$v_{xt} + v_{xx} + av_x^m v_{xx} + bv_{xxxt} = 0. \tag{2}$$

The authors claimed that the conserved vectors constructed by partial Noether's theorem failed to satisfy the divergence relation and need to be adjusted to satisfy the divergence relationship. For the "partial Noether approach," in [13] Sarlet had given some negative comments (see Section 4 in [13]). So we will give some conservation laws of (1) by other methods.

In this paper, we investigate the Lagrangian and the self-adjointness and construct some new conservation laws for the generalized regularized long-wave equation by Ibragimov's theorem [14]. We should point out that the exact Lagrangian of (1) obtained in Section 2 is a nonlocal Lagrangian with an auxiliary function Ψ. In Section 3, we show that (1) is strictly self-adjoint, (2) is nonlinearly self-adjoint, and the minimal order of the differential substitution of (2) is equal to one.

Based on these facts, some new conservation laws for the GRLW equation (1) are derived.

2. The Auxiliary Lagrangian

In this section, we will study the Lagrangian for the generalized regularized long-wave equation (1). First, we briefly present some notations to be used in what follows. Let $x = (x_1, x_2, \ldots, x_n)$ be n independent variables, let $u = u(x)$ be dependent variables, let

$$D_i = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_i} + \cdots \tag{3}$$

be the total differentiation operator, and let

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{m \geq 1} (-1)^m D_{i_1} \cdots D_{i_m} \frac{\partial}{\partial u_{i_1 \cdots i_m}} \tag{4}$$

denote the Euler-Lagrange operator.

It is well known that a partial differential equation with odd order does not often admit a Lagrangian. However, noting the structure of (1), we can rewrite the equation as

$$\left[u + bu_{xx} \right]_t + \left[u + \frac{a}{m+1} u^{m+1} \right]_x = 0; \tag{5}$$

thus we find an auxiliary Lagrangian of (1) by the semi-inverse method [15]. Consequently some conserved quantities of (1) can be derived from Noether's theorem.

According to (5), we introduce an auxiliary function Ψ defined as

$$\Psi_x = u + bu_{xx}, \tag{6}$$

$$\Psi_t = -\left(u + \frac{a}{m+1} u^{m+1} \right), \tag{7}$$

so that (5) is automatically satisfied.

By the semi-inverse method [15], we construct a trial Lagrangian in the form

$$L(u, \Psi) = (u + bu_{xx}) \Psi_t - \left(u + \frac{a}{m+1} u^{m+1} \right) \Psi_x + F, \tag{8}$$

where F is an unknown function to be determined. The Euler-Lagrange equations of (8) are

$$-D_t (u + bu_{xx}) + D_x \left(u + \frac{a}{m+1} u^{m+1} \right) + \frac{\delta F}{\delta \Psi} = 0,$$
$$\Psi_t + bD_x^2 \Psi_t - \Psi_x (1 + au^m) + \frac{\delta F}{\delta u} = 0. \tag{9}$$

Substituting (6) and (7) into the above equations, we obtain that

$$\frac{\delta F}{\delta \Psi} = 2\Psi_{xt},$$

$$\frac{\delta F}{\delta u} = 2u + \frac{a(m+2)}{m+1} u^{m+1} + 2b(1 + au^m) u_{xx} \tag{10}$$

$$+ abmu^{m-1} u_x^2.$$

Thus, we can take F as

$$F = -\Psi_x \Psi_t + u^2 + \frac{a}{m+1} u^{m+2} - b(1 + au^m) u_x^2, \tag{11}$$

which implies that the Lagrangian of (1) reads

$$L = (u + bu_{xx}) \Psi_t - \left(u + \frac{au^{m+1}}{m+1} \right) \Psi_x - \Psi_x \Psi_t$$
$$+ u^2 \left(1 + \frac{au^m}{m+1} \right) - b(1 + au^m) u_x^2, \tag{12}$$

which is subjected to (6). Therefore, we demonstrate the following result.

Theorem 1. *Equation* (1) *admits a Lagrangian as follows:*

$$L = (u + bu_{xx}) \Psi_t - \left(u + \frac{au^{m+1}}{m+1} \right) \Psi_x - \Psi_x \Psi_t$$
$$+ u^2 \left(1 + \frac{au^m}{m+1} \right) - b(1 + au^m) u_x^2 \tag{13}$$

with constraint $\Psi_x = u + bu_{xx}$.

