Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 170397, 9 pages
http://dx.doi.org/10.1155/2014/170397

Research Article

Asymptotic Behaviors of the Eigenvalues of Schrödinger Operator with Critical Potential

Xiaoyao Jia,[1] Yan Zhao,[2] and Haoyu Zhai[3]

[1] *Mathematics and Statistics School, Henan University of Science and Technology, No. 263 Kaiyuan Road, Luolong District, Luoyang, Henan 471023, China*
[2] *College of Civil Engineering and Architecture, Zhejiang University, B505 Anzhong Building, 866 Yuhangtang Road, Hangzhou, Zhejiang 310058, China*
[3] *College of Food and Bioengineering, Henan University of Science and Technology, No. 263 Kaiyuan Road, Luo-Long District, Luolong, Henan 471023, China*

Correspondence should be addressed to Xiaoyao Jia; jiaxiaoyao@gmail.com

Received 29 October 2013; Revised 8 January 2014; Accepted 9 January 2014; Published 13 March 2014

Academic Editor: Manuel Maestre

We study the asymptotic behaviors of the discrete eigenvalue of Schrödinger operator $P(\lambda) = P_0 + \lambda V$ with $P_0 = -\Delta + q(\theta)/r^2$. We obtain the leading terms of discrete eigenvalues of $P(\lambda)$ when the eigenvalues tend to 0. In particular, we obtain the asymptotic behaviors of eigenvalues when $(P_0 - \alpha)^{-1}$ has singularity at $\alpha = 0$.

1. Introduction

This paper is devoted to the study of the asymptotic behaviors of the eigenvalues of a class of Schrödinger operators. This problem is related to low-energy spectral analysis for Schrödinger operators, which has been studied in many works (see [1–7]). References [1–3] are concerned with perturbation of a constant elliptic differential operator by a term decaying like $O(|x|^{-2-\epsilon})$, $\epsilon > 0$, as $|x| \to +\infty$. The spectral analysis of Schrödinger operators with potentials of critical decay (decaying like $O(|x|^{-2})$, as $|x| \to \infty$) is studied in [4–7]. The complex potentials are considered in [7].

The asymptotic behaviors of the eigenvalues of Schrödinger operator have been studied by many authors. In [8], Klaus and Simon got the leading term of the eigenvalue of Schrödinger operator with fast decaying potential. The asymptotic behavior of the eigenvalue of Schrödinger operator with periodic potential has been studied by Fassari and Klaus [9]. We have studied the asymptotic behavior of the smallest eigenvalue of Schrödinger operator with potential of critical decay [6]. In this paper, we will consider the asymptotic behavior of all other eigenvalues. The main tool we used in this paper is Birman-Schwinger kernel which was originated in the seventies. But Birman-Schwinger kernel is still a very important tool for the spectrum problem of Schrödinger operators (see [10–14]).

In this paper, we want to study the asymptotic behavior of discrete eigenvalue of Schrödinger operator

$$P(\lambda) = P_0 + \lambda V, \quad \text{for } \lambda \geq 0. \tag{1}$$

$L^2(\mathbb{R}^d)$, $d \geq 2$. Here, $P_0 = -\Delta + q(\theta)/r^2$. (r, θ) is the polar coordinate on $\mathbb{R}^d$, and $q(\theta)$ is a real continuous function.

The assumptions used in this paper are as follows. Suppose that $V \leq 0$ is a nonzero real continuous function and satisfies

$$|V(x)| \leq C\langle x\rangle^{-\rho_0}, \quad \text{for some } \rho_0 > 2. \tag{2}$$

Here, $\langle x\rangle = (1 + |x|^2)^{1/2}$. The assumption on $q(\theta)$ is that

$$-\Delta_s + q(\theta) \geq -\frac{1}{4}(d-2)^2, \quad \text{on } L^2\left(\mathbb{S}^{d-1}\right). \tag{3}$$

Here, Δ_s denotes Laplace operator on the sphere $\mathbb{S}^{d-1}$. If (3) holds, then $P_0 \geq 0$ in $L^2(\mathbb{R}^d)$ (see [5]).

In Section 2 [6], it is shown that, under the assumption on V, $P(\lambda)$ has discrete eigenvalues when λ is large enough, and each discrete eigenvalue tends to zero at some λ_0. Set

$$\sigma_\infty = \left\{\nu; \nu = \sqrt{\lambda + \frac{(d-2)^2}{4}}, \lambda \in \sigma\left(-\Delta_s + q(\theta)\right)\right\}, \quad \sigma_k = \sigma_\infty \cap [0,k], \; k \in \mathbb{N}. \tag{4}$$

If $0 \notin \sigma_\infty$, then the Birman-Schwinger kernel $|V|^{1/2}(P_0 - \alpha)^{-1}|V|^{1/2}$ is a bounded operator for $\alpha < 0$ (see [6]). Since there is a one-to-one correspondence between the eigenvalues of $P(\lambda)$ and the eigenvalues of $|V|^{1/2}(P_0-\alpha)^{-1}|V|^{1/2}$, the asymptotic expansion of the smallest eigenvalue of $P(\lambda)$ has been obtained through the asymptotic expansion of the eigenvalue of $|V|^{1/2}(P_0-\alpha)^{-1}|V|^{1/2}$ in [6], if (2) holds with $\rho_0 > 6$. In this paper, we will get the asymptotic behaviors of all discrete eigenvalues of $P(\lambda)$ in the case of $0 \notin \sigma_\infty$. The eigenfunction corresponding to the smallest eigenvalue is a positive function; we can obtain the leading term of the smallest eigenvalue easily. The eigenfunctions corresponding to all other eigenvalues may not be positive. Therefore, it is much more difficult to obtain the leading term of the eigenvalue.

If $0 \in \sigma_\infty$, $(P_0-\alpha)^{-1}$ has singularity at $\alpha = 0$ (see [5]). Thus, the Birman-Schwinger kernel $|V|^{1/2}(P_0-\alpha)^{-1}|V|^{1/2}$ also has singularity at $\alpha = 0$. The other goal of this paper is to obtain the asymptotic behavior of discrete eigenvalue when $0 \in \sigma_\infty$. The main difficulty of this situation is the singularity of $|V|^{1/2}(P_0-\alpha)^{-1}|V|^{1/2}$. In Section 4, the discrete eigenvalue of $|V|^{1/2}(P_0-\alpha)^{-1}|V|^{1/2}$ will be studied through the operator $(z - |V|^{1/2}(P_0-\alpha)^{-1}|V|^{1/2})^{-1}$ for some fixed z. Finally, we obtain the leading term of discrete eigenvalue of $P(\lambda)$, when the assumption (2) and (3) with $\rho_0 > 6$.

The plan of this work is as follows. In Section 2, we recall some known results for P_0, especially the asymptotic expansion of $(P_0-z)^{-1}$ for z near 0. We obtain the asymptotic behavior of discrete eigenvalue of $P(\lambda)$ for the case of $0 \notin \sigma_\infty$ in Section 3. Section 4 concentrates on the asymptotic expansion of discrete eigenvalue in the case of $0 \in \sigma_\infty$.

Let us introduce some notations first.

Notation 1. The scalar product on $L^2(\mathbb{R}^+; r^{d-1}dr)$ and $L^2(\mathbb{R}^d)$ is denoted by $\langle \cdot,\cdot\rangle$ and that on $L^2(\mathbb{S}^{d-1})$ by $(\cdot,\cdot)$. $H^{r,s}(\mathbb{R}^d)$, $r \in \mathbb{Z}$, $s \in \mathbb{R}$, denotes the weighted Sobolev space of order r with volume element $\langle x\rangle^{2s}dx$. The duality between $H^{1,s}$ and $H^{-1,-s}$ is identified with L^2 product. Denote $H^{0,s} = L^{2,s}$. Notation $\mathscr{L}(H^{r,s}, H^{r',s'})$ stands for the space of continuous linear operators from $H^{r,s}$ to $H^{r',s'}$. The complex plans $\mathbb{C}$ is slit along positive real axis so that $z^\nu = e^{\nu \ln z}$ and $\ln z = \ln|z| + i\arg z$ with $0 < \arg z < 2\pi$ are holomorphic there.

