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Research Article

Classification of the Quasifiliform Nilpotent Lie Algebras of Dimension 9

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On the basis of the family of quasifiliform Lie algebra laws of dimension 9 of 16 parameters and 17 constraints, this paper is devoted to identify the invariants that completely classify the algebras over the complex numbers except for isomorphism. It is proved that the nullification of certain parameters or of parameter expressions divides the family into subfamilies such that any couple of them is nonisomorphic and any quasifiliform Lie algebra of dimension 9 is isomorphic to one of them. The iterative and exhaustive computation with Maple provides the classification, which divides the original family into 263 subfamilies, composed of 157 simple algebras, 77 families depending on 1 parameter, 24 families depending on 2 parameters, and 5 families depending on 3 parameters.

1. Introduction

The interest in classifying nilpotent Lie algebras is broad both within the academic community and the industrial engineering community, since they are applied in classical mechanical problems and current research in scientific disciplines as modern geometry, solid state physics, or particle physics [1–5]. Lie algebras classification consists in determining equivalence relations that subdivide the original set in equivalence classes defined by at least one element in each set, and it is usual to classify the algebras except for isomorphisms. The solvable Lie algebras classification problem comes down in a sense to the nilpotent Lie algebras classification [6] and computer algebra has been indispensable. However, the more the dimension increases, the more and more complex is the determination of exhaustive lists of Lie algebras, so new computation methodologies are a present field of research [7, 8] with current symbolic manipulation programs such as Reduce, Mathematica, or Maple [9].

The classification of nilpotent Lie algebras over the complex numbers experimented an important advance based on the works of Ancochéa-Bermúdez and Goze [10] introducing an invariant more potent than the previously existing: the characteristic sequence or Goze's invariant (defined in Section 2.1). Those authors were able, by using the characteristic sequence as an invariant, to classify the nilpotent Lie algebras of dimension 7 [11] and the filiform Lie algebras of dimension 8 [12]. Later, by using that invariant, Gomez and Echarte [13] classify the filiform Lie algebras of dimension 9. Afterward, Castro et al. [14] develop an algorithm for symbolic language for finding the generic families of filiform Lie algebras in any dimension with the restrictions required to the parameters.

Subsequent works about quasifiliform Lie algebras classification were centered on specific types of families or subclasses, obtaining results applicable to higher dimensions. For instance, the classifications of naturally graded [15] and graded by derivations [16] quasifiliform Lie algebras. These works extended to other algebras, with a high nilindex, the classification of graded filiform Lie algebras, studied initially by Vergne [17, 18], obtained from the gradation related to the filtration produced in a natural way by the descending central sequence.

In this paper we focus on a method of identification of the invariants that completely classify the nilpotent Lie algebras of dimension 9 over the complex numbers except for isomorphisms. With this aim, the dimensions of the subalgebras of its derived series, of its descending central series, and of its descending central series centralizers are used as class invariants. The exhaustive analysis has been developed with significant computational effort; the total code is 2820 pages in 37 files, summing more than 12000 lines of Maple code, and these programs have provided 3038 pages of results [19]. We strongly recommend the reading of Bäuerle and de Kerf [20], Benjumea et al. [21], and Sendra et al. [22] to become familiar with Lie algebras terminology and symbolic computation with Maple.

2. The Subfamilies of Laws

2.1. Preliminaries. Let $\mathfrak{g}$ be a nilpotent Lie algebra; the characteristic sequence of $ad(X)$ is denoted by $c(X) = (c_1, \ldots, c_k, 1)$, and for the lexicographic order $c(\mathfrak{g}) = \mathrm{Max}_{X \in \mathfrak{g}-[\mathfrak{g},\mathfrak{g}]} c(X)$ is known as the Goze's invariant or characteristic sequence [23]. Obviously $c(\mathfrak{g})$ is an invariant for the isomorphisms and, by construction, there is at least one vector $X \in \mathfrak{g} - [\mathfrak{g}, \mathfrak{g}]$ such that $c(\mathfrak{g}) = c(X)$; all vector verifying this condition is called characteristic vector of the algebra.

The abelian algebra of dimension n is the only one with Goze's invariant $(1, \ldots, 1)$, in metabelian algebras the characteristic series is $(2, \ldots 2, 1, \ldots 1)$, in Heisenberg algebras it is $(2, 1, \ldots, 1)$, in filiform algebras it is $(n-1, 1)$, and in quasifiliform algebras it is $(n-2, 1, 1)$. A Lie algebra $\mathfrak{g}$ is nilpotent if and only if the characteristic polynomial of the matrix $ad(x)$ is λ^9, for every vector x of $\mathfrak{g}$. Anyway this condition is often difficult to be applied, so the moment in the process, when the nilpotence condition should be applied or, much better, when the condition should be applied for each vector, has to be chosen carefully. The condition of being quasifiliform can be also interpreted in terms of matrices. Thus the vectors candidate to characteristic vectors, that is, the vectors in $\mathfrak{g} - [\mathfrak{g}, \mathfrak{g}]$, have to satisfy that the respective adjoint matrices do not have nonnull minors of order $\leqslant 7$. As in the case of the nilpotence, this condition has to be applied with caution and in several stages.

Every quasifiliform Lie algebra of dimension 9 can have an adapted base $\{x_0, x_1, \ldots, x_8\}$ such that

$$[x_0, x_i] = x_{i+1}, \quad 1 \leqslant i \leqslant 6; \qquad [x_0, x_i] = 0, \quad 7 \leqslant i \leqslant 8. \tag{1}$$

On the whole all the bracket products can be described by

$$[x_i, x_j] = \sum_{k=0}^{n-1} C_{ij}^k \cdot x_k, \quad 0 \leqslant i, j \leqslant n-1, \tag{2}$$

where C_{ij}^k are the algebra structure constants.

The laws of every complex quasifiliform Lie algebra (QFLA) of dimension 9 can be described by the following family with 16 parameters and 17 polynomial restriction equations [19] derived from the Jacobi identity:

$$[x_0, x_i] = x_{i+1}, \quad 1 \leqslant i \leqslant 6, \tag{3a}$$

$$[x_1, x_2] = \alpha_1 x_4 + \alpha_2 x_5 + \alpha_3 x_6 + \alpha_4 x_7 + \alpha_5 x_8, \tag{3b}$$

$$[x_1, x_3] = \alpha_1 x_5 + \alpha_2 x_6 + \alpha_3 x_7, \tag{3c}$$

$$[x_1, x_4] = \alpha_6 x_5 + \alpha_7 x_6 + \alpha_8 x_7 + \alpha_9 x_8, \tag{3d}$$

$$[x_1, x_5] = 2\alpha_6 x_6 + (2\alpha_7 - \alpha_1) x_7, \tag{3e}$$

$$[x_1, x_6] = \alpha_{10} x_7 + \alpha_{11} x_8, \tag{3f}$$

$$[x_1, x_8] = \alpha_{12} x_3 + \alpha_{13} x_4 + \alpha_{14} x_5 + \alpha_{15} x_6 + \alpha_{16} x_7, \tag{3g}$$

$$[x_2, x_3] = -\alpha_6 x_5 + (\alpha_1 - \alpha_7) x_6 + (\alpha_2 - \alpha_8) x_7 - \alpha_9 x_8, \tag{3h}$$

$$[x_2, x_4] = -\alpha_6 x_6 + (\alpha_1 - \alpha_7) x_7, \tag{3i}$$

$$[x_2, x_5] = (2\alpha_6 - \alpha_{10}) x_7 - \alpha_{11} x_8, \tag{3j}$$

$$[x_2, x_8] = \alpha_{12} x_4 + \alpha_{13} x_5 + \alpha_{14} x_6 + \alpha_{15} x_7, \tag{3k}$$

$$[x_3, x_4] = (-3\alpha_6 + \alpha_{10}) x_7 + \alpha_{11} x_8, \tag{3l}$$

$$[x_3, x_8] = \alpha_{12} x_5 + \alpha_{13} x_6 + \alpha_{14} x_7, \tag{3m}$$

$$[x_4, x_8] = \alpha_{12} x_6 + \alpha_{13} x_7, \tag{3n}$$

$$[x_5, x_8] = \alpha_{12} x_7 \tag{3o}$$

subject to

$$\alpha_5 \alpha_{12} = 0, \tag{4a}$$

$$\alpha_6 \alpha_{12} = 0, \tag{4b}$$

$$\alpha_6 \alpha_{13} = 0, \tag{4c}$$

$$\alpha_9 \alpha_{12} = 0, \tag{4d}$$

$$\alpha_9 \alpha_{13} = 0, \tag{4e}$$

$$\alpha_9 \alpha_{14} = 0, \tag{4f}$$

$$\alpha_{10} \alpha_{12} = 0, \tag{4g}$$

$$\alpha_{11} \alpha_{12} = 0, \tag{4h}$$

$$\alpha_{11} \alpha_{13} = 0, \tag{4i}$$

$$\alpha_{11} \alpha_{14} = 0, \tag{4j}$$

$$\alpha_{11} \alpha_{15} = 0, \tag{4k}$$

$$\alpha_{11} \alpha_{16} = 0, \tag{4l}$$

$$\alpha_{11} (3\alpha_1 - \alpha_7) = 0, \tag{4m}$$

$$\alpha_{12} (\alpha_1 - \alpha_7) = 0, \tag{4n}$$

$$\alpha_5 \alpha_{13} - 2\alpha_6^2 - \alpha_9 \alpha_{15} = 0, \tag{4o}$$

$$2(\alpha_2 - \alpha_8)\alpha_{12} + 3(\alpha_1 - \alpha_7)\alpha_{13} + 2(\alpha_6 - \alpha_{10})\alpha_{14} = 0, \tag{4p}$$

$$\alpha_5 \alpha_{14} - 2(2\alpha_1 + \alpha_7)\alpha_6 - \alpha_9 \alpha_{16} + (3\alpha_1 - \alpha_7)\alpha_{10} = 0 \tag{4q}$$

with the application of the Jacobi identity to the 3-tuple (x_0, x_i, x_j), where x_i, x_j are base vectors different from

Table 1: Notation for the QFLA parameters.