Proof. A direct calculation yields the Euler-Lagrangian equations of $L(u, \Psi)$ as follows:

$$-D_t (u + bu_{xx}) + D_x \left(u + \frac{a}{m+1} u^{m+1} \right) + 2\Psi_{xt} = 0, \tag{14}$$

$$\Psi_t - (1 + au^m) \Psi_x + 2u + \frac{a(m+2)}{m+1} u^{m+1}$$
$$+ mabu^{m-1} u_x^2 + b\Psi_{txx}$$
$$+ 2b(1 + au^m) u_{xx} = 0. \tag{15}$$

Considering the constraint equation (6), it is obvious that (14) is equivalent to (5) and (15) can be rewritten in the form of

$$\left(\Psi_t + u + \frac{a}{m+1} u^{m+1} \right) + b\left(\Psi_t + u + \frac{a}{m+1} u^{m+1} \right)_{xx} = 0. \tag{16}$$

Solving the last equation we obtain that

$$\Psi_t = -u - \frac{a}{m+1} u^{m+1} + P, \tag{17}$$

where $P = P(x, t)$ satisfies

$$bP_{xx} + P = 0. \tag{18}$$

Noting that $\Psi_x = u + bu_{xx}$, we deduce from $\Psi_{xt} = \Psi_{tx}$ that

$$u_t + bu_{xxt} + u_x + au^m u_x - P_x = 0, \tag{19}$$

which combining with (5) yields $P_x = 0$. Thus (18) implies $P = 0$ and (17) implies that (7) holds. $\square$

Remark 2. The Lagrangian (13) of (1) with constraint (6) implies that auxiliary equation (7) holds.

3. Self-Adjointness and Conservation Laws

In this section, we will investigate the self-adjointness and conservation laws of (1) by Ibragimov's theorem.

3.1. Self-Adjointness for (1) and (2). Let

$$H = u_t + u_x + au^m u_x + bu_{xxt};$$ (20)

then we have the following formal Lagrangian for (1):

$$\mathcal{L}_1 = w \left(u_t + u_x + au^m u_x + bu_{xxt} \right),$$ (21)

where $w = w(x,t)$ is a new function. Computing the variational derivative of this formal Lagrangian, we obtain the following adjoint equation of (1):

$$H^* = -w_t - w_x - aw_x u^m - bw_{xxt} = 0.$$ (22)

Assume that $H^*|_{w=\phi(t,x,u)} = \lambda H$, for a certain function λ. Then it is easy to obtain that $\phi = c_1 u + c_2$ and $\lambda = -\phi_u = -c_1$, where c_1, c_2 are two arbitrary constants. This implies that (1) is quasiself-adjoint; especially, if $c_1 = 1$ and $c_2 = 0$; that is, $\phi = u$, then (1) is strictly self-adjoint (see [16] for the definitions). Thus, we have demonstrated the following statement.

Theorem 3. *Equation* (1) *is quasiself-adjoint with the substitution* $w = c_1 u + c_2$, *where* $c_1 \neq 0$, c_2 *are two arbitrary constants. Especially,* (1) *is strictly self-adjoint with the substitution* $w = u$.

Next, we discuss the self-adjointness of (2), and then by the nonlinear self-adjointness and Ibragimov's theorem on conservation laws we construct some conserved quantities of (1).

Similar to the above, we set

$$E = v_{xt} + v_{xx} + av_x^m v_{xx} + bv_{xxxt};$$ (23)

then the formal Lagrangian for (2) is

$$\mathcal{L}_2 = w \left(v_{xt} + v_{xx} + av_x^m v_{xx} + bv_{xxxt} \right)$$ (24)

and the adjoint equation of (1) is

$$E^* = w_{xt} + w_{xx} + aw_{xx}v_x^m + amw_x v_x^{m-1} v_{xx} + bw_{xxxt} = 0.$$ (25)

According to [16], (2) will be nonlinearly self-adjoint if there exists a differential substitution

$$w = h \left(x, t, v, v_x, v_t, v_{xx}, \dots, \right), \quad h \neq 0,$$ (26)

involving a finite number of partial derivatives of v with respect to x and t, such that the equation

$$E^*|_{w=h} = \mu_0 E + \mu_1 D_x E + \mu_2 D_t E + \mu_3 D_x^2 E + \cdots$$ (27)

holds identically in the variables $t, x, u, u_x, u_t, \dots$, where $\mu_0, \mu_1, \dots$ are undetermined variable coefficients different from ∞ on the solutions of (2). The highest order of derivatives involved in u is called the *order of the differential substitution* (26). The calculation provides the following result.

Theorem 4. *Equation* (2) *is nonlinearly self-adjoint. The minimal order of the differential substitution* (26) *satisfying* (27) *is equal to one and is given by the function* $h = c_1 v_x + c_2 v_t + g(t)$, *where* c_1 *and* c_2 *are two constants and* $g(t)$ *is a differentiable function of* t.