2. Some Results for P_0

Consider the operator

$$P_0 = -\Delta + \frac{q(\theta)}{r^2} \tag{5}$$

on $L^2(\mathbb{R}^d)$, $d \ge 2$. Assume that (r,θ) is the polar coordinate on $\mathbb{R}^d$. Then, the condition

$$-\Delta_s + q(\theta) \ge -\frac{1}{4}(d-2)^2, \quad \text{on } L^2\left(\mathbb{S}^{d-1}\right) \tag{6}$$

implies that $P_0 \ge 0$, in $L^2(\mathbb{R}^d)$ (see [5]). First, we recall some results on the resolvent of Schrödinger operator P_0. Define

$$\sigma_\infty = \left\{\nu; \nu = \sqrt{\lambda + \frac{(d-2)^2}{4}}, \lambda \in \sigma\left(-\Delta_s + q(\theta)\right)\right\}; \quad \sigma_k = \sigma_\infty \cap [0,k], \; k \in \mathbb{N}. \tag{7}$$

For $\nu \in \sigma_\infty$, let n_ν denote the multiplicity of $\lambda_\nu = \nu^2 - (d-2)^2/4$ as the eigenvalue of $-\Delta_s + q(\theta)$. Let $\varphi_\nu^{(j)}$, $\nu \in \sigma_\infty$, $1 \le j \le n_\nu$ denote an orthogonal basis of $L^2(\mathbb{S}^{d-1})$ consisting of eigenfunctions of $-\Delta_s + q(\theta)$:

$$\left(-\Delta_s + q(\theta)\right)\varphi_\nu^{(j)} = \lambda_\nu \varphi_\nu^{(j)}, \qquad \left(\varphi_\nu^{(i)}, \varphi_\nu^{(j)}\right) = \delta_{ij}. \tag{8}$$

Let π_ν denote the orthogonal projection in $L^2(\mathbb{S}^{d-1})$ onto the subspace spanned by the eigenfunctions of $-\Delta_s + q(\theta)$ associated with the eigenvalue λ_ν:

$$\pi_\nu f = \sum_{j=1}^{n_\nu}\left(f, \varphi_\nu^{(j)}\right) \otimes \varphi_\nu^{(j)}, \quad f \in L^2\left(\mathbb{S}^{n-1}\right). \tag{9}$$

Defined by $\nu \in \sigma_\infty$,

$$z_\nu = \begin{cases} z^{\nu'}, & \text{if } \nu \notin \mathbb{N}, \\ z\ln z, & \text{if } \nu \in \mathbb{N}. \end{cases} \tag{10}$$

Here, $\nu' = \nu - [\nu]$ and $[\nu]$ is the largest integer which is not larger than ν. For $\nu > 0$, let $[\nu]_-$ be the largest integer strictly less than ν. When $\nu = 0$, set $[\nu]_- = 0$. Define δ_ν by $\delta_\nu = 1$, if $\nu \in \sigma_\infty \cap \mathbb{N}$, $\delta_\nu = 0$, otherwise. One has $[\nu] = [\nu]_- + \delta_\nu$.

Theorem 1 ([5] Theorem 2.2). *The following asymptotic expansion holds for z near 0: with* $\Im z > 0$,

$$R_0(z) = \delta_0 \ln z G_{0,0}\pi_0 + \sum_{j=0}^{N} z^j F_j + \sum_{\nu\in\sigma_N} z_\nu \sum_{j=[\nu]_-}^{N-1} z^j G_{\nu,j+\delta_\nu}\pi_\nu + R_0^{(N)}(z) \tag{11}$$

in $\mathscr{L}(-1,s;1,-s)$, $s > 2N+1$. *Here,*

$$G_{\nu,j}(r,\tau) = \begin{cases} b_{\nu',j}(r\tau)^{j+\nu'} f_{j-[\nu]}\left(r,\tau;\nu'\right), & \nu \notin \mathbb{N}; \\ -\dfrac{(ir\tau)^j}{j!} f_{j-[\nu]}(r,\tau;0), & \nu \in \mathbb{N}; \end{cases}$$
$$F_j \in \mathscr{L}(-1,s;1,-s), \quad s > 2j+1;$$
$$R_0^{(N)}(z) = O\left(|z|^{N+\epsilon}\right) \in \mathscr{L}(-1,s;1,-s), \quad s > 2N+1, \; \epsilon > 0. \tag{12}$$

Here,

$$b_{\nu',j} = -\frac{i^j e^{-i\nu'\pi/2}\Gamma\left(1-\nu'\right)}{\nu'\left(\nu'+1\right)\cdots\left(\nu'+j\right)}, \tag{13}$$

for $0 \le \nu' < 1$*, and*

$$f_j\left(r,\tau,\nu\right) = \left(r\tau\right)^{-(1/2)(n-2)} P_{j,\nu}\left(\rho\right), \tag{14}$$

with $P_{j,\nu}(\rho)$ *being a polynomial in* ρ *of degree* j*:*

$$P_{j,\nu}\left(\rho\right) = \frac{i^j a_\nu}{j!}\int_{-1}^{1}\left(\rho+\frac{1}{2}\theta\right)^j\left(1-\theta^2\right)^{\nu-1/2}d\theta,$$
$$a_\nu = -\frac{e^{-i\pi\nu/2}}{2^{2\nu+1}\pi^{1/2}\Gamma\left(\nu+1/2\right)}. \tag{15}$$

Let $e_1(\lambda) \le e_2(\lambda) \le \cdots \le e_n(\lambda) \le \cdots$ denote the negative eigenvalues of $P(\lambda)$ (counting multiplicity). Suppose that $e_{n+1}(\lambda) = 0$, for convenience, if there are at most n negative eigenvalues. Using the same discussion as in [6], we know that each negative eigenvalue tends to 0, as λ tends to some value. Suppose that $e_i(\lambda) \to 0$, at $\lambda = \lambda_i$. Then, from the definition of $e_n(\lambda)$, one has $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$. In [6], we have studied the asymptotic behavior of the smallest eigenvalue of $P(\lambda)$, namely, $e_1(\lambda)$, as $e_1(\lambda) \to 0$. In this paper, we investigate the asymptotic behaviors of all eigenvalues of $P(\lambda)$. Suppose that there are exactly m eigenvalues of $P(\lambda)$ tending to 0 at λ_k. Without loss of generality, suppose that $\lambda_{k-1} < \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+m-1} < \lambda_{k+m}$.

As in [6], we define a family of Birman-Schwinger kernel operators, which are used to study the eigenvalues of $P(\lambda)$. For $z \notin \sigma(P_0)$, set

$$K\left(z\right) = |V|^{1/2}\left(P_0 - z\right)^{-1}|V|^{1/2};$$
$$K\left(0\right) = |V|^{1/2}F_0|V|^{1/2}. \tag{16}$$

The following result shows the relation between the eigenvalues of $P(\lambda)$ and the eigenvalues of $K(\alpha)$.