$\alpha_1 = C_{12}^4$	$\alpha_2 = C_{12}^5$	$\alpha_3 = C_{12}^6$	$\alpha_4 = C_{12}^7$
$\alpha_5 = C_{12}^8$	$\alpha_6 = C_{14}^5$	$\alpha_7 = C_{14}^6$	$\alpha_8 = C_{14}^7$
$\alpha_9 = C_{14}^8$	$\alpha_{10} = C_{16}^7$	$\alpha_{11} = C_{16}^8$	$\alpha_{12} = C_{18}^3$
$\alpha_{12} = C_{18}^4$	$\alpha_{14} = C_{18}^5$	$\alpha_{15} = C_{18}^6$	$\alpha_{16} = C_{18}^7$

x_0 vector. Table 1 shows the structure constants corresponding with the 16 parameters. From here forward the Lie Algebra Families will be denoted as $\mu(\alpha_{1_1}, \ldots, \alpha_{16})$.

Our objective is to study exhaustively the case of dimension 9; therefore the coefficients identification is tackled in an iterative and interactive way by imposing the Jacobi identity. Maple programs have been developed so that all the equations resulting from the application of the abovementioned conditions are obtained, the simplest conditions are applied, and the process is repeated until there are no restrictions of simple application.

The exhaustiveness of the classification is developed by analyzing *all* the possible combinations of values of the 16 parameters $(\alpha_1, \ldots, \alpha_{16})$, which is summarized within the cases shown in the following subsections: case A.1 ($\alpha_{11} \neq 0$ and $\alpha_1 \neq 0$), A.2 ($\alpha_{11} \neq 0$ and $\alpha_1 = 0$), B.1.1 ($\alpha_{11} = 0$, $\alpha_9 \neq 0$, and $\alpha_6 \neq 0$), B.1.2 ($\alpha_{11} = 0$, $\alpha_9 \neq 0$, and $\alpha_6 = 0$), B.2.1 ($\alpha_{11} = 0$, $\alpha_9 = 0$, and $\alpha_5 \neq 0$), and B.2.2 ($\alpha_{11} = 0$, $\alpha_9 = 0$, and $\alpha_5 = 0$). In all the cases the nonisomorphism is proved in the corresponding propositions.

2.2. General Case

Proposition 1. *The nilpotent QFLA of dimension 9 and $\alpha_{11} \neq 0$ are nonisomorphic to the algebras with $\alpha_{11} = 0$.*

Proof. For the family described by (3a)–(3o) and (4a)–(4q), its descending central series is $\mathscr{C}^1\mathfrak{g} = \langle x_2, x_3, x_4, x_5, x_6, x_7, \alpha_5 x_8, \alpha_9 x_8, \alpha_{11} x_8\rangle$, $\mathscr{C}^2\mathfrak{g} = \langle x_3, x_4, x_5, x_6, x_7, \alpha_5 x_8, \alpha_9 x_8, \alpha_{11} x_8\rangle$, $\mathscr{C}^3\mathfrak{g} = \langle x_4, x_5, x_6, x_7, \alpha_9 x_8, \alpha_{11} x_8\rangle$, $\mathscr{C}^4\mathfrak{g} = \langle x_5, x_6, x_7, \alpha_9 x_8, \alpha_{11} x_8\rangle$, $\mathscr{C}^5\mathfrak{g} = \langle x_6, x_7, \alpha_{11} x_8\rangle$, and so forth. Thus $\mathrm{Dim}[\mathscr{C}^5\mathfrak{g}] = 3$ if $\alpha_{11} \neq 0$ and $\mathrm{Dim}[\mathscr{C}^5\mathfrak{g}] = 2$ if $\alpha_{11} = 0$. Therefore the nullity of α_{11} constitutes the first classification criterion. □

2.3. Case A. $\alpha_{11} \neq 0$.

Proposition 2. *The nilpotent QFLA of dimension 9 with $\alpha_{11} \neq 0 \wedge \alpha_1 \neq 0$ are nonisomorphic to the algebras with $\alpha_{11} \neq 0 \wedge \alpha_1 = 0$.*

Proof. If $\alpha_{11} \neq 0$, from restrictions (4a)–(4q) it can be deduced that $\alpha_6 = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = \alpha_{16} = 0$ and $\alpha_7 = 3\alpha_1$. By computing the Jacobi equations, the family of laws is reduced to

$$[x_0, x_i] = x_{i+1}, \quad 1 \leqslant i \leqslant 6, \tag{5a}$$

$$[x_1, x_2] = \alpha_1 x_4 + \alpha_2 x_5 + \alpha_3 x_6 + \alpha_4 x_7 + \alpha_5 x_8, \tag{5b}$$

$$[x_1, x_3] = \alpha_1 x_5 + \alpha_2 x_6 + \alpha_3 x_7, \tag{5c}$$

$$[x_1, x_4] = 3\alpha_1 x_6 + \alpha_8 x_7 + \alpha_9 x_8, \tag{5d}$$

$$[x_1, x_5] = 5\alpha_1 x_7, \tag{5e}$$

$$[x_1, x_6] = \alpha_{10} x_7 + \alpha_{11} x_8, \tag{5f}$$

$$[x_2, x_3] = -2\alpha_1 x_6 + (\alpha_2 - \alpha_8) x_7 - \alpha_9 x_8, \tag{5g}$$

$$[x_2, x_4] = -2\alpha_1 x_7, \tag{5h}$$

$$[x_2, x_5] = -\alpha_{10} x_7 - \alpha_{11} x_8, \tag{5i}$$

$$[x_3, x_4] = \alpha_{10} x_7 + \alpha_{11} x_8 \tag{5j}$$

without restrictions derived from the Jacobi identity (4a)–(4q). It can be observed that x_7 and x_8 are now central; thus the application of the elementary change of base

$$\begin{aligned} y_i &= x_i, \quad i \neq 8, \\ y_8 &= \alpha_{10} \cdot x_7 + \alpha_{11} \cdot x_8 \end{aligned} \tag{6}$$

permits us to suppose that $\alpha_{10} = 0$ and $\alpha_{11} = 1$. Then (5f), (5i), and (5j) are simplified and the derived series is $\mathscr{D}^1\mathfrak{g} = \langle x_2, x_3, x_4, x_5, x_6, x_7, x_8\rangle$, $\mathscr{D}^2\mathfrak{g} = \langle -2\alpha_1 x_6, (\alpha_2 - \alpha_8)x_7, -\alpha_9 x_8, -2\alpha_1 x_7, x_8\rangle$, and so forth. Thus $\mathrm{Dim}[\mathscr{D}^2\mathfrak{g}] = 3$ if $\alpha_1 \neq 0$ and $\mathrm{Dim}[\mathscr{D}^2\mathfrak{g}] \leqslant 2$ if $\alpha_1 = 0$. Therefore the nullity of α_1 constitutes a new classification criterion. □

In this subsection (case A), the notation to describe the parameters of the subfamily i is reduced to μ_i $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7, \alpha_8, \alpha_9)$ for simplification. Figure 1 shows the classification in 19 subfamilies in the case A. They are classified with the criteria summarized in Figure 2 and detailed in the following cases.

2.3.1. Case A.1. One has $\alpha_{11} \neq 0$ and $\alpha_1 \neq 0$.

Proposition 3. *Case A.1 permits us to suppose that $\alpha_1 = 1$.*

Proof. With the elementary change of base CB,

$$\begin{aligned} y_0 &= x_0, \\ y_i &= \frac{x_i}{\alpha_1}, \quad 1 \leqslant i \leqslant 7, \\ y_8 &= \frac{x_8}{\alpha_1^2}, \end{aligned} \tag{7}$$

$|\mathrm{CB}| = 1/\alpha_1^9 \neq 0$ and then $\alpha_1 = 1$. □

The subfamilies of laws with the structure (5a)–(5j) are $\mu_i(1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, 3, \alpha_8, \alpha_9)$ with $i = 1, 2, 3$.

Let us denote, from here forward, by δ the new parameters obtained from the changes of base and $\overline{\mu}$ the Lie algebra families depending on these new parameters δ (which in general depend on the 16 parameters α), in order to differentiate the new representation $\overline{\mu}_i(\delta_j)$ from the representation of the families depending in general on the 16 parameters $\mu_i(\alpha_1, \ldots, \alpha_{16})$.