3.2. Conservation Laws of (1). Now, we study the conservation laws of (1). We will first construct the conservation laws of (2) and then reduce them to the ones of (1). From the classical Lie group theory [17], we assume that a Lie point symmetry of (2) is a vector field

$$X = \xi^x (t,x,v) \frac{\partial}{\partial x} + \xi^t (t,x,v) \frac{\partial}{\partial t} + \rho (t,x,v) \frac{\partial}{\partial v}$$ (28)

on $\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ such that $X^{(4)} E = 0$ when $E = 0$, where E is given by (23). Taking into account (2), the operator $X^{(4)}$ is given as follows:

$$X^{(4)} = X + \rho_x^{(1)} \frac{\partial}{\partial v_x} + \rho_{xt}^{(2)} \frac{\partial}{\partial v_{xt}} + \rho_{xx}^{(2)} \frac{\partial}{\partial v_{xx}} + \rho_{xxxt}^{(4)} \frac{\partial}{\partial v_{xxxt}},$$ (29)

where

$$\rho_x^{(1)} = D_x \rho - \left(D_x \xi^x \right) v_x - \left(D_x \xi^t \right) v_t,$$

$$\rho_{xt}^{(2)} = D_t \rho_x^{(1)} - \left(D_t \xi^x \right) v_{xx} - \left(D_t \xi^t \right) v_{xt},$$

$$\rho_{xx}^{(2)} = D_x \left(\rho_x^{(1)} \right) - \left(D_x \xi^x \right) v_{xx} - \left(D_x \xi^t \right) v_{xt},$$ (30)

$$\rho_{xxxt}^{(4)} = D_x \left(\rho_{xxx}^{(3)} \right) - \left(D_x \xi^x \right) v_{xxxx} - \left(D_x \xi^t \right) v_{xxxt}.$$

The condition $X^{(4)} E|_{E=0} = 0$ yields determining equations. Solving these determining equations, we can obtain the symmetries of (5) as follows:

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = f(t) \frac{\partial}{\partial v}$$ (31)

with $f(t)$ as an arbitrary function. Ibragimov's theorem on conservation laws yields the following conserved quantities of (2) for a general Lie symmetry $X = \xi^x (\partial/\partial x) + \xi^t (\partial/\partial t) + \rho (\partial/\partial v)$. Consider the following:

$$C^t = \xi^t \mathcal{L}_2 - W \left(\frac{w_x}{2} + \frac{bw_{xxx}}{4} \right) + W_x \left(\frac{w}{2} + \frac{bw_{xx}}{4} \right)$$

$$- W_{xx} \left(\frac{bw_x}{4} \right) + W_{xxx} \left(\frac{bw}{4} \right),$$

$$C^x = \xi^x \mathcal{L}_2 + W \left(amv_x^{m-1} v_{xx} w - D_x \left[w \left(1 + av_x^m \right) \right] \right.$$

$$\left. - \frac{w_t}{2} - \frac{3bw_{xxt}}{4} \right)$$

$$+ W_x \left(\left(1 + av_x^m \right) w + \frac{bw_{xt}}{2} \right) + W_t \left(\frac{w}{2} + \frac{bw_{xx}}{4} \right)$$

$$- W_{xx} \left(\frac{bw_t}{4} \right)$$

$$- W_{xt} \frac{bw_x}{2} + W_{xxt} \left(\frac{3bw}{4} \right),$$

(32)

where $W = \rho - \xi^x v_x - \xi^t v_t$ is the Lie characteristic function and w is given by the differential substitution $w = c_1 v_x + c_2 v_t + g(t)$. Let us construct the conserved vector corresponding to the space translation group with the generator

$$X_1 = \frac{\partial}{\partial x}. \tag{33}$$

In this case $W = -v_x$. Taking account of the differential substitution $w = c_1 v_x + c_2 v_t + g(t)$ and $v_x = u$, from (32), we obtain the conserved quantities of (2), which can be transformed to the ones of (1) as follows:

$$
\begin{aligned}
C_1^t &= \frac{1}{4} c_2 \left(2uu_t + buu_{xxt} - 2u_x \int u_t dx \right.\\
&\qquad \left. - bu_x u_{xt} + bu_t u_{xx} - bu_{xxx} \int u_t dx \right) \\
&\quad - \frac{1}{4} bg(t) u_{xxx} - \frac{1}{2} g(t) u_x \\
&= \frac{1}{2} c_2 \left(2uu_t + buu_{xxt} + bu_t u_{xx} \right) \\
&\quad - \frac{1}{4} c_2 D_x \left(buu_{xt} + 2u \int u_t dx + bu_{xx} \int u_t dx \right) \\
&\quad - \frac{1}{4} D_x \left[bg(t) u_{xx} + 2g(t) u \right],
\end{aligned}
$$