Proposition 2 ([6] Proposition 2.2). *Let* $\alpha < 0$*. Then,*

(a) *let*

$$A = \left\{\psi \in L^2\left(\mathbb{R}^d\right); \left(P\left(\lambda\right)-\alpha\right)\psi = 0\right\},$$
$$B = \left\{\phi \in L^2\left(\mathbb{R}^d\right); K\left(\alpha\right)\phi = \lambda^{-1}\phi\right\}. \tag{17}$$

Then, $|V|^{1/2}$ *is injective from* A *to* B*, and* $(P_0 - \alpha)^{-1}|V|^{1/2}$ *is injective from* B *to* A*.*

(b) *The multiplicity of* α *as the eigenvalue of* $P(\lambda)$ *is exactly the multiplicity of* λ^{-1} *as the eigenvalue of* $K(\alpha)$*.*

Proposition 3. *(a) Let* $\lambda > 0$*. The negative eigenvalue of the* $P(\lambda)$ *is monotone decreasing and continuous about* λ*.*

(b) Let $\alpha < 0$*. The eigenvalue of the* $K(\alpha)$ *is monotone increasing and continuous about* α*.*

Proof. (a) Suppose λ_1, λ_2 are two arbitrary positive numbers. By Lemma 3.4 [6], one has

$$\left|e_i\left(\lambda_1\right) - e_i\left(\lambda_2\right)\right| \le \left\|P\left(\lambda_1\right) - P\left(\lambda_2\right)\right\| \le \left|\lambda_1 - \lambda_2\right| \cdot \|V\|. \tag{18}$$

This means that the negative eigenvalue of $P(\lambda)$ is continuous about λ.

By min-max principle (Theorem XIII.1 [15]), the negative eigenvalue of $P(\lambda)$ has the following form:

$$e_n\left(\lambda\right) = \sup_{\phi_1,\ldots,\phi_{n-1}} \inf_{\|\psi\|=1,\psi\in[\phi_1,\ldots,\phi_{n-1}]^\perp} \left\langle \psi, P\left(\lambda\right)\psi\right\rangle. \tag{19}$$

Since $e_n(\lambda) < 0$ and $P_0 \ge 0$, it is easy to see that $\langle\psi, \lambda V\psi\rangle < 0$. Hence, $\langle\psi, P(\lambda)\psi\rangle$ is monotone decreasing about λ. It follows that the eigenvalue of $P(\lambda)$ is monotone decreasing about λ.

(b) For $\alpha_1 < \alpha_2 < 0$,

$$K\left(\alpha_1\right) - K\left(\alpha_2\right) = \left(\alpha_1 - \alpha_2\right)|V|^{1/2}\left(P_0 - \alpha_1\right)^{-1}\left(P_0 - \alpha_2\right)^{-1}|V|^{1/2} < 0. \tag{20}$$

This shows that $K(\alpha_1) < K(\alpha_2)$. Hence, using min-max principle again, one has that the eigenvalue of $K(\alpha)$ is monotone increasing about α.

Note that, for any $\alpha < 0$, $(P_0 - \alpha)^{-1}$ is a bounded operator in $L^2(\mathbb{R}^d)$. By (20) and Lemma 3.4 [6], we can get that, for any $\alpha_1 < 0$, $\alpha_2 < 0$, $|\mu_i(K(\alpha_1)) - \mu_i(K(\alpha_2))| \le C|\alpha_1 - \alpha_2|$ with some C large enough. Here, $\mu_i(K(\alpha))$ is the eigenvalue of $K(\alpha)$. It implies that the eigenvalue of $K(\alpha)$ is continuous about α. □

We give the definition of resonance which will be used to investigate the asymptotic behavior of the eigenvalue of $P(\lambda)$ later.

Definition 4. Set $\mathcal{N}(\lambda) = \{u; P(\lambda)u = 0, u \in H^{1,-s}, \forall s > 1\}$. If $\mathcal{N}(\lambda) \setminus L^2 \ne \{0\}$, one says that 0 is the resonance of $P(\lambda)$. A nonzero function $u \in \mathcal{N}(\lambda) \setminus L^2$ is called a resonant state of $P(\lambda)$ at 0.

3. The Case $0 \notin \sigma_\infty$

Proposition 5. *Assume that* $0 \notin \sigma_\infty$*. Then,* λ_k^{-1} *is the eigenvalue of* $K(0)$*, and the multiplicity of* λ_k^{-1} *is* m*.*

Proof. $e_k(\lambda)$ is the eigenvalue of $P(\lambda)$; thus, λ^{-1} is the eigenvalue of $K(e_k(\lambda))$ by Proposition 2. Notice that $e_k(\lambda) \to 0$, as $\lambda \to \lambda_k$, and one gets that λ_k^{-1} is the eigenvalue of $K(0)$, by Lemma 3.5 [6].

Suppose that the multiplicity of λ_k^{-1} is m_0. We will prove that $m_0 = m$ in the following. First, using Lemma 3.5 [6] again, we know that there are m_0 eigenvalues (counting multiplicity) of $K(\alpha)$ tending to λ_k^{-1}. Suppose that these m_0 eigenvalues are $\mu_1(\alpha), \mu_2(\alpha), \ldots, \mu_{m_0}(\alpha)$. Since the eigenvalue of $K(\alpha)$ is continuous and monotone increasing, for a fixed λ near λ_k with $\lambda > \lambda_k$, there exists a unique $\alpha_i < 0$ $(1 \le i \le m_0)$

such that $\mu_i(\alpha_i) = \lambda^{-1}$. It follows that α_i is the eigenvalue of $P(\lambda)$. This fact shows that $m_0 \le m$, since there are m eigenvalues of $P(\lambda)$ that tend to 0 at $\lambda = \lambda_k$. On the other hand, note that, for any $\lambda > 0$, $e_i(\lambda) \to 0$ and $e_i(\lambda)$ is continuous and monotone decreasing. Therefore, for a fixed $\alpha < 0$, there exists a unique $\widetilde{\lambda_i}$ such that $e_i(\widetilde{\lambda_i}) = \alpha$ $(k \le i \le k+m-1)$. It follows that $\widetilde{\lambda_i}^{-1}$ is the eigenvalue of $K(\alpha)$. One has that $K(\alpha) \to K(0)$ and $\widetilde{\lambda_i}^{-1} \to \lambda_k^{-1}$ $(k \le i \le k+m-1)$, as $\alpha \to 0$. It means that there are at least m eigenvalues (counting multiplicity) of $K(\alpha)$ tending to λ_k^{-1}. Thus, $m \le m_0$. Therefore, $m = m_0$. This ends the proof. □

Since $K(0)$ is a compact operator, then we can suppose that all of the eigenvalues of $K(0)$ can be denoted by $\mu_1, \mu_2, \dots, \mu_n, \dots$ (counting multiplicity), and the corresponding eigenvectors are $\phi_1, \phi_2, \dots, \phi_n, \dots$, respectively. Set

$$\psi_i = F_0 |V|^{1/2} \phi_i. \tag{21}$$

Then,

$$|V|^{1/2} \psi_i = |V|^{1/2} F_0 |V|^{1/2} \phi_i = \mu_i \phi_i. \tag{22}$$

By Lemma 3.5 [6] and Proposition 5, we know that there are m eigenvalues (counting multiplicity) of $K(\alpha)$ that tend to λ_k^{-1}, as $\alpha \to 0$. Without loss of generality, we can suppose that $\mu_1 = \mu_2 = \cdots = \mu_m = \lambda_k^{-1}$ and the eigenvalues $\mu_1(\alpha), \mu_2(\alpha), \dots, \mu_m(\alpha)$ tend to λ_k^{-1}, as $\alpha \to 0$. Moreover, we can choose a set of eigenvectors, $\{\phi_j(\alpha)\ (1 \le j \le m)\}$, of $K(\alpha)$ such that $\langle \phi_j(\alpha), \phi_l(\alpha)\rangle = \delta_{jl}$, $K(\alpha)\phi_j(\alpha) = \mu_j(\alpha)\phi_j(\alpha)$, and $\phi_j(\alpha) \to \phi_j$, as $\alpha \to 0$. Then, by Lemma 3.5 [6], ϕ_j is the eigenvector of $K(0)$ corresponding to λ_k^{-1}. Thus, $K(0)\phi_j = \lambda_k^{-1}\phi_j$ $(1 \le j \le m)$. It follows that

$$P_0 \psi_j = P_0 F_0 |V|^{1/2} \phi_j = |V|^{1/2} \phi_j = \lambda_k |V| \psi_j \tag{23}$$

by (22). In the second step of the last equality, we use Proposition 3.2 [6]. Thus, $P(\lambda_k)\psi_j = 0$. By Theorem 4.1 [4],

$$\psi_j(r\theta) = \sum_{0<\nu\le 1} \sum_{s=1}^{n_\nu} -\frac{1}{2\nu} \left\langle \lambda_k V u, |y|^{-(d-2)/2+\nu} \varphi_\nu^{(s)} \right\rangle \frac{\varphi_\nu^{(s)}(\theta)}{r^{(d-2)/2+\nu}} + u, \tag{24}$$

where $u \in L^2$. $\psi_j \in \mathcal{N}(\lambda_k) \setminus L^2$ will be called a ζ_j-resonant state of $P(\lambda_k)$, if

$$\psi_j(r\theta) = \frac{\phi(\theta)}{r^{(d-2)/2+\nu}} + o\left(\frac{1}{r^{(d-2)/2+\nu+\epsilon}}\right), \quad r \longrightarrow \infty,\ \epsilon > 0, \tag{25}$$

for some $\phi \neq 0$.