μ	α_1	α_2	α_3	α_4	α_5	α_7	α_8	α_9
1	1	0	0	0	0	3	1	λ
2	1	0	0	0	0	3	0	1
3	1	0	0	0	0	3	0	0
4	0	λ	1	0	0	0	1	0
5	0	1/4	1	λ	0	0	1	0
6	0	1	1	λ	0	0	0	0
7	0	λ	0	1	0	0	1	0
8	0	1	0	1	0	0	4	0
9	0	1	0	0	0	0	4	0
10	0	1	0	1	0	0	0	0
11	0	1	0	0	0	0	0	0
12	0	0	1	0	0	0	1	0
13	0	0	0	1	0	0	1	0
14	0	1	1	0	0	0	1	0
15	0	1	0	0	0	0	1	0
16	0	0	1	1	0	0	0	0
17	0	0	1	0	0	0	0	0
18	0	0	0	1	0	0	0	0
19	0	0	0	0	0	0	0	0

Aa Classification criterion
Aa Classifcation criterion from previous cases
Aa Additional result
Aa Additional result from previous cases
Aa Direct result

Figure 1: Case A: $\alpha_{11} \neq 0$; classification of the nonisomorphic QFLA of dimension 9 (the last 7 values, 0, 1, 0, 0, 0, 0, 0, common to the 19 families have been omitted from the figure for simplicity).

Proposition 4. *The nilpotent QFLA of dimension 9 with $\alpha_{11} \neq 0$ and $\alpha_1 \neq 0$ can be classified in three nonisomorphic subfamilies μ_i with i from 1 to 3, described in Figure 1, according to the conditions described in Figure 2.*

Proof. Let us apply the change of base:

$$\begin{aligned} y_0 &= P_0x_0 + P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 \\ &\quad + P_5x_5 + P_6x_6 + P_7x_7 + P_8x_8, \\ y_1 &= Q_0x_0 + Q_1x_1 + Q_2x_2 + Q_3x_3 + Q_4x_4 \\ &\quad + Q_5x_5 + Q_6x_6 + Q_7x_7 + Q_8x_8, \\ y_{i+1} &= [y_0, y_i], \quad 1 \leqslant i \leqslant 6, \\ y_8 &= [y_1, y_6]. \end{aligned} \tag{8}$$

The subfamilies of laws are $\overline{\mu}_i(\delta_2, \delta_3, \delta_4, \delta_5, \delta_8, \delta_9)$. The determinant of the change matrix is $P_0^{18}(P_0Q_1 - P_1Q_0)^9$; thus $P_0 \neq 0$ and $(P_0Q_1 - P_1Q_0) \neq 0$. Since the coefficient of y_3 and y_4 in $[y_1, y_2]$ must be null, then $Q_0 = 0$ and $Q_1 = P_0^2$. Let us apply (8) again and the only restriction is $P_0 \neq 0$. Thus, with the coefficient identifications, the new parameters are

$$\delta_2 = \frac{(-2P_1 + \alpha_2 P_0)}{P_0^2}, \tag{9}$$

$$\delta_3 = \frac{(-2Q_2^2 + 4P_0^2Q_3 - 7P_1\alpha_2P_0^3 + 7P_1^2P_0^2 + \alpha_3P_0^4)}{P_0^6}, \tag{10}$$

$$\begin{aligned} \delta_4 = \big(&\alpha_4P_0^6 - 2\alpha_2^2P_0^5P_1 - 2\alpha_2P_0^4Q_3 + \alpha_2P_0^2Q_2^2 \\ &- \alpha_2P_0^5P_1\alpha_8 + 2Q_2^3 - 12P_1\alpha_3P_0^5 \\ &- 42P_1P_0^3Q_3 - 44P_1^3P_0^3 + 56P_0^4P_1^2\alpha_2 \\ &+ 18P_1Q_2^2P_0 - 6P_0^2Q_3Q_2 + 6P_0^4Q_4 \\ &+ 6P_2P_0^3Q_2 + P_0^4P_1^2\alpha_8 - 6P_0^5P_3 \\ &+ 2\alpha_8P_0^4Q_3 - Q_2^2P_0^2\alpha_8\big) \times \left(P_0^9\right)^{-1}, \end{aligned} \tag{11}$$

$$\begin{aligned} \delta_5 = \big(&2P_0P_1Q_3Q_2 + 2P_2\alpha_2P_0^3Q_2 + 6P_1^2\alpha_3P_0^4 \\ &- 2P_2P_0Q_2^2 - 2P_0^2Q_4Q_2 + 6P_0^4P_3P_1 \\ &- 8P_1^2Q_2^2 - 2P_0P_1Q_2^2\alpha_2 - 4P_1P_2P_0^2Q_2 \\ &+ 2P_2P_0^3Q_3 + 2P_3P_0^3Q_2 - 4P_1P_0^3Q_4 \\ &+ 20P_0^2P_1^2Q_3 - Q_2^2P_0^2\alpha_9 + 2Q_3P_0^4\alpha_9 \\ &+ P_1Q_2^2P_0\alpha_8 - P_1P_0^5\alpha_4 + P_0^2Q_3^2 + 2P_0^4Q_5 \\ &+ P_0^6\alpha_5 + 2P_1P_0^3Q_3\alpha_2 - 2P_1\alpha_8P_0^3Q_3 \\ &+ \alpha_2P_0^4P_1^2\alpha_8 - \alpha_2P_0^5P_1\alpha_9 - 2P_0^5P_3\alpha_2 \\ &- 2P_0^5P_4 + 15P_0^2P_1^4 - P_0^4P_2^2 + P_0^4P_1^2\alpha_9 \\ &- 20P_0^3P_1^3\alpha_2 - P_0^3P_1^3\alpha_8\big) \times \left(P_0^{10}\right)^{-1}, \end{aligned} \tag{12}$$

$$\delta_8 = \frac{(-20P_1 + \alpha_8P_0)}{P_0^2}, \tag{13}$$

$$\delta_9 = \frac{(8P_1^2P_0^2 - P_0^3P_1\alpha_8 + P_0^4\alpha_9 + 2P_0^2Q_3 + 2P_1\alpha_2P_0^3 - Q_2^2)}{P_0^6}. \tag{14}$$

Let us select P_1, Q_3, P_3, and P_4 appropriately and the subfamilies of laws result in $\overline{\mu}_i(0, 0, 0, 0, \delta_8, \delta_9)$

$$\begin{aligned} [y_0, y_i] &= y_{i+1}, \quad 1 \leqslant i \leqslant 6, \\ [y_1, y_2] &= y_4, \\ [y_1, y_3] &= y_5, \\ [y_1, y_4] &= 3y_6 + \delta_8y_7 + \delta_9y_8, \end{aligned}$$

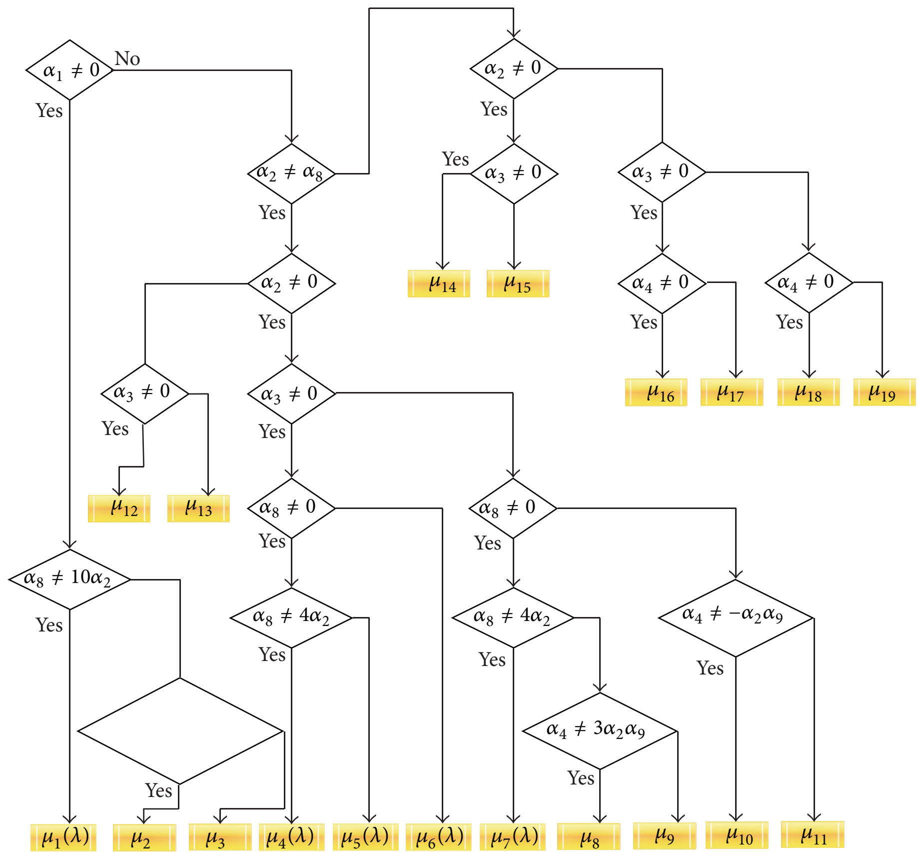

FIGURE 2: Case A: $\alpha_{11} \neq 0$; nonisomorphic subfamilies determination.

$$
\begin{aligned}
[y_1, y_5] &= 5y_7, \\
[y_1, y_6] &= y_8, \\
[y_2, y_3] &= -2y_6 - \delta_8 y_7 - \delta_9 y_8, \\
[y_2, y_4] &= -2y_7, \\
[y_2, y_5] &= -y_8, \\
[y_3, y_4] &= y_8.
\end{aligned}
\tag{15}
$$