$$
\begin{aligned}
C_1^x &= \frac{1}{4} c_2 \left(3buu_{xtt} - 2bu_x u_{tt} + bu_t u_{xt} + bu_{xx} \int u_{tt} dx \right.\\
&\qquad + 2u \int u_{tt} dx + 4au^{m+1} u_t \\
&\qquad \left. + 4uu_t + 2u_t \int u_t dx + bu_{xxt} \int u_t dx \right) \\
&\quad + \frac{b}{4} \left[g(t) u_{xxt} + g'(t) u_{xx} \right] \\
&\quad + c_1 u \left(u_x + u_t + au^m u_x + bu_{xxt} \right) + \frac{1}{2} \left[ug'(t) + g(t) u_t \right] \\
&= c_1 u \left(u_x + u_t + au^m u_x + bu_{xxt} \right) \\
&\quad + \frac{1}{2} c_2 \left(buu_{xtt} - bu_x u_{tt} + 2au^{m+1} u_t + 2uu_t \right) \\
&\quad + \frac{1}{4} c_2 D_t \left(buu_{xt} + 2u \int u_t dx + bu_{xx} \int u_t dx \right) \\
&\quad + \frac{1}{4} D_t \left[bg(t) u_{xx} + 2g(t) u \right],
\end{aligned}
\tag{34}
$$

which can be reduced to the form

$$
\begin{aligned}
C_1^t &= \frac{1}{2} c_2 \left(2uu_t + buu_{xxt} + bu_t u_{xx} \right), \\
C_1^x &= \frac{c_2}{2} \left(buu_{xtt} - bu_x u_{tt} + 2au^{m+1} u_t + 2uu_t \right).
\end{aligned}
\tag{35}
$$

Here we have used (1).

By the procedure analogous to that used above, we obtain the conserved quantities of (2) corresponding to the generators X_2 and X_3, which can be reduced by the differential substitutions $w = c_1 v_x + c_2 v_t + g(t)$ and $v_x = u$ to the conserved quantities of (1) as follows:

$$
\begin{aligned}
C_2^t &= \frac{c_1 b}{2} \left(uu_{xxt} - u_t u_{xx} \right) + ag(t) u^m u_x, \\
C_2^x &= \frac{c_1 b}{2} \left(u_x u_{tt} - uu_{xtt} \right) - ag(t) u^m u_t \\
&\quad + g'(t) \left(\int u_t dx + u + bu_{xt} \right), \\
C_3^t &= f(t) \left(c_1 u_x + c_2 u_t \right), \\
C_3^x &= f(t) \left(c_1 u_x + c_2 u_t \right) \left(1 + au^m \right) \\
&\quad + bf(t) \left(c_1 u_{xxt} + c_2 u_{xtt} \right) \\
&\quad - f'(t) \left(c_1 u + c_2 \int u_t dx + g(t) \right).
\end{aligned}
\tag{36}
$$

Here we have used (1).

Remark 5. In fact, (2) admits a Lagrangian

$$L_2 = -\frac{1}{2} v_x v_t - \frac{1}{2} v_x^2 - \frac{a}{(m+1)(m+2)} v_x^{m+2} + \frac{b}{2} v_{xt} v_{xx} \tag{37}$$

and has the following Noether symmetry generators:

$$X_5 = \frac{\partial}{\partial x}, \qquad X_6 = \frac{\partial}{\partial t},$$

$$X_7 = f(t) \frac{\partial}{\partial v} \quad \text{with gangue function } A^t = 0, \tag{38}$$

$$A^x = -\frac{1}{2} f'(t) v,$$

where $f(t)$ is a differentiable function in t. Noether's theorem on conservation laws [18] can yield the conserved quantities of (2), which are the same as the ones in [11]. Those were constructed by the so-called "partial Noether approach."

4. Conclusions

In this paper, we investigate the Lagrangians and self-adjointnesses of the GRLW equation (1) and its transformed equation (2). We show that (1) has a nonlocal Lagrangian with an auxiliary function and is strictly self-adjoint; its transformed equation is nonlinearly self-adjoint and the minimal order of the differential substitution is equal to one. Then by Ibragimov's theorem on conservation laws we obtain some conserved qualities of the generalized regularized long-wave equation.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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