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Research Article

Impulsive Antiperiodic Boundary Value Problems for Nonlinear q_k-Difference Equations

Lihong Zhang,[1] Bashir Ahmad,[2] and Guotao Wang[1]

[1] *School of Mathematics and Computer Science, Shanxi Normal University, Linfen, Shanxi 041004, China*
[2] *Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Lihong Zhang; zhanglih149@126.com

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We show the existence and uniqueness of solutions for an antiperiodic boundary value problem of nonlinear impulsive q_k-difference equations by applying some well-known fixed point theorems. An example is presented to illustrate the main results.

1. Introduction

The subject of q-calculus (also known as quantum calculus) is regarded as ordinary calculus without the idea of limit. The systematic development of q-calculus started with the work of Jackson [1] at the beginning of the twentieth century. The application of q-calculus covers a variety of topics such as special functions, particle physics and supersymmetry, combinatorics, initial and boundary value problems of q-difference equations, operator theory, and control theory. For details of the advancement of q-calculus, we refer the reader to the texts [2–4] and papers [5–13].

One of the advantages for considering q-difference equations is that these equations are always completely controllable and appear in the q-optimal control problem [14]. The variational q-calculus is regarded as a generalization of the continuous variational calculus due to the presence of an extra parameter q whose nature may be physical or economical. The variational calculus on the q-uniform lattice includes the study of the q-Euler equations and its applications to the isoperimetric and Lagrange problem and commutation equations. In other words, it suffices to solve the q-Euler-Lagrange equation for finding the extremum of the functional involved instead of solving the Euler-Lagrange equation [15]. Further details can be found in [16–22].

Impulsive differential equations have extensively been studied in the past two decades. In particular, initial and boundary value problems of impulsive fractional differential equations have attracted the attention of many researchers; for instance, see [23–34] and references therein. In a recent work [32], the authors discussed the existence and uniqueness of solutions for impulsive q_k-difference equations.

Motivated by [32], we investigate the existence and uniqueness of solutions for an antiperiodic boundary value problem of nonlinear impulsive q_k-difference equation in this paper. Precisely, we consider

$$\begin{aligned} D_{q_k} u(t) &= f(t, u(t)), \quad 0 < q_k < 1, \ t \in J', \\ \Delta u(t_k) &= I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ u(0) &= -u(T), \end{aligned} \tag{1}$$

where D_{q_k} are q_k-derivatives ($k = 0, 1, 2, \ldots, m$), $f \in C(J \times \mathbb{R}, \mathbb{R})$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $J = [0, T]$ ($T > 0$), $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, and $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limits of $u(t)$ at $t = t_k$ ($k = 1, 2, \ldots, m$), respectively.

Here, we remark that the classical q-difference equations cannot be considered with impulses as the definition of q-derivative fails to work when an impulse point $t_k \in (qt, t)$ for some $k \in \mathbb{N}$. On the other hand, this situation does not arise for impulsive problems on q-time scale as the points t and qt are consecutive points. In quantum calculus on finite intervals, the points t and $q_k t + (1 - q_k)t_k$ are considered only in an interval $[t_k, t_{k+1}]$. Therefore, the problems with impulses at fixed times can be considered in the framework of q_k-calculus. For more details, see [32].

2. Preliminaries

Let us set $J_0 = [0,t_1]$, $J_1 = (t_1,t_2],\ldots,J_{m-1} = (t_{m-1},t_m]$, $J_m = (t_m,T]$ and introduce the space as follows:

$$\mathrm{PC}(J,\mathbb{R}) = \{u : J \to \mathbb{R} \mid u \in C(J_k), k = 0,1,\ldots,m, \text{ and } u(t_k^+) \text{ exist, } k = 1,2,\ldots,m\} \tag{2}$$

with the norm $\|u\| = \sup_{t\in J}|u(t)|$. Then, $\mathrm{PC}(J,\mathbb{R})$ is a Banach space.

Let us recall some basic concepts of q_k-calculus [32].