The nullity of $-10\alpha_2 + \alpha_8$ and the nullity of $31\alpha_2^2 + 8\alpha_9 - 4\alpha_3 - 4\alpha_2\alpha_8$ are invariants. It can be proved by substituting (9) and (13) in the expressions $-10\delta_2 + \delta_8$ and $31\delta_2^2 + 8\delta_9 - 4\delta_3 - 4\delta_2\delta_8$; then P_0 is in the denominator and since it must be nonnull, the nullity of those expressions is invariant for the change (8). Then four subcases are determined by the nullity of δ_8 and δ_9. If $\delta_8 \neq 0$, the subcases corresponding to δ_9 nullity or nonnullity can be reduced to a subfamily of algebras $\boldsymbol{\mu}_1^{\boldsymbol{\lambda}}(\mathbf{1,0,0,0,0,0,3,1,\boldsymbol{\lambda},0,1})$ with one parameter $\boldsymbol{\lambda} \in \mathbf{C}$, with $\lambda = (1/8)(-31\alpha_2^2 + 8\alpha_9 + 4\alpha_3 + 4\alpha_2\alpha_8)/(-31\alpha_2 + 31\alpha_8)^2$. If $\delta_8 = 0$, in the subcase corresponding to $\delta_9 \neq 0$, P_0 can be selected as $(-18\alpha_2^2 + 16\alpha_9 - 8\alpha_3)^{1/2}/4$ since $\alpha_9 \neq (-31\alpha_2^2 + 4\alpha_3 + 4\alpha_2\alpha_8)/8$ and $\alpha_8 = 10\alpha_2$ imply $-9\alpha_2^2 + 8\alpha_9 - 4\alpha_3 \neq 0$. Thus $\delta_9 = 1$, and the algebra is described by $\boldsymbol{\mu}_2(\mathbf{1,0,0,0,0,0,3,0,1,0,1})$. Finally, the forth subcase is the algebra $\boldsymbol{\mu}_3(\mathbf{1,0,0,0,0,0,3,0,0,0,1})$. □

2.3.2. Case A.2. One has $\alpha_{11} \neq 0$ and $\alpha_1 = 0$.

Proposition 5. *The nilpotent QFLA of dimension 9 with $\alpha_{11} \neq 0$, $\alpha_1 = 0$, and $\alpha_2 = \alpha_8$ are nonisomorphic to the algebras with $\alpha_{11} \neq 0$, $\alpha_1 = 0$, and $\alpha_2 \neq \alpha_8$.*

Proof. The derived series is $\mathcal{D}^1\mathfrak{g} = \langle x_2, x_3, x_4, x_5, x_6, x_7, x_8 \rangle$, $\mathcal{D}^2\mathfrak{g} = \langle (\alpha_2 - \alpha_8)x_7, x_8 \rangle$, and so forth; thus $\mathrm{Dim}[\mathcal{D}^2\mathfrak{g}] = 2$ if $\alpha_2 \neq \alpha_8$ and $\mathrm{Dim}[\mathcal{D}^2\mathfrak{g}] = 1$ if $\alpha_2 = \alpha_8$. The nullity of $\alpha_2 - \alpha_8$ constitutes a new classification criterion. □

Proposition 6. *The nilpotent QFLA of dimension 9 with $\alpha_{11} \neq 0$, $\alpha_1 = 0$, and $\alpha_2 \neq \alpha_8$ can be classified in ten nonisomorphic subfamilies μ_j with j from 4 to 13, described in Figure 1, according to the conditions described in Figure 2.*

Proof. Let us apply the change of base (8). The subfamilies of laws are $\overline{\mu}_j(\delta_2, \delta_3, \delta_4, \delta_5, \delta_8, \delta_9)$ and the restrictions $P_0 \neq 0$ and $Q_1 \neq 0$. Thus, with the coefficient identifications, the new parameters are

$$\delta_2 = \frac{Q_1\alpha_2}{P_0^3}, \tag{16}$$

$$\delta_3 = \frac{Q_1\alpha_3}{P_0^3}, \tag{17}$$

$$\begin{aligned}\delta_4 = &-\left(2Q_1^2P_1\alpha_2^2 + 2Q_3P_0Q_1\alpha_2 - Q_2^2P_0\alpha_2\right.\\ &+ \alpha_2Q_1^2P_1\alpha_8 - Q_1^2P_0\alpha_4\\ &\left.-2Q_1P_0Q_3\alpha_8 + P_0Q_2^2\alpha_8\right) \times \left(Q_1P_0^6\right)^{-1},\end{aligned} \tag{18}$$

$$\begin{aligned}\delta_5 = &\left(-\alpha_2Q_1^2P_1P_0\alpha_9 - 2\alpha_2Q_1^2P_3P_0 + \alpha_2Q_1^2P_1^2\alpha_8\right.\\ &+ 2P_0P_2Q_1\alpha_2Q_2 - P_1P_0\alpha_4Q_1^2 - 2P_1P_0Q_1Q_3\alpha_8\\ &- 2P_1P_0\alpha_2Q_2^2 + P_1P_0Q_2^2\alpha_8 + 2P_1P_0\alpha_2Q_1Q_3\\ &+ Q_1^2P_0^2\alpha_5 + 2Q_1P_0^2Q_3\alpha_9 + 2Q_5P_0^2Q_1\\ &\left.-Q_2^2P_0^2\alpha_9 - 2Q_4P_0^2Q_2 + Q_3^2P_0^2\right) \times \left(Q_1^2P_0^6\right)^{-1},\end{aligned} \tag{19}$$

$$\delta_8 = \frac{\alpha_8Q_1}{P_0^3}, \tag{20}$$

$$\delta_9 = \frac{\left(-Q_1^2P_1\alpha_8 + Q_1^2P_0\alpha_9 + 2P_0Q_3Q_1 + 2Q_1^2P_1\alpha_2 - P_0Q_2^2\right)}{P_0^3Q_1^2}. \tag{21}$$