Proposition 6. *(I) Assume f is a ν_0-resonant state of $P(\lambda_k)$. Then*

(a) $\langle Vf, G_{\nu_0,\delta_{\nu_0}} \pi_{\nu_0} Vf\rangle = c_{\nu_0} \sum_{s=1}^{n_{\nu_0}} |\langle Vf, |y|^{-(d-2)/2+\nu_0} \varphi_{\nu_0}^{(s)}\rangle|^2 \neq 0$ with c_{ν_0} be a nonzero constant depending only on ν_0;

(b) $\forall g \in H^{1,-s}$, $\forall \nu \in \sigma_1$, with $\nu < \nu_0$, $\langle Vg, G_{\nu,\delta_\nu} \pi_\nu Vf\rangle = 0$.

(II) Assume f is an eigenvector of $P(\lambda_k)$. Then, $\forall \nu \in \sigma_1$, $\forall g \in H^{-1,s}$, $\langle Vg, G_{\nu,\delta_\nu} \pi_\nu Vf\rangle = 0$.

Proof. (I) By the definition of G_{ν,δ_ν} and π_ν, $\forall f, g \in H^{1,-s}$, $\nu \in \sigma_1$,

$$\left\langle Vg, G_{\nu,\delta_\nu} \pi_\nu Vf \right\rangle = c_\nu \sum_{s=1}^{n_\nu} \left\langle Vg, |y|^{-(d-2)/2+\nu} \varphi_\nu^{(s)} \right\rangle \overline{\left\langle Vf, |y|^{-(d-2)/2+\nu} \varphi_\nu^{(s)} \right\rangle}. \tag{26}$$

Here, c_ν is a constant depending only on ν. If f is a resonant state of $P(\lambda_k)$, then, by (24),

$$f(r\theta) = \sum_{0<\nu\le 1} \sum_{s=1}^{n_\nu} -\frac{1}{2\nu} \left\langle \lambda_k V f, |y|^{-(d-2)/2+\nu} \varphi_\nu^{(s)} \right\rangle \times \frac{\varphi_\nu^{(s)}(\theta)}{r^{(d-2)/2+\nu}} + u, \tag{27}$$

where $u \in L^2$. If f is a ν_0-resonant state of $P(\lambda_k)$, then the leading part of f is $\phi(\theta)/r^{(d-2)/2+\nu_0}$. It follows that, for $\nu \in \sigma_1$ with $\nu < \nu_0$,

$$\left\langle Vf, |y|^{-(d-2)/2+\nu} \varphi_\nu^{(s)} \right\rangle = 0, \tag{28}$$

and there exists at least one s with $1 \le s \le n_{\nu_0}$, such that

$$\left\langle Vf, |y|^{-(d-2)/2+\nu_0} \varphi_{\nu_0}^{(s)} \right\rangle \neq 0. \tag{29}$$

Thus, (26) shows that

$$\begin{gathered}\left\langle Vg, G_{\nu,\delta_\nu} \pi_\nu Vf \right\rangle = 0, \quad \nu \in \sigma_1,\ \nu < \nu_0; \\ \left\langle Vf, G_{\nu_0,\delta_{\nu_0}} \pi_{\nu_0} Vf \right\rangle = c_{\nu_0} \sum_{s=1}^{n_{\nu_0}} \left| \left\langle Vf, |y|^{-(d-2)/2+\nu_0} \varphi_{\nu_0}^{(s)} \right\rangle \right|^2 \neq 0.\end{gathered} \tag{30}$$

(II) By (24) and $f \in L^2$, it is easy to see that $\langle Vf, |y|^{-(d-2)/2+\nu} \varphi_\nu^{(s)}\rangle = 0$ for $\nu \in \sigma_1$. Thus, for $\nu \in \sigma_1$,

$$\left\langle Vg, G_{\nu,\delta_\nu} \pi_\nu Vf \right\rangle = c_\nu \sum_{s=1}^{n_\nu} \left\langle Vg, |y|^{-(d-2)/2+\nu} \varphi_\nu^{(s)} \right\rangle \times \overline{\left\langle Vf, |y|^{-(d-2)/2+\nu} \varphi_\nu^{(s)} \right\rangle} = 0. \tag{31}$$

□

Theorem 7. *Assume $0 \notin \sigma_\infty$. ϕ_j, ψ_j, and ζ_j are defined as above. If $\rho_0 > 6$, one of three situations holds.*

(a) $e_l(\lambda) = c(\lambda - \lambda_k)^{1/\zeta_j} + o(|\lambda - \lambda_k|^{1/\zeta_j})$. *c is a nonzero constant independent of λ, $\zeta_j \in \sigma_1$, and $\zeta_j < 1$.*

(b) $e_l(\lambda) = c((\lambda - \lambda_k)/\ln(\lambda - \lambda_k)) + o(|(\lambda - \lambda_k)/\ln(\lambda - \lambda_k)|)$. *c is a nonzero constant independent of λ.*

(c) $e_l(\lambda) = c(\lambda - \lambda_k) + o(|\lambda - \lambda_k|)$ *c is a nonzero constant independent of λ.*

Proof. Applying Theorem 1 to $R_0(\alpha) = (P_0 - \alpha)^{-1}$, we obtain the asymptotic expansion of $R_0(\alpha)$ for α near 0, with $\alpha < 0$,

$$R_0(\alpha) = F_0 + \sum_{0<\nu\le 1} \alpha_\nu G_{\nu,\delta_\nu}\pi_\nu + \alpha F_1 + O\left(|\alpha|^{1+\epsilon}\right) \quad (32)$$

in $\mathscr{L}(-1, s;\ 1, -s)$, $s > 3$, $\epsilon > 0$. It follows that, if $\rho_0 > 6$,

$$\begin{aligned} K(\alpha) = K(0) &+ \sum_{0<\nu\le 1} \alpha_\nu |V|^{1/2} G_{\nu,\delta_\nu}\pi_\nu |V|^{1/2} \\ &+ \alpha|V|^{1/2}F_1|V|^{1/2} + O\left(|\alpha|^{1+\epsilon}\right) \end{aligned} \quad (33)$$

in $\mathscr{L}(0,0;0,0)$. λ_k^{-1} is the eigenvalue of $K(0)$ with multiplicity m; then, there are m eigenvalues of $K(\alpha)$ (counting multiplicity) tending to λ_k^{-1}, as $\alpha \to 0$. Suppose that the m eigenvalues of $K(\alpha)$ are $\mu_1(\alpha), \mu_2(\alpha), \ldots, \mu_m(\alpha)$. We study the asymptotic expansion of $\mu_i(\alpha)$ first.