Selecting Q_5 and Q_3 appropriately the subfamilies of laws result in $\overline{\mu}_j(\delta_2, \delta_3, \delta_4, 0, \delta_8, 0)$. From (16), (17), and (20) the invariance of the nullities of α_2, α_3, and α_8 is clear. The nullity of $4\alpha_2 - \alpha_8$ is also invariant. It can be proved by substituting (16) and (20) in the expression $4\delta_2 - \delta_8$ considering the change restrictions. If $\boldsymbol{\alpha_{11} \neq 0}$, $\alpha_1 = \mathbf{0}$, $\boldsymbol{\alpha_2 \neq \alpha_8}$, $\boldsymbol{\alpha_2 \neq 0}$, $\alpha_3 \neq \mathbf{0}$, $\alpha_8 \neq \mathbf{0}$, and $\alpha_8 \neq \mathbf{4}\boldsymbol{\alpha_2}$, selecting $Q_1 = P_0^3/\alpha_8 \neq 0$, $P_0 = \alpha_3/\alpha_8 \neq 0$, and $P_1 = \alpha_3(-\alpha_8\alpha_9 + \alpha_4 + \alpha_2\alpha_9)/(\alpha_8^2(-\alpha_8 + 4\alpha_2))$, the subfamily is $\boldsymbol{\mu_4^\lambda}(\mathbf{0}, \boldsymbol{\lambda}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0})$, with $\delta_2 = \boldsymbol{\alpha_2/\alpha_8} = \boldsymbol{\lambda} \in \mathbf{C} - \{\mathbf{0}, \mathbf{1}, \mathbf{1/4}\}$. If $\boldsymbol{\alpha_{11} \neq 0}$, $\alpha_1 = \mathbf{0}$, $\boldsymbol{\alpha_2 \neq \alpha_8}$, $\boldsymbol{\alpha_2 \neq 0}$, $\alpha_3 \neq \mathbf{0}$, $\alpha_8 \neq \mathbf{0}$, and $\alpha_8 = \mathbf{4}\boldsymbol{\alpha_2}$, then δ_4 is invariant, and the subfamily is $\boldsymbol{\mu_5^\lambda}(\mathbf{0}, \mathbf{1/4}, \mathbf{1}, \boldsymbol{\lambda}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0})$, $\boldsymbol{\lambda} \in \mathbf{C}$. If $\boldsymbol{\alpha_{11} \neq 0}$, $\alpha_1 = \mathbf{0}$, $\alpha_2 \neq \boldsymbol{\alpha_8}$, $\boldsymbol{\alpha_2 \neq 0}$, $\alpha_3 \neq \mathbf{0}$, and $\boldsymbol{\alpha_8} = \mathbf{0}$, selecting $Q_1 = P_0^4/\alpha_3 \neq 0$ and $P_0 = \alpha_3/\alpha_2 \neq 0$, the subfamily is $\boldsymbol{\mu_6^\lambda}(\mathbf{0}, \mathbf{1}, \mathbf{1}, \boldsymbol{\lambda}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$, $\boldsymbol{\lambda} \in \mathbf{C}$. If $\boldsymbol{\alpha_{11} \neq 0}$, $\alpha_1 = \mathbf{0}$, $\alpha_2 \neq \boldsymbol{\alpha_8}$, $\alpha_2 \neq \mathbf{0}$, $\alpha_3 = \mathbf{0}$, $\boldsymbol{\alpha_8 \neq 0}$, and $\boldsymbol{\alpha_8 \neq 4\alpha_2}$, selecting $Q_1 = P_0^3/\alpha_8 \neq 0$ and $P_1 = P_0(-\alpha_9\alpha_8 + \alpha_2\alpha_9 + \alpha_4 - P_0^2\alpha_8)/(\alpha_8(-\alpha_8 + 4\alpha_2))$, the subfamily is $\boldsymbol{\mu_7^\lambda}(\mathbf{0}, \boldsymbol{\lambda}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0})$, $\boldsymbol{\lambda} \in \mathbf{C} - \{\mathbf{0}, \mathbf{1}, \mathbf{1/4}\}$. If $\boldsymbol{\alpha_{11} \neq 0}$, $\alpha_1 = \mathbf{0}$, $\boldsymbol{\alpha_2 \neq \alpha_8}$, $\alpha_2 \neq \mathbf{0}$, $\alpha_3 = \mathbf{0}$, $\boldsymbol{\alpha_8 \neq 0}$, and $\boldsymbol{\alpha_8} = \mathbf{4}\boldsymbol{\alpha_2}$, selecting $Q_1 = P_0^3/\alpha_2$, the nullity of $-\alpha_4 + 3\alpha_2\alpha_9$ is invariant. If $\alpha_4 \neq 3\alpha_2\alpha_9$, selecting $P_0^2 = -\alpha_2(-\alpha_4 + 3\alpha_2\alpha_9)/\alpha_2^2$, the subfamily is $\boldsymbol{\mu_8}(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{4}, \mathbf{0})$, else the subfamily is $\boldsymbol{\mu_9}(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{4}, \mathbf{0})$. If $\boldsymbol{\alpha_{11} \neq 0}$, $\alpha_1 = \mathbf{0}$, $\alpha_2 \neq \boldsymbol{\alpha_8}$, $\boldsymbol{\alpha_2 \neq 0}$, $\alpha_3 = \mathbf{0}$, and $\boldsymbol{\alpha_8} = \mathbf{0}$, selecting $Q_1 = P_0^3/\alpha_2$, the nullity of $\alpha_2\alpha_9 + \alpha_4$ is invariant. If $\boldsymbol{\alpha_4 \neq -\alpha_2\alpha_9}$, selecting $P_0^2 = \alpha_2(\alpha_2\alpha_9 + \alpha_4)/\alpha_2^2 \neq 0$, the subfamily is $\boldsymbol{\mu_{10}}(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$; else with $Q_1 = P_0^3/\alpha_2$ the subfamily is $\boldsymbol{\mu_{11}}(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$. If $\boldsymbol{\alpha_{11} \neq 0}$, $\alpha_1 = \mathbf{0}$, $\alpha_2 \neq \boldsymbol{\alpha_8}$, $\boldsymbol{\alpha_2} = \mathbf{0}$, and $\alpha_3 \neq \mathbf{0}$, selecting $Q_1 = P_0^4/\alpha_3 \neq 0$, $P_0 = \alpha_3/\alpha_8 \neq 0$, and $P_1 = -\alpha_3(\alpha_4 - \alpha_9\alpha_8)/\alpha_8^3$, the subfamily is $\boldsymbol{\mu_{12}}(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0})$. If $\boldsymbol{\alpha_{11} \neq 0}$, $\alpha_1 = \mathbf{0}$, $\alpha_2 \neq \boldsymbol{\alpha_8}$, $\boldsymbol{\alpha_2} = \mathbf{0}$, and $\alpha_3 = \mathbf{0}$, selecting $Q_1 = P_0^3/\alpha_8 \neq 0$ and $P_1 = P_0(-\alpha_4 + \alpha_9\alpha_8 + P_0^2\alpha_8)/\alpha_8^2 \neq 0$, the subfamily is $\boldsymbol{\mu_{13}}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0})$. □

Proposition 7. *The nilpotent QFLA of dimension 9 with $\alpha_{11} \neq 0$, $\alpha_1 = 0$, and $\alpha_2 = \alpha_8$ can be classified in six nonisomorphic subfamilies μ_k with k from 14 to 19, described in Figure 1, according to the conditions described in Figure 2.*

Proof. Let us apply the change of base (8). The subfamilies of laws are $\overline{\mu}_k(\delta_2, \delta_3, \delta_4, \delta_5, \delta_8, \delta_9)$ and the restrictions $P_0 \neq 0$ and $Q_1 \neq 0$. Thus, with the coefficient identifications, the new parameters are

$$\delta_2 = \frac{Q_1\alpha_2}{P_0^3}, \tag{22}$$

$$\delta_3 = \frac{Q_1\alpha_3}{P_0^4}, \tag{23}$$

$$\delta_4 = \frac{Q_1\left(-3\alpha_2^2P_1 + P_0\alpha_4\right)}{P_0^6}, \tag{24}$$

$$\begin{aligned}\delta_5 = &\left(-\alpha_2Q_1^2P_1P_0\alpha_9 - 2\alpha_2Q_1^2P_3P_0 + \alpha_2^2Q_1^2P_1^2\right.\\ &+ 2P_2P_0Q_1\alpha_2Q_2 - P_0Q_1^2P_1\alpha_4 + Q_1^2P_0^2\alpha_5\\ &+ 2Q_1P_0^2Q_3\alpha_9 + 2Q_5P_0^2Q_1 - Q_2^2P_0^2\alpha_9\\ &\left.-2Q_4P_0^2Q_2 - P_0P_1Q_2^2\alpha_2 + Q_3^2P_0^2\right) \times \left(Q_1^2P_0^6\right)^{-1},\end{aligned} \tag{25}$$

$$\delta_9 = \frac{\left(P_1Q_1^2\alpha_2 + Q_1^2P_0\alpha_9 + 2P_0Q_3Q_1 - P_0Q_2^2\right)}{\left(P_0^3Q_1^2\right)}. \tag{26}$$

Let us select Q_5 and Q_3 appropriately and the subfamilies of laws result in $\overline{\mu}_k(\delta_2, \delta_3, \delta_4, 0, 0)$. From (22), (23), and (24) the invariance of the nullities of α_2, α_3, and α_4 is clear. If $\boldsymbol{\alpha_{11} \neq 0}$, $\alpha_1 = \mathbf{0}$, $\alpha_2 = \boldsymbol{\alpha_8}$, $\boldsymbol{\alpha_2 \neq 0}$, and $\alpha_3 \neq \mathbf{0}$, selecting $Q_1 = P_0^3/\alpha_2 \neq 0$, $P_0 = \alpha_3/\alpha_2 \neq 0$, and $P_1 = \alpha_3\alpha_4/(3\alpha_2^3)$, the subfamily is $\boldsymbol{\mu_{14}}(\mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0})$. If $\boldsymbol{\alpha_{11} \neq 0}$, $\alpha_1 = \mathbf{0}$, $\alpha_2 = \boldsymbol{\alpha_8}$, $\boldsymbol{\alpha_2 \neq 0}$, and $\alpha_3 = \mathbf{0}$, selecting $Q_1 = P_0^3/\alpha_2 \neq 0$ and $P_1 = P_0\alpha_4/(3\alpha_2^2)$, the subfamily is $\boldsymbol{\mu_{15}}(\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0})$. If $\boldsymbol{\alpha_{11} \neq 0} \wedge \boldsymbol{\alpha_1} = \mathbf{0}$, $\alpha_2 = \boldsymbol{\alpha_8}$, $\boldsymbol{\alpha_2} = \mathbf{0}$, and $\alpha_3 \neq \mathbf{0}$, selecting $Q_1 = P_0^4/\alpha_3 \neq 0$, the nullity of α_4 is invariant. If $\alpha_4 \neq 0$ and $P_0 = \alpha_4/\alpha_3 \neq 0$, the subfamily is $\boldsymbol{\mu_{16}}(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$. If $\alpha_4 = 0$ the subfamily is $\boldsymbol{\mu_{17}}(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$. If $\boldsymbol{\alpha_{11} \neq 0}$, $\alpha_1 = \mathbf{0}$, $\alpha_2 = \boldsymbol{\alpha_8}$, $\boldsymbol{\alpha_2} = \mathbf{0}$, and $\alpha_3 = \mathbf{0}$, selecting $Q_1 = P_0^5/\alpha_4 \neq 0$, the subfamilies are $\boldsymbol{\mu_{18}}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$ with $\alpha_4 \neq 0$, and $\boldsymbol{\mu_{19}}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0}, \mathbf{0})$ with $\alpha_4 = 0$. □

2.4. Case B. One has $\alpha_{11} = 0$.

Proposition 8. *The nilpotent QFLA of dimension 9 and $\alpha_{11} = 0$ and $\alpha_9 \neq 0$ are nonisomorphic to the algebras with $\alpha_{11} = 0$ and $\alpha_9 = 0$.*

Proof. For the family described by (3a)–(3o) and (4a)–(4q), $\mathscr{C}^3\mathfrak{g} = \langle x_4, x_5, x_6, x_7, \alpha_9x_8\rangle$; therefore its dimension is $\mathrm{Dim}[\mathscr{C}^3\mathfrak{g}] = 5$, if $\alpha_9 \neq 0$ or $\mathrm{Dim}[\mathscr{C}^3\mathfrak{g}] = 4$, if $\alpha_9 = 0$, and the nullity of α_9 constitutes a new classification criterion. □