Set $K_1(\alpha) = \sum_{0<\nu\le 1} \alpha_\nu |V|^{1/2} G_{\nu,\delta_\nu}\pi_\nu |V|^{1/2} + \alpha|V|^{1/2}F_1|V|^{1/2} + O(|\alpha|^{1+\epsilon})$ and $K_{sj}(\alpha) = \langle \phi_s, K_1(\alpha)\phi_j\rangle$. Then, by (22),

$$\begin{aligned} &K_{sj}(\alpha) \\ &= \lambda_k \left\langle \phi_s, K_1(\alpha)|V|^{1/2}\psi_j \right\rangle \\ &= -\lambda_k \left\langle |V|^{1/2}\phi_s, \right. \\ &\qquad \left. \left(\sum_{0<\nu\le 1}\alpha_\nu G_{\nu,\delta_\nu}\pi_\nu + \alpha F_1 + o\left(|\alpha|^{1+\epsilon}\right)\right) V\psi_j \right\rangle. \end{aligned} \quad (34)$$

By Lemma 3.6 [6], the eigenvalue of $K(\alpha)$, $\mu_j(\alpha)$ $(1 \le j \le m)$, has the following form:

$$\mu_j(\alpha) = \lambda_k^{-1} + \frac{\sum_{n=0}^{\infty} a_n^{(j)}(\alpha)}{\sum_{n=0}^{\infty} b_n^{(j)}(\alpha)}. \quad (35)$$

Here,

$$\begin{aligned} a_0^{(j)}(\alpha) &= K_{jj}(\alpha), \\ a_1^{(j)}(\beta) &= -\sum_{\{s;\mu_s \ne \lambda_k^{-1}\}} \left(\mu_s - \lambda_k^{-1}\right)^{-1} K_{js}(\alpha)K_{sj}(\alpha), \\ a_2^{(j)}(\alpha) &= \sum_{\mu_s \ne \lambda_k^{-1} \ne \mu_l} \left(\mu_s - \lambda_k^{-1}\right)^{-1} \\ &\quad \times \left(\mu_l - \lambda_k^{-1}\right)^{-1} K_{js}(\alpha)K_{sl}(\alpha)K_{lj}(\alpha) \\ &\quad - 2\sum_{\{s;\mu_s \ne \lambda_k^{-1}\}} \left(\mu_s - \lambda_k^{-1}\right)^{-1} \\ &\quad \times K_{js}(\alpha)K_{sj}(\alpha)K_{jj}(\alpha), \\ a_n^{(j)}(\alpha) &= -\frac{(-1)^n}{2\pi i} \\ &\quad \times \oint_{|E-\lambda_k^{-1}|=\delta} \left(\lambda_k^{-1} - E\right)^{-1} \\ &\quad \times \sum_{i_1,i_2,\ldots,i_n} \left(\mu_{i_1} - E\right)^{-1} \cdots \left(\mu_{i_n} - E\right)^{-1} \\ &\quad \times K_{ji_1}(\alpha)K_{i_1i_2}(\alpha)\cdots K_{i_nj}(\alpha)\,dE, \\ &\qquad \text{for } n > 2, \\ b_0^{(j)}(\alpha) &= 1, \\ b_1^{(j)}(\alpha) &= 0, \\ b_2^{(j)}(\alpha) &= \sum_{\{s;\mu_s \ne \lambda_k^{-1}\}} \left(\mu_s - \lambda_k^{-1}\right)^{-2} K_{js}(\alpha)K_{sj}(\alpha), \\ b_n^{(j)}(\alpha) &= -\frac{(-1)^n}{2\pi i} \\ &\quad \times \oint_{|E-\lambda_k^{-1}|=\delta} \left(\mu_s - \lambda_k^{-1}\right)^{-2} \\ &\quad \times \sum_{i_1,i_2,\ldots,i_{n-1}} \left(\mu_{i_1} - E\right)^{-1} \cdots \left(\mu_{i_{n-1}} - E\right)^{-1} \\ &\quad \times K_{ji_1}(\alpha)K_{i_1i_2}(\alpha)\cdots K_{i_{n-1}j}(\alpha)\,dE, \\ &\qquad \text{for } n > 2. \end{aligned} \quad (36)$$

If ψ_j is ζ_j-resonant state of $K(0)$ with $\zeta_j < 1$, then by Proposition 6,

$$\begin{aligned} K_{jj}(\alpha) &= c_{\zeta_j}\alpha^{\zeta_j}\lambda_k^2 \sum_{s=1}^{n_{\zeta_j}} \left| \left\langle V\psi_j, |y|^{-(d-2)/2+\nu_0} \varphi_{\zeta_j}^{(s)} \right\rangle \right|^2 \\ &\quad + o\left(|\alpha|^{\zeta_j}\right) = \tilde{c}_{\zeta_j}\alpha^{\zeta_j} + o\left(|\alpha|^{\zeta_j}\right); \end{aligned}$$

$$K_{js}(\alpha) = \sum_{0<\nu\leq 1} \lambda_k \left\langle |V|^{1/2}\phi_s, G_{\nu,\delta_\nu}\pi_\nu V\psi_j \right\rangle + O(|\alpha|) = O\left(|\alpha|^{\zeta_j}\right). \tag{37}$$

Hence,

$$a_2^{(j)} = O\left(|\alpha|^{2\zeta_j}\right). \tag{38}$$

Similarly,

$$a_n^{(j)} = O\left(|\alpha|^{2\zeta_j}\right); \qquad b_n^{(j)} = O\left(|\alpha|^{2\zeta_j}\right), \tag{39}$$

for $n \geq 2$. It follows that

$$\sum_{n=0}^{\infty} a_n^{(j)} = \tilde{c}_{\zeta_j}\alpha^{\zeta_j} + o\left(|\alpha|^{\zeta_j}\right); \qquad \sum_{n=0}^{\infty} b_n^{(j)} = 1 + O\left(|\alpha|^{2\zeta_j}\right). \tag{40}$$

Thus, $\mu_j(\alpha) = \lambda_k^{-1} + \tilde{c}_{\zeta_j}\alpha^{\zeta_j} + o(|\alpha|^{\zeta_j})$. By Proposition 2, for $1 \leq j \leq m$, $\exists l$ with $k \leq l \leq k+m-1$ such that $\mu_j(e_l(\lambda)) = \lambda^{-1}$. Thus, $\lambda^{-1} = \lambda_k^{-1} + \tilde{c}_{\zeta_j} e_l(\lambda)^{\zeta_j} + o(|e_l(\lambda)|^{\zeta_j})$. Since

$$\lambda^{-1} = \lambda_k^{-1} - \lambda_k^{-2}(\lambda - \lambda_k) + O\left(|\lambda - \lambda_k|^2\right), \tag{41}$$

we can get that the leading term of $e_l(\lambda)$ is $c(\lambda - \lambda_k)^{1/\zeta_j}$, with $c \neq 0$ being a constant independent of λ. This is in case (a).

If ψ_j is 1-resonant state of $K(0)$, computing similarly, we obtain that $\mu_j(\alpha) = \lambda_k^{-1} + \tilde{c}\alpha\ln\alpha + O(|\alpha|)$. Using again $\mu_j(e_l(\lambda)) = \lambda^{-1}$ and (41), we can get that the leading part of $e_l(\lambda)$ is $c(\lambda - \lambda_k)/\ln(\lambda - \lambda_k)$, with $c \neq 0$ being a constant independent of λ. This is in case (b).

If ψ_j is the eigenfunction of $K(0)$ corresponding to eigenvalue λ_k^{-1}, then, for $k \leq j \leq k+m-1$,

$$R_0(\alpha)V\psi_j = -\lambda_k^{-1}\left(\psi_j + \alpha R_0(\alpha)\psi_j\right);$$
$$F_0V\psi_j = -F_0|V|^{1/2}|V|^{1/2}\psi_j = -\lambda_k^{-1}F_0|V|^{1/2}\phi_j = -\lambda_k^{-1}\psi_j. \tag{42}$$

In the second equation, we use (22) and the definition of ψ_j. Proposition 6 shows that for $\nu \in \sigma_1$, $\langle V\psi_s, G_{\nu,\delta_\nu}\pi_\nu V\psi_j\rangle = 0$. Thus, by (42),

$$\begin{aligned}
&\left\langle V\psi_j, F_1V\psi_j\right\rangle \\
&= \lim_{\alpha\to 0}\alpha^{-1}\left\langle V\psi_j, \left(R_0(\alpha) - F_0 - \sum_{\nu\in\sigma_1} G_{\nu,\delta_\nu}\pi_\nu - o(|\alpha|)\right)V\psi_j\right\rangle \\
&= \lim_{\alpha\to 0}\alpha^{-1}\left\langle V\psi_j, \left(R_0(\alpha) - F_0\right)V\psi_j\right\rangle \\
&= -\lim_{\alpha\to 0}\lambda_k^{-1}\left\langle V\psi_j, R_0(\alpha)\psi_j\right\rangle \\
&= -\lambda_k^{-2}\left\|\psi_j\right\|^2.
\end{aligned} \tag{43}$$

Computing similarly as before, we obtain that $\mu_j(\alpha) = \lambda_k^{-1} + \tilde{c}\alpha + o(|\alpha|)$. Thus, the leading part of $e_l(\lambda)$ is $c(\lambda - \lambda_k)$, with $c \neq 0$ being a constant independent of λ. This is in case (c). This completes the proof. □

4. The Case $0 \in \sigma_\infty$

Assume that $0 \in \sigma_\infty$ in this section. By Theorem 1,

$$R_0(\alpha) = \delta_0\ln\alpha G_{0,0}\pi_0 + F_0 + \sum_{\nu\in\sigma_1}\alpha_\nu G_{\nu,\delta_\nu}\pi_\nu + \alpha F_1 + O\left(|\alpha|^{1+\epsilon}\right) \tag{44}$$

in $\mathscr{L}(-1,s;1,-s)$, $s > 3$. Here,

$$G_{0,0}\pi_0 = \frac{1}{2}\langle\cdot,\phi_0\rangle\phi_0 \tag{45}$$

with $\phi_0 = \varphi_0^{(1)}(\theta)r^{-(d-2)/2}$. (r,θ) is the polar coordinate on $\mathbb{R}^d$. Hence, $K(\alpha)$ ban will be written as

$$K(\alpha) = l(\alpha)P + K(0) + K_1(\alpha). \tag{46}$$

Here, P is rank 1 projection with $P = \langle\cdot,\tilde{\phi}_0\rangle\tilde{\phi}_0$, $\tilde{\phi}_0 = (\int|V|\phi_0^2dx)^{-1/2}|V|^{1/2}\phi_0$, $l(\alpha) = (1/2)\ln\alpha\int|V|\phi_0^2dx$, and $K_1(\alpha) = \sum_{0<\nu\leq 1}\alpha_\nu|V|^{1/2}G_{\nu,\delta_\nu}\pi_\nu|V|^{1/2} + \alpha|V|^{1/2}F_1|V|^{1/2} + O(|\alpha|^{1+\epsilon})$.

Note that as $\alpha \to 0$, $|l(\alpha)| \to +\infty$. It follows that $K(\alpha)$ has singularity at $\alpha = 0$. Hence, it is more complicated than the case $0 \notin \sigma_\infty$. We first study the operator

$$\widetilde{K}(\alpha) = l(\alpha)P + K(0), \tag{47}$$

as $\alpha \to 0$. Set $Q = 1 - P$.

We can obtain Propositions 8, 10, and 11, in the same way that Lemmas 7.1 and 7.2 [8] were obtained.

Proposition 8. (a) *For α small enough with $|l(\alpha)| > 2\|K(0)\|$, and for a fixed z with $\|K(0)\| < z < |l(\alpha)| - \|K(0)\|$, $(z - \widetilde{K}(\alpha))^{-1}$ converges to $Q(z - QK(0)Q)^{-1}Q$ in norm, as $\alpha \to 0$.*

(b) *For α small enough, and for fixed z with $z > \|K(0)\|$,*

$$\begin{aligned}
\left(z - \widetilde{K}(\alpha)\right)^{-1} &= (z - K(0))^{-1} \\
&\quad - \omega(z)^{-1}(z - K(0))^{-1}P(z - K(0))^{-1} \\
&\quad - l(\alpha)^{-1}\omega(z)^{-2}(z - K(0))^{-1}P(z - K(0))^{-1} \\
&\quad + O\left(|l(\alpha)|^{-2}\right).
\end{aligned} \tag{48}$$

Here, $\omega(z) = \langle(z - K(0))^{-1}\tilde{\phi}_0, \tilde{\phi}_0\rangle$.

Remark 9. Proposition 8 shows that, for fixed z, with $\|K(0)\| < z < |l(\alpha)| - \|K(0)\|$,

$$Q(z - QK(0)Q)^{-1}Q = (z - K(0))^{-1} - \omega(z)^{-1}(z - K(0))^{-1}P(z - K(0))^{-1}. \tag{49}$$

Proposition 10. *For fixed z, with $\|K(0)\| < z < |l(\alpha)| - \|K(0)\|$, $(z-\lambda_k^{-1})^{-1}$ is the eigenvalue of $Q(z-QK(0)Q)^{-1}Q$.*

This proposition follows that λ_k^{-1} is the eigenvalue of $QK(0)Q$.

Proposition 11. *Suppose that the multiplicity of λ_k^{-1} as the eigenvalue of $QK(0)Q$ is m_0. Set $U = \{f; QK(0)Qf = \mu_0 f\}$. Then, $\exists W \sqsubset U$ with $\dim W = m_0$ or $m_0 - 1$, so that*

$$\left(z-\widetilde{K}(\alpha)\right)^{-1} f \equiv (z-K(0))^{-1} f \tag{50}$$

for $f \in W$. And for $f \in W$, $K(0)f = \lambda_k^{-1} f$.

Suppose that all of the eigenvalues of $QK(0)Q$ can be denoted by $\mu_1, \mu_2, \ldots, \mu_n, \ldots$ (counting multiplicity), and the corresponding eigenvectors are $\phi_1, \phi_2, \ldots, \phi_n, \ldots$, respectively. By Lemma 3.5 [6] and Proposition 5, we know that there are m_0 eigenvalues (counting multiplicity) of $K(\alpha)$ tending to λ_k^{-1}, as $\alpha \to 0$. Without loss of generality, we can suppose that $\mu_1 = \mu_2 = \cdots = \mu_{m_0} = \lambda_k^{-1}$ and the eigenvalues $\mu_1(\alpha), \mu_2(\alpha), \ldots, \mu_{m_0}(\alpha)$ tend to λ_k^{-1}, as $\alpha \to 0$. Moreover, we can choose a set of eigenvectors, $\{\phi_j(\alpha)\ (1 \le j \le m_0)\}$, of $K(\alpha)$ such that $\langle \phi_j(\alpha), \phi_l(\alpha)\rangle = \delta_{jl}$, $K(\alpha)\phi_j(\alpha) = \mu_j(\alpha)\phi_j(\alpha)$, and $\phi_j(\alpha) \to \phi_j$, as $\alpha \to 0$. Then, ϕ_j is the eigenvector of $QK(0)Q$ corresponding to λ_k^{-1}, by Lemma 3.5 [6]. Thus, $QK(0)Q\phi_j = \lambda_k^{-1}\phi_j$.