2.4.1. Case B.1. One has $\alpha_{11} = 0$ and $\alpha_9 \neq 0$.

Proposition 9. *The nilpotent QFLA of dimension 9 with $\alpha_{11} = 0$, $\alpha_9 \neq 0$, and $\alpha_6 \neq 0$ are nonisomorphic to the algebras with $\alpha_{11} = 0$, $\alpha_9 \neq 0$, and $\alpha_6 = 0$.*

Proof. If $\alpha_9 \neq 0$, it can be deduced that $\alpha_{12} = \alpha_{13} = \alpha_{14} = 0$. By computing the Jacobi equations, the family of laws is reduced to

$$[x_0, x_i] = x_{i+1}, \quad 1 \leqslant i \leqslant 6, \tag{27}$$

$$[x_1, x_2] = \alpha_1 x_4 + \alpha_2 x_5 + \alpha_3 x_6 + \alpha_4 x_7 + \alpha_5 x_8, \tag{28}$$

$$[x_1, x_3] = \alpha_1 x_5 + \alpha_2 x_6 + \alpha_3 x_7, \tag{29}$$

$$[x_1, x_4] = \alpha_6 x_5 + \alpha_7 x_6 + \alpha_8 x_7 + \alpha_9 x_8, \tag{30}$$

$$[x_1, x_5] = 2\alpha_6 x_6 + (2\alpha_7 - \alpha_1) x_7, \tag{31}$$

$$[x_1, x_6] = \alpha_{10} x_7, \tag{32}$$

$$[x_1, x_8] = \alpha_{15} x_6 + \alpha_{16} x_7, \tag{33}$$

$$[x_2, x_3] = -\alpha_6 x_5 + (\alpha_1 - \alpha_7) x_6 + (\alpha_2 - \alpha_8) x_7 - \alpha_9 x_8, \tag{34}$$

$$[x_2, x_4] = -\alpha_6 x_6 + (\alpha_1 - \alpha_7) x_7, \tag{35}$$

$$[x_2, x_5] = (2\alpha_6 - \alpha_{10}) x_7, \tag{36}$$

$$[x_2, x_8] = \alpha_{15} x_7, \tag{37}$$

$$[x_3, x_4] = (-3\alpha_6 + \alpha_{10}) x_7 \tag{38}$$

with two restrictions

$$\begin{gathered} -2\alpha_6^2 - \alpha_9 \alpha_{15} = 0, \\ -4\alpha_1\alpha_6 + 3\alpha_1\alpha_{10} - 2\alpha_6\alpha_7 - \alpha_7\alpha_{10} - \alpha_9\alpha_{16} = 0. \end{gathered} \tag{39}$$

Since $\alpha_9 \neq 0$, the application of the elementary change of base CB

$$\begin{aligned} y_0 &= y_1 \\ y_i &= \frac{x_i}{\alpha_9}, \quad 1 \leqslant i \leqslant n-1 \\ &\text{with } |\mathrm{CB}| = \frac{1}{\alpha_9^8} \neq 0 \end{aligned} \tag{40}$$

permits us to suppose that $\alpha_9 = 1$. Then from (39), $\alpha_{15} = -2\alpha_6^2$ and $\alpha_{16} = -4\alpha_1\alpha_6 + 3\alpha_1\alpha_{10} - 2\alpha_6\alpha_7 - \alpha_7\alpha_{10}$. This implies that (33) and (37) are changed to $[x_1, x_8] = -2\alpha_6^2 x_6 + (\alpha_1(-4\alpha_6 + 3\alpha_{10}) - \alpha_7(2\alpha_6 + \alpha_{10}))x_7$ and $[x_2, x_8] = -2\alpha_6^2 x_7$, respectively, and the subfamily of laws $\mu_l(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, 1, \alpha_{10}, 0, 0, 0, 0, 0, 0)$, with l from 20 to 63, has no restrictions (4a)–(4q). Its derived series is $\mathscr{D}^1\mathfrak{g} = \langle x_2, x_3, x_4, x_5, x_6, x_7, x_8\rangle$, $\mathscr{D}^2\mathfrak{g} = \langle -\alpha_6 x_5 + (\alpha_1 - \alpha_7)x_6 + (\alpha_2 - \alpha_8)x_7, x_8, -\alpha_6 x_6 + (\alpha_1 - \alpha_7)x_7, (2\alpha_6 - \alpha_{10})x_7, -2\alpha_6^2 x_7, (-3\alpha_6 + \alpha_{10})x_7\rangle$, and so forth. Thus $\mathrm{Dim}[\mathscr{D}^2\mathfrak{g}] = 4$ if $\alpha_6 \neq 0$ and $\mathrm{Dim}[\mathscr{D}^2\mathfrak{g}] \leqslant 3$ if $\alpha_6 = 0$. Therefore the nullity of α_6 constitutes a new classification criterion. □

Case B.1.1: $\alpha_{11} = 0$, $\alpha_9 \neq 0$ and $\alpha_6 \neq 0$. Figure 3 provides the classification in 18 subfamilies in this case.

Case B.1.2: $\alpha_{11} = 0$, $\alpha_9 \neq 0$, and $\alpha_6 = 0$. Figure 4 provides the classification in 26 subfamilies in this case.

2.4.2. Case B.2. One has $\alpha_{11} = 0$ and $\alpha_9 = 0$.

The restrictions in the family (4a)–(4q) are reduced to

$$\alpha_5\alpha_{12} = 0, \tag{41}$$

$$\alpha_6\alpha_{12} = 0, \tag{42}$$

$$\alpha_6\alpha_{13} = 0, \tag{43}$$

$$\alpha_{10}\alpha_{12} = 0, \tag{44}$$

$$\alpha_{12}(\alpha_1 - \alpha_7) = 0, \tag{45}$$

$$\alpha_5\alpha_{13} - 2\alpha_6^2 = 0, \tag{46}$$

$$2(\alpha_2 - \alpha_8)\alpha_{12} + 3(\alpha_1 - \alpha_7)\alpha_{13} + 2(\alpha_6 - \alpha_{10})\alpha_{14} = 0, \tag{47}$$

$$\alpha_5\alpha_{14} - 2(2\alpha_1 + \alpha_7)\alpha_6 - \alpha_9\alpha_{16} + (3\alpha_1 - \alpha_7)\alpha_{10} = 0. \tag{48}$$

Proposition 10. *The nilpotent QFLA of dimension 9 with $\alpha_{11} = 0$, $\alpha_9 = 0$, and $\alpha_5 \neq 0$ are nonisomorphic to the algebras with $\alpha_{11} = 0$, $\alpha_9 = 0$, and $\alpha_5 = 0$.*

Proof. Equations (43) and (46) imply that $\alpha_6 = 0$. By computing the Jacobi equations, the subfamily of laws is $\mu_m(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, 0, \alpha_7, \alpha_8, 0, \alpha_{10}, 0, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15}, \alpha_{16})$, with m from 64 to 263, and the restrictions are reduced to 6. Its descending central series is $\mathscr{C}^1\mathfrak{g} = \langle x_2, x_3, x_4, x_5, x_6, x_7, \alpha_5 x_8\rangle$, and so forth. Thus the nullity of α_5 constitutes a new classification criterion. □

An exhaustive and extensive process of analysis with the same methodology shown in the previous subsections leads to the final subclassification, which is summarized in the following Figures.

Case B.2.1: $\alpha_{11} = 0$, $\alpha_9 = 0$, and $\alpha_5 \neq 0$. Figure 5 provides the classification in 55 subfamilies in this case.

Case B.2.2: $\alpha_{11} = 0$, $\alpha_9 = 0$, and $\alpha_5 = 0$. Figures 6 and 7 provide the classification in 145 subfamilies in this case.

3. Concluding Remarks

Computational aid has been indispensable in this piece of research. A PC Pentium 4 of 2.4 GHz and the programming language Maple 6 have been used in the process. The library modules developed represent approximately 12,000 lines of code. In some cases, in this massive application of computational resources and looking for the simplification of some laws, procedures that perhaps can be considered of "inverse engineering" have been used in order to find some very complex changes of base, which have allowed us to

μ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}
20	1	0	0	0	λ_1	1	λ_2	0	1	1	0	0	0	0	-2	$-1-3\lambda_2$
21	1	0	0	0	0	1	λ	0	1	3	0	0	0	0	-2	$5-5\lambda$
22	1	0	0	0	2/3	1	0	0	1	3	0	0	0	0	-2	5
23	1	0	0	0	2/3	1	7/9	0	1	3	0	0	0	0	-2	10/9
24	0	0	0	0	1	1	λ	0	1	$\neq 3$	0	0	0	0	-2	-3λ
25	0	0	0	0	1	1	1	0	1	3	0	0	0	0	-2	-5
26	0	0	0	0	1	1	0	0	1	3	0	0	0	0	-2	0
27	0	0	0	0	0	1	1	0	1	1	0	0	0	0	-2	-3
28	0	0	0	0	0	1	0	0	1	1	0	0	0	0	-2	0
29	0	0	0	0	0	1	1	0	1	3	0	0	0	0	-2	-5
30	0	0	0	0	0	1	0	0	1	3	0	0	0	0	-2	0
31	1	0	0	0	0	1	λ	0	1	0	0	0	0	0	-2	$-4-2\lambda$
32	1	0	0	0	2/3	1	0	0	1	0	0	0	0	0	-2	-4
33	1	0	0	0	2/3	1	$-1/3$	0	1	0	0	0	0	0	-2	$-10/3$
34	0	0	0	0	1	1	1	0	1	0	0	0	0	0	-2	-2
35	0	0	0	0	1	1	0	0	1	0	0	0	0	0	-2	0
36	0	0	0	0	0	1	1	0	1	0	0	0	0	0	-2	-2
37	0	0	0	0	0	1	0	0	1	0	0	0	0	0	-2	0

Aa Classifcation criterion
Aa Classifcation criterion from previous cases
Aa Additional result from previous cases
Aa Direct result

Figure 3: Case B.1.1: $\alpha_{11} = 0$, $\alpha_9 \neq 0$, and $\alpha_6 \neq 0$; classification of the QFLA of dimension 9.