Theorem 12. *Assume $0 \in \sigma_\infty$. If $\rho_0 > 6$, one of the four situations holds.*

(a) $e_l(\lambda) = c_1 e^{-1/c_2(\lambda-\lambda_k)} + o(e^{-1/c_2(\lambda-\lambda_k)})$. *$c_1$ and c_2 are two nonzero constants independent of λ.*

(b) $e_l(\lambda) = c(\lambda-\lambda_k)^{1/\zeta_j} + o(|\lambda-\lambda_k|^{1/\zeta_j})$. *$c$ is a nonzero constant independent of λ, $\zeta_j \in \sigma_1$ and $\sigma_1 < 1$.*

(c) $e_l(\lambda) = c((\lambda-\lambda_k)/\ln(\lambda-\lambda_k)) + o(|(\lambda-\lambda_k)/\ln(\lambda-\lambda_k)|)$. *$c$ is a nonzero constant independent of λ.*

(d) $e_l(\lambda) = c(\lambda-\lambda_k) + o(|\lambda-\lambda_k|)$ *c is a nonzero constant independent of λ.*

Proof. By Proposition 10,

$$(z-K(\alpha))^{-1} = Q(z-QK(0)Q)^{-1}Q + \widetilde{K}_1(\alpha) \tag{51}$$

with

$$\begin{aligned}\widetilde{K}_1(\alpha) &= l(\alpha)^{-1}\omega(z)^{-2}(z-K(0))^{-1} \\ &\quad \times P(z-K(0))^{-1} + O\left(|l(\alpha)|^{-2}\right) \\ &\quad + \left(z-\widetilde{K}(\alpha)\right)^{-1}\sum_{n=1}^{\infty}\left[K_1(\alpha)\left(z-\widetilde{K}(\alpha)\right)^{-1}\right]^n.\end{aligned} \tag{52}$$

Lemma 3.6 [6] shows that the eigenvalue of $(z-K(\alpha))^{-1}$, $(z-\mu_j(\alpha))^{-1}$ $(1 \le j \le m_0)$ has the form

$$\left(z-\mu_j(\alpha)\right)^{-1} = \left(z-\lambda_k^{-1}\right)^{-1} + \frac{\sum_{n=0}^{\infty} a_n^{(j)}(\alpha)}{\sum_{n=0}^{\infty} b_n^{(j)}(\alpha)}. \tag{53}$$

As in Theorem 7, we should compute $K_{sj}(\alpha) = \langle \phi_s, \widetilde{K}_1(\alpha)\phi_j\rangle$, $(1 \le j \le m_0)$.

If $P(z-K(0))^{-1}\phi_j \ne 0$, then

$$\begin{aligned}\left\langle \phi_j, \widetilde{K}_1(\alpha)\phi_j\right\rangle &= \left\|P(z-K(0))^{-1}\phi_j\right\|^2 \\ &\quad \times \omega(z)^{-2} l(\alpha)^{-1} + o\left(|l(\alpha)|^{-1}\right).\end{aligned} \tag{54}$$

It follows that $K_{jj}(\alpha) = \tilde{c}l(\alpha)^{-1} + o(|l(\alpha)|^{-1})$. Note that $K_{sj}(\alpha) = O(|l(\alpha)|^{-1})$. It is easy to get that

$$\begin{aligned}\sum_{n=0}^{\infty} a_n^{(j)} &= \tilde{c}l(\alpha)^{-1} + o\left(|l(\alpha)|^{-1}\right); \\ \sum_{n=0}^{\infty} b_n^{(j)} &= 1 + O\left(|l(\alpha)|^{-2}\right).\end{aligned} \tag{55}$$

It follows that $(z-\mu_j(\alpha))^{-1} = (z-\lambda_k^{-1})^{-1} + \tilde{c}l(\alpha)^{-1} + o(|l(\alpha)|^{-1})$ with some $c_1 \ne 0$. Analyzing similarly as in Theorem 7, we can get the leading term of $e_l(\lambda) = c_1 e^{-1/c_2(\lambda-\lambda_k)}$. We are in case (a).

If $P(z-K(0))^{-1}\phi_j = 0$, then by Proposition 11 and the definition of W, $\phi_j \in W$. Thus,

$$\begin{aligned}\widetilde{K}_1(\alpha)\phi_j &= \left(z-\widetilde{K}(\alpha)\right)^{-1} K_1(\alpha)\left(z-\widetilde{K}(\alpha)\right)^{-1}\phi_j \\ &\quad + \left(z-\widetilde{K}(\alpha)\right)^{-1}\sum_{n=2}^{\infty}\left[K_1(\alpha)\left(z-\widetilde{K}(\alpha)\right)^{-1}\right]^n\phi_j \\ &= \left(z-\widetilde{K}(\alpha)\right)^{-1} K_1(\alpha)(z-K(0))^{-1}\phi_j \\ &\quad + \left(z-\widetilde{K}(\alpha)\right)^{-1}\sum_{n=2}^{\infty}\left[K_1(\alpha)\left(z-\widetilde{K}(\alpha)\right)^{-1}\right]^n\phi_j \\ &= \left(z-\lambda_k^{-1}\right)^{-1}\left(z-\widetilde{K}(\alpha)\right)^{-1} K_1(\alpha)\phi_j \\ &\quad + \left(z-\widetilde{K}(\alpha)\right)^{-1}\sum_{n=2}^{\infty}\left[K_1(\alpha)\left(z-\widetilde{K}(\alpha)\right)^{-1}\right]^n\phi_j.\end{aligned} \tag{56}$$

In the second and third steps, we use Proposition 11 and the fact that $\phi_j \in W$. Then,

$$\begin{aligned}&\left\langle \phi_j, \left(z-\lambda_k^{-1}\right)^{-1}\left(z-\widetilde{K}(\alpha)\right)^{-1} K_1(\alpha)\phi_j\right\rangle \\ &\quad = \left(z-\lambda_k^{-1}\right)^{-2}\left\langle \phi_j, K_1(\alpha)\phi_j\right\rangle \\ &\quad = \left(z-\lambda_k^{-1}\right)^{-2} \\ &\qquad \times \left[\sum_{\nu\in\sigma_1}\alpha_\nu c_\nu \sum_{s=1}^{n_\nu}\left|\left\langle |V|^{1/2}\phi_j, |y|^{-(d-2)/2+\nu}\varphi_\nu^{(s)}\right\rangle\right|^2 \right. \\ &\qquad \left. + \alpha\left\langle |V|^{1/2}\phi_j, F_1|V|^{1/2}\phi_j\right\rangle\right] + o(|\alpha|).\end{aligned} \tag{57}$$

If $\{\nu \in \sigma_1;\ \exists 1 \le s \le n_\nu$, such that $\langle |V|^{1/2}\phi_j, |y|^{-(d-2)/2+\nu} \varphi_\nu^{(s)}\rangle \neq 0\} \neq \emptyset$, set $\nu_j = \min\{\nu \in \sigma_1; \langle |V|^{1/2}\phi_j, |y|^{-(d-2)/2+\nu} \varphi_\nu^{(s)}\rangle \neq 0\}$. If $\nu_j < 1$, then, by (57),

$$\left\langle \phi_j, \left(z-\lambda_k^{-1}\right)^{-1}\left(z-\widetilde{K}(\alpha)\right)^{-1}K_1(\alpha)\phi_j\right\rangle = c_1\alpha^{\nu_j} + o\left(|\alpha|^{\nu_j}\right);$$
$$\left\langle \phi_s, \left(z-\lambda_k^{-1}\right)^{-1}\left(z-\widetilde{K}(\alpha)\right)^{-1}K_1(\alpha)\phi_j\right\rangle = O\left(|\alpha|^{\nu_j}\right). \tag{58}$$

Computing similarly as before, we also can obtain that

$$\left\langle \phi_s, \left(z-\widetilde{K}(\alpha)\right)^{-1}\left[K_1(\alpha)\left(z-\widetilde{K}(\alpha)\right)^{-1}\right]^n\phi_j\right\rangle = o\left(|\alpha|^{\nu_j}\right) \quad \text{for } n \ge 2. \tag{59}$$

By (53), one has

$$\left(z-\mu_j(\alpha)\right)^{-1} = \left(z-\lambda_k^{-1}\right)^{-1} + c_1\alpha^{\nu_j} + o\left(|\alpha|^{\nu_j}\right). \tag{60}$$

Computing similarly as in Theorem 7, we can get that the leading term of $e_l(\lambda)$ is $c(\lambda-\lambda_k)^{\nu_j}$. We are in case (b).

If $\nu_j = 1$, computing similarly as in Theorem 7, we can get the leading term of $e_l(\lambda) = c(\lambda-\lambda_k)/\ln(\lambda-\lambda_k)$, and we are in case (c).

If $\{\nu \in \sigma_1; \langle |V|^{1/2}\phi_j, |y|^{-(d-2)/2+\nu}\varphi_\nu^{(s)}\rangle \neq 0\} = \emptyset$, then for $\nu \in \sigma_1$,

$$\begin{aligned} &G_{\nu,\delta_\nu}\pi_\nu|V|^{1/2}\phi_j \\ &= c_\nu\sum_{s=1}^{n_\nu}\left\langle |V|^{1/2}\phi_j, |y|^{-(d-2)/2+\nu}\varphi_\nu^{(s)}\right\rangle |x|^{-(d-2)/2+\nu}\varphi_\nu^{(s)} = 0. \end{aligned} \tag{61}$$

It follows that

$$\begin{aligned} &\left\langle \phi_j, |V|^{1/2}\alpha F_1|V|^{1/2}\phi_j\right\rangle \\ &= \left\langle \phi_j, |V|^{1/2}\left(R_0(\alpha) - F_0 - \sum_{\nu\in\sigma_1}\alpha_\nu G_{\nu,\delta_\nu}\pi_\nu - o(|\alpha|)\right)|V|^{1/2}\phi_j\right\rangle \\ &= \left\langle \phi_j, |V|^{1/2}\left(R_0(\alpha) - F_0 - o(|\alpha|)\right)|V|^{1/2}\phi_j\right\rangle \\ &= \left\langle \phi_j, \lim_{\beta\to 0}|V|^{1/2}\left[R_0(\alpha) - \left(R_0(\beta) - \ln\beta G_{0,0}\pi_0\right) - o(|\alpha|)\right]|V|^{1/2}\phi_j\right\rangle \\ &= \left\langle \phi_j, \lim_{\beta\to 0}\left[|V|^{1/2}(\alpha-\beta)R_0(\alpha)R_0(\beta)|V|^{1/2}\phi_j - |V|^{1/2}\ln\beta G_{0,0}\pi_0|V|^{1/2}\phi_j\right]\right\rangle - o(|\alpha|) \\ &= \left\langle \phi_j, \lim_{\beta\to 0}\left[(\alpha-\beta)|V|^{1/2}R_0(\alpha) \times \left(F_0 + \ln\beta G_{0,0}\pi_0\right)|V|^{1/2}\phi_j\right]\right\rangle - o(|\alpha|) \\ &= \left\langle \phi_j, \lim_{\beta\to 0}(\alpha-\beta)|V|^{1/2}R_0(\alpha)F_0|V|^{1/2}\phi_j\right\rangle - o(|\alpha|) \\ &= \left\langle \phi_j, \alpha|V|^{1/2}R_0(\alpha)F_0|V|^{1/2}\phi_j\right\rangle - o(|\alpha|) \\ &= \left\langle \phi_j, \alpha|V|^{1/2}R_0(\alpha)F_0|V|^{1/2}\phi_j - o(|\alpha|)\right\rangle. \end{aligned} \tag{62}$$

Thus,

$$\begin{aligned} &\left\langle \phi_j, |V|^{1/2}F_1|V|^{1/2}\phi_j\right\rangle \\ &= \lim_{\alpha\to 0}\left\langle F_0|V|^{1/2}\phi_j, R_0(\alpha)F_0|V|^{1/2}\phi_j\right\rangle \\ &= \left\|F_0|V|^{1/2}\phi_j\right\|^2 \neq 0. \end{aligned} \tag{63}$$

Computing as before, we obtain

$$\left\langle \phi_j, \left(z-\lambda_k^{-1}\right)^{-1}\left(z-\widetilde{K}(\alpha)\right)^{-1}K_1(\alpha)\phi_j\right\rangle = c_1\alpha + o(|\alpha|);$$
$$\left\langle \phi_s, \left(z-\lambda_k^{-1}\right)^{-1}\left(z-\widetilde{K}(\alpha)\right)^{-1}K_1(\alpha)\phi_j\right\rangle = O(|\alpha|);$$
$$\left\langle \phi_s, \left(z-\widetilde{K}(\alpha)\right)^{-1}\left[K_1(\alpha)\left(z-\widetilde{K}(\alpha)\right)^{-1}\right]^n\phi_j\right\rangle = o(|\alpha|) \quad \text{for } n \ge 2. \tag{64}$$

By (53), one has

$$\left(z-\mu_j(\alpha)\right)^{-1} = \left(z-\lambda_k^{-1}\right)^{-1} + c_1\alpha + o(|\alpha|). \tag{65}$$

Computing similarly as in Theorem 7, we can get the leading term of $e_l(\lambda) = c(\lambda-\lambda_k)$. We are in case (d). This completes the proof. □

Conflict of Interests

The authors declare that they have no conflict of interests.

Authors' Contribution

All authors contributed equally to the paper and read and approved the final paper.

Acknowledgment

This research is supported by the Natural Science Foundation of China (11301153, 11101127, and 11271110).

References

[1] A. Jensen and G. Nenciu, "A unified approach to resolvent expansions at thresholds," *Reviews in Mathematical Physics*, vol. 13, no. 6, pp. 717–754, 2001.

[2] M. Murata, "Asymptotic expansions in time for solutions of Schrödinger-type equations," *Journal of Functional Analysis*, vol. 49, no. 1, pp. 10–56, 1982.

[3] J. Rauch, "Local decay of scattering solutions to Schrödinger's equation," *Communications in Mathematical Physics*, vol. 61, no. 2, pp. 149–168, 1978.

[4] X. Wang, "Thresholdenergy resonance in geometric scattering," in *Proceedings of the Symposium Scattering and Spectral Theory*, vol. 26, pp. 135–164, Matem atica Contempor anea, Recife, Brazil, 2004.

[5] X. Wang, "Asymptotic expansion in time of the Schrödinger group on conical manifolds," *AnnaLes de L'Institut Fourier*, vol. 56, no. 6, pp. 1903–1946, 2006.

[6] X. Jia and Y. Zhao, "Coupling constant limits of Schrödinger operators with critical potentials," *Boundary Value Problems*, vol. 2013, no. 1, 15 pages, 2013.

[7] X. Wang, "Number of eigenvalues for dissipative Schrödinger operators under perturbation," *Journal de Mathématiques Pures et Appliquées*, vol. 96, no. 5, pp. 409–422, 2011.

[8] M. Klaus and B. Simon, "Coupling constant thresholds in nonrelativistic quantum mechanics, I. Short-range two-body case," *Annals of Physics*, vol. 130, no. 2, pp. 251–281, 1980.

[9] S. Fassari and M. Klaus, "Coupling constant thresholds of perturbed periodic Hamiltonians," *Journal of Mathematical Physics*, vol. 39, no. 9, pp. 4369–4416, 1998.

[10] C. Hainzl and R. Seiringer, "Asymptotic behavior of eigenvalues of Schrödinger type operators with degenerate kinetic energy," *Mathematische Nachrichten*, vol. 283, no. 3, pp. 489–499, 2010.

[11] V. Vougalter, "On threshold eigenvalues and resonances for the linearized NLS equation," *Mathematical Modelling of Natural Phenomena*, vol. 5, no. 4, pp. 448–469, 2010.

[12] A. Laptev and M. Solomyak, "On the negative spectrum of the two-dimensional Schrödinger operator with radial potential," *Communications in Mathematical Physics*, vol. 314, no. 1, pp. 229–241, 2012.

[13] G. Rozenblum and M. Solomyak, "Spectral estimates for Schrödinger operators with sparse potentials on graphs," *Journal of Mathematical Sciences*, vol. 176, no. 3, pp. 458–474, 2011.

[14] A. Pushnitski, "The Birman-Schwinger principle on the essential spectrum," *Journal of Functional Analysis*, vol. 261, no. 7, pp. 2053–2081, 2011.

[15] M. Reed and B. Simon, *Methods of Modern Mathematical Physics: Analysis of Operators*, vol. 4, Academic Press, New York, NY, USA, 1978.