μ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}
38	1	λ_1	λ_2	0	0	0	1	0	1	1	0	0	0	0	0	2
39	1	1	λ	0	0	0	3	0	1	1	0	0	0	0	0	0
40	1	-2	1	0	0	0	3	0	1	1	0	0	0	0	0	0
41	1	-2	5	0	0	0	3	0	1	1	0	0	0	0	0	0
42	0	1	1	0	0	0	λ	0	1	1	0	0	0	0	0	$-\lambda$
43	0	1	0	0	0	0	1	0	1	1	0	0	0	0	0	-1
44	0	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
45	0	0	1	0	0	0	1	0	1	1	0	0	0	0	0	-1
46	0	0	1	0	0	0	0	0	1	1	0	0	0	0	0	0
47	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	-1
48	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
49	1	0	1	0	0	0	λ	0	1	0	0	0	0	0	0	0
50	1	0	0	0	0	0	λ	0	1	0	0	0	0	0	0	0
51	1	0	1	λ	0	0	1	0	1	0	0	0	0	0	0	0
52	1	0	0	1	0	0	1	0	1	0	0	0	0	0	0	0
53	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0
54	0	1	1	0	0	0	1	0	1	0	0	0	0	0	0	0
55	0	1	1	0	0	0	0	0	1	0	0	0	0	0	0	0
56	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0
57	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0
58	0	0	1	0	0	0	1	0	1	0	0	0	0	0	0	0
59	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0
60	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0
61	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0
62	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0
63	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0

Aa Classification criterion
Aa Classifcation criterion from previous cases
Aa Additional result
Aa Additional result from previous cases
Aa Direct result

Figure 4: Case B.1.2: $\alpha_{11} = 0$; $\alpha_9 \neq 0$, and $\alpha_6 = 0$; classification of the QFLA of dimension 9.

μ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}
64	1	1	$\lambda 1$	0	1	0	3	0	0	1	0	0	0	0	$\lambda 2 \neq 0, 3$	$\lambda 3$
65	1	0	$\lambda 1$	0	1	0	3	0	0	1	0	0	0	0	3	$\lambda 2$
66	1	$\lambda 1 - 2$	$\lambda 2$	0	1	0	3	0	0	1	0	0	0	0	$\lambda 1 \neq 0, 3$	0
67	1	$\lambda - 2$	0	0	1	0	3	0	0	1	0	0	0	0	$\lambda \neq 0, 3, 31/8$	-5λ
68	1	15/8	1	0	1	0	3	0	0	1	0	0	0	0	31/8	$-155/8$
69	1	$\lambda - 2$	$31 - 8\lambda$	0	1	0	3	0	0	1	0	0	0	0	$\lambda <> 0, 3$	-5λ
70	1	1	λ	0	1	0	3	0	0	1	0	0	0	0	3	0
71	1	1	0	0	1	0	3	0	0	1	0	0	0	0	3	-15
72	1	1	7	0	1	0	3	0	0	1	0	0	0	0	3	-15
73	1	$\lambda 1$	$\lambda 2$	0	1	0	3	0	0	1	0	0	0	0	0	1
74	1	0	λ	0	1	0	3	0	0	1	0	0	0	0	0	0
75	1	-2	0	0	1	0	3	0	0	1	0	0	0	0	0	0
76	1	-2	31	0	1	0	3	0	0	1	0	0	0	0	0	0
77	0	$\lambda 1$	$\lambda 2$	0	1	0	0	0	0	1	0	0	0	0	1	1
78	0	λ	1	0	1	0	0	0	0	1	0	0	0	0	1	0
79	0	0	0	0	1	0	0	0	0	1	0	0	0	0	1	0
80	0	1	0	0	1	0	0	0	0	1	0	0	0	0	1	0
81	0	1	λ	0	1	0	0	0	0	1	0	0	0	0	0	1
82	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	1
83	0	0	-1	0	1	0	0	0	0	1	0	0	0	0	0	1
84	0	1	1	0	1	0	0	0	0	1	0	0	0	0	0	0
85	0	1	0	0	1	0	0	0	0	1	0	0	0	0	0	0
86	0	0	1	0	1	0	0	0	0	1	0	0	0	0	0	0
87	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0
88	1	0	0	0	1	0	$\lambda 1 \neq 1/2$	$\lambda 2$	0	0	0	0	0	0	$\lambda 3 \neq 2$	1
89	1	0	0	0	1	0	$\lambda 1 \neq 1/2$	1	0	0	0	0	0	0	$\lambda 2 \neq 2$	0
90	1	1	0	0	1	0	$\lambda 1 \neq 0$	$\lambda 2$	0	0	0	0	0	0	2	0
91	1	1	0	0	1	0	0	0	0	0	0	0	0	0	2	λ
92	1	0	0	0	1	0	$\lambda \neq 0$	1	0	0	0	0	0	0	2	0
93	1	0	0	0	1	0	$\lambda \neq 0$	0	0	0	0	0	0	0	2	0
94	1	0	0	0	1	0	0	0	0	0	0	0	0	0	2	1
95	1	0	0	0	1	0	0	0	0	0	0	0	0	0	2	0
96	0	1	0	0	1	0	1	0	0	0	0	0	0	0	$\lambda 1$	$\lambda 2$
97	0	0	0	0	1	0	1	0	0	0	0	0	0	0	λ	1
98	0	0	0	0	1	0	1	0	0	0	0	0	0	0	λ	0
99	1	$\lambda 1$	$\lambda 2$	0	1	0	1	0	0	0	0	0	0	0	$\lambda 3$	1
100	1	$\lambda 1$	1	0	1	0		0	0	0	0	0	0	0	$\lambda 2$	0
101	1	1	0	0	1	0	1	0	0	0	0	0	0	0	λ	0
102	1	0	0	0	1	0	1	0	0	0	0	0	0	0	λ	0
103	1	$\lambda 1$	0	0	1	0	1/2	0	0	0	0	0	0	0	$\lambda 2$	1
104	1	1	0	0	1	0	1/2	0	0	0	0	0	0	0	λ	0
105	1	0	0	0	1	0	1/2	0	0	0	0	0	0	0	λ	0
106	0	$\lambda 1$	0	0	1	0	0	1	0	0	0	0	0	0	$\lambda 2$	1
107	0	0	λ	0	1	0	0	1	0	0	0	0	0	0	1	0
108	0	λ	1	0	1	0	0	1	0	0	0	0	0	0	0	0
109	0	λ	0	0	1	0	0	1	0	0	0	0	0	0	0	0
110	0	0	λ	0	1	0	0	0	0	0	0	0	0	0	1	1
111	0	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0
112	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0
113	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	1
114	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0
115	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0
116	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1
117	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0
118	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0

Aa Classification criterion
Aa Classifcation criterion from previous cases
Aa Additional result
Aa Additional result from previous cases
Aa Direct result

FIGURE 5: Case B.2.1: $\alpha_{11} = 0$, $\alpha_9 = 0$, $\alpha_5 \neq 0$; classification of the QFLA of dimension 9.

μ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}
119	1	0	0	0	0	0	3	0	0	1	0	0	1	−3	1	λ
120	1	0	0	0	0	0	3	0	0	1	0	0	1	−3	9/2	0
121	1	0	0	0	0	0	3	0	0	1	0	0	1	−3	9/2	15/2
122	1	0	0	0	0	0	3	$\lambda 1$	0	1	0	0	0	0	1	$\lambda 2$
123	1	−2	0	0	0	0	3	λ	0	1	0	0	0	0	1	0
124	1	−2	0	0	0	0	3	0	0	1	0	0	0	0	1	−5
125	1	−2	0	0	0	0	3	−31/2	0	1	0	0	0	0	1	−5
126	1	0	0	0	0	0	3	λ	0	1	0	0	0	0	0	1
127	1	−2	0	0	0	0	3	0	0	1	0	0	0	0	0	1
128	1	−2	0	0	0	0	3	−31/2	0	1	0	0	0	0	0	1
129	1	0	0	0	0	0	3	λ	0	1	0	0	0	0	0	0
130	1	−2	0	0	0	0	3	0	0	1	0	0	0	0	0	0
131	1	−2	0	0	0	0	3	−31/2	0	1	0	0	0	0	0	0
132	0	0	1	0	0	0	0	0	0	1	0	0	1	−3	0	0
133	0	0	1	0	0	0	0	0	0	1	0	0	1	−3	0	-1
134	0	0	0	0	0	0	0	0	0	1	0	0	1	−3	0	1
135	0	0	0	0	0	0	0	0	0	1	0	0	1	−3	0	0
136	0	1	1	0	0	0	0	0	0	1	0	0	0	0	1	λ
137	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0	1
138	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0	0
139	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1	1
140	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1	0
141	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	1
142	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0
143	0	0	1	0	0	0	0	0	0	1	0	0	0	0	1	1
144	0	0	1	0	0	0	0	0	0	1	0	0	0	0	1	0
145	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	1
146	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0
147	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1
148	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0
149	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1
150	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
151	$\lambda 1 \neq 0$	0	0	0	0	0	$\lambda 1 - 1$	1	0	0	0	0	0	1	$\lambda 2$	$\lambda 3$
152	$\lambda 1 \neq 0$	0	0	0	0	0	$\lambda 1 - 1$	0	0	0	0	0	0	1	1	$\lambda 2$
153	$\lambda \neq 0$	0	0	0	0	0	$\lambda - 1$	0	0	0	0	0	0	1	0	1
154	$\lambda \neq 0$	0	0	0	0	0	$\lambda - 1$	0	0	0	0	0	0	1	0	0
155	$\lambda 1 \neq 0$	0	0	0	0	0	$\lambda 1 - 1$	$\lambda 2$	0	0	0	0	0	0	1	1
156	$\lambda \neq 0$	0	0	0	0	0	$\lambda - 1$	1	0	0	0	0	0	0	1	0
157	$\lambda \neq 0$	0	0	0	0	0	$\lambda - 1$	0	0	0	0	0	0	0	1	0
158	$\lambda \neq 0$	0	0	0	0	0	$\lambda - 1$	1	0	0	0	0	0	0	0	1
159	$\lambda \neq 0$	0	0	0	0	0	$\lambda - 1$	0	0	0	0	0	0	0	0	1
160	$\lambda \neq 0$	0	0	0	0	0	$\lambda - 1$	1	0	0	0	0	0	0	0	0
161	$\lambda \neq 0$	0	0	0	0	0	$\lambda - 1$	0	0	0	0	0	0	0	0	0
162	0	1	0	0	0	0	−1	0	0	0	0	0	0	1	$\lambda 1$	$\lambda 2$
163	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	1	λ
164	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	0	1
165	0	1	0	0	0	0	−1	0	0	0	0	0	0	0	0	0
166	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	1	λ
167	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	1
168	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	0
169	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	1	1
170	0	0	0	0	0	0	−1	0	0	0	0	0	0	0	1	0
171	0	0	0	0	0	0	−1	0	0	0	0	0	0	1	0	1
172	0	0	0	0	0	0	−1	0	0	0	0	0	0	2	0	0
173	1	0	0	0	0	0	1	−1	0	0	0	0	1	0	$\lambda 1$	$\lambda 2$
174	1	0	0	0	0	0	1	−1	0	0	0	0	0	1	$\lambda 1$	$\lambda 2$
175	1	0	$\lambda 1$	0	0	0	1	−1	0	0	0	0	0	0	1	$\lambda 2$
176	1	0	λ	0	0	0	1	−1	0	0	0	0	0	0	0	1
177	1	0	λ	0	0	0	1	−1	0	0	0	0	0	0	0	0
178	0	0	λ	0	0	0	0	−1	0	0	0	0	1	1	0	0
179	0	0	2/3	0	0	0	0	−1	0	0	0	0	1	1	0	λ
180	0	0	1	0	0	0	0	−1	0	0	0	0	1	0	0	0
181	0	0	0	0	0	0	0	−1	0	0	0	0	1	0	0	1
182	0	0	0	0	0	0	0	−1	0	0	0	0	1	0	0	0
183	0	$\lambda \neq 1/2$	0	0	0	0	0	$\lambda - 1$	0	0	0	0	0	1	1	0
184	0	1/2	0	0	0	0	0	−1/2	0	0	0	0	0	1	1	λ
185	0	$\lambda \neq 1/2$	0	0	0	0	0	$\lambda - 1$	0	0	0	0	0	1	0	0
186	0	1/2	0	0	0	0	0	−1/2	0	0	0	0	0	1	0	1
187	0	1/2	0	0	0	0	0	−1/2	0	0	0	0	0	1	0	0
188	0	$\lambda 1$	$\lambda 2$	0	0	0	0	$\lambda 1 - 1$	0	0	0	0	0	0	1	1
189	0	λ	1	0	0	0	0	$\lambda - 1$	0	0	0	0	0	0	1	0
190	0	λ	0	0	0	0	0	$\lambda - 1$	0	0	0	0	0	0	1	0

Aa Classification criterion
Aa Classifcation criterion from previous cases
Aa Additional result
Aa Additional result from previous cases
Aa Direct result

FIGURE 6: Case B.2.2. First part: $\alpha_{11} = 0$, $\alpha_9 = 0$, and $\alpha_5 = 0$. Classification of the QFLA of dimension 9.

μ	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}	α_{15}	α_{16}
191	0	λ	1	0	0	0	0	$\lambda-1$	0	0	0	0	0	0	0	1
192	0	λ	0	0	0	0	0	$\lambda-1$	0	0	0	0	0	0	0	1
193	0	λ	1	0	0	0	0	$\lambda-1$	0	0	0	0	0	0	0	0
194	0	λ	0	0	0	0	0	$\lambda-1$	0	0	0	0	0	0	0	0
195	0	1	1	$\lambda 1$	0	0	0	1	0	0	0	1	0	$\lambda 2$	0	$\lambda 3$
196	0	1	0	1	0	0	0	1	0	0	0	1	0	0	$\lambda 1$	$\lambda 2$
197	0	1	0	0	0	0	0	1	0	0	0	1	0	0	1	λ
198	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	1
199	0	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0
200	0	0	1	1	0	0	0	0	0	0	0	1	0	$\lambda 1$	$\lambda 2$	0
201	0	0	1	0	0	0	0	0	0	0	0	1	0	1	0	λ
202	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1
203	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0
204	0	0	0	1	0	0	0	0	0	0	0	1	0	1	λ	0
205	0	0	0	1	0	0	0	0	0	0	0	1	0	0	1	0
206	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0
207	0	0	0	0	0	0	0	0	0	0	0	1	0	1	$\lambda 1$	$\lambda 2$
208	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	λ
209	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
210	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
211	1	0	1	$\lambda 1$	0	0	1	0	0	0	0	0	1	0	0	$\lambda 2$
212	1	0	0	1	0	0	1	0	0	0	0	0	1	0	0	λ
213	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	1
214	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0
215	0	0	1	λ	0	0	0	0	0	0	0	0	1	1	0	1
216	0	0	0	1	0	0	0	0	0	0	0	0	1	1	0	λ
217	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	λ
218	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0
219	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0	0
220	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	1
221	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0
222	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1
223	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
224	1	0	0	0	0	0	1	0	0	0	0	0	0	1	1	λ
225	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1
226	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
227	0	1	0	0	0	0	0	1	0	0	0	0	0	1	1	λ
228	0	0	0	1	0	0	0	0	0	0	0	0	0	1	1	λ
229	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	λ
230	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	1
231	0	1	0	0	0	0	0	1	0	0	0	0	0	1	0	0
232	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	1
233	0	0	0	1	0	0	0	0	0	0	0	0	0	1	0	0
234	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
235	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
236	1	0	λ	0	0	0	1	0	0	0	0	0	0	0	1	1
237	1	0	1	0	0	0	1	0	0	0	0	0	0	0	1	0
238	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
239	1	0	1	λ	0	0	1	0	0	0	0	0	0	0	0	1
240	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	1
241	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1
242	1	0	1	λ	0	0	1	0	0	0	0	0	0	0	0	0
243	1	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0
244	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
245	0	1	λ	0	0	0	0	1	0	0	0	0	0	0	1	1
246	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1
247	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
248	0	1	1	0	0	0	0	1	0	0	0	0	0	0	1	0
249	0	1	0	0	0	0	0	1	0	0	0	0	0	0	1	0
250	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0
251	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
252	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	1
253	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	1
254	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	1
255	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1
256	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1
257	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
258	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	0
259	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0
260	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
261	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
262	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
263	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Aa Classification criterion
Aa Classifcation criterion from previous cases
Aa Additional result
Aa Additional result from previous cases
Aa Direct result

Figure 7: Case B.2.2. Second part: $\alpha_{11} = 0$, $\alpha_9 = 0$, and $\alpha_5 = 0$. Classification of the QFLA of dimension 9.

eliminate some parameters in the laws involved. In any case, the massive application of changes of base and characteristic vector has allowed us to obtain the complete classification in 263 subfamilies of the QFLA laws of dimension 9.

The 263 families have been represented in the paper, consisting of 157 simple algebras, 77 families depending on 1 parameter, 24 families depending on 2 parameters, and 5 families depending on 3 parameters. The classification is complete since any couple of the obtained 263 families is nonisomorphic and any quasifiliform Lie algebra of dimension 9 is isomorphic to one of them. The nonisomorphism of the 263 Lie algebra families has been proved in the 10 propositions of the paper, and the completeness of the classification is proved by the "exhaustive" analysis of all the possible cases, depending on the combination of the values of the 16 parameters ($a1 \cdots a16$).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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