For $0 < q_k < 1$ and $t \in J_k$, we define the q_k-derivatives of a real valued continuous function f as

$$\begin{aligned} D_{q_k} f(t) &= \frac{f(t) - f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}, \\ D_{q_k} f(t_k) &= \lim_{t\to t_k} D_{q_k} f(t). \end{aligned} \tag{3}$$

Higher order q_k-derivatives are given by

$$\begin{aligned} D_{q_k}^0 f(t) &= f(t), \\ D_{q_k}^n f(t) &= D_{q_k} D_{q_k}^{n-1} f(t), \\ & n \in \mathbb{N},\ t \in J_k. \end{aligned} \tag{4}$$

The q_k-integral of a function f is defined by

$$\begin{aligned} {}_{t_k}I_{q_k} f(t) &:= \int_{t_k}^t f(s)\, d_{q_k}s \\ &= (1-q_k)(t-t_k) \times \sum_{n=0}^\infty q_k^n f(q_k^n t + (1-q_k^n)t_k), \\ & \qquad t \in J_k, \end{aligned} \tag{5}$$

provided that the series converges. If $a \in (t_k,t)$ and f is defined on the interval (t_k,t), then

$$\int_a^t f(s)\, d_{q_k}s = \int_{t_k}^t f(s)\, d_{q_k}s - \int_{t_k}^a f(s)\, d_{q_k}s. \tag{6}$$

Observe that

$$\begin{aligned} D_{q_k}\left({}_{t_k}I_{q_k} f(t)\right) &= D_{q_k}\int_{t_k}^t f(s)\, d_{q_k}s = f(t), \\ {}_{tk}I_{qk}\left(D_{qk} f(t)\right) &= \int_{tk}^t D_{qk} f(s)\, d_{qk}s = f(t), \\ {}_aI_{qk}\left(D_{qk} f(t)\right) &= \int_a^t D_{qk} f(s)\, d_{qk}s = f(t) - f(a), \\ & \qquad a \in (t_k,t). \end{aligned} \tag{7}$$

Note that if $t_k = 0$ and $q_k = q$ in (3) and (5), then $D_{q_k} f = D_q f$ and ${}_{t_k}I_{q_k} f = {}_0I_q f$, where D_q and ${}_0I_q$ are the well-known q-derivative and q-integral of the function $f(t)$ defined by

$$\begin{aligned} D_q f(t) &= \frac{f(t) - f(qt)}{(1-q)t}, \\ {}_0I_q f(t) &= \int_0^t f(s)\, d_qs = \sum_{n=0}^\infty t(1-q)q^n f(tq^n). \end{aligned} \tag{8}$$

Lemma 1. *A function $u \in C(J,\mathbb{R})$ is a solution of the impulsive antiperiodic boundary value problem* (1) *if and only if it is a solution of the following impulsive q_k-integral equation:*

$$u(t) = \begin{cases} \displaystyle\int_0^t f(s,u(s))\, d_{q_0}s \\ \displaystyle -\frac{1}{2}\left[\sum_{i=0}^m \int_{t_i}^{t_{i+1}} f(s,u(s))\, d_{q_i}s + \sum_{i=1}^m I_i(u(t_i))\right], & t \in J_0; \\ \displaystyle\int_{t_k}^t f(s,u(s))\, d_{q_k}s \\ \displaystyle + \sum_{0<t_k<t}\left[\int_{t_{k-1}}^{t_k} f(s,u(s))\, d_{q_{k-1}}s + I_k(u(t_k))\right] \\ \displaystyle -\frac{1}{2}\left[\sum_{i=0}^m \int_{t_i}^{t_{i+1}} f(s,u(s))\, d_{q_i}s + \sum_{i=1}^m I_i(u(t_i))\right], & t \in J_k. \end{cases} \tag{9}$$

Proof. Let u be a solution of (1). For $t \in J_0$, q_0-integrating both sides of (1), we get

$$u(t) = u(0) +_{t_0} I_{q_0} f(t,u(t)) = u(0) + \int_0^t f(s,u(s))\, d_{q_0}s. \tag{10}$$

Thus, we have

$$u(t_1^-) = u(0) + \int_0^{t_1} f(s,u(s))\, d_{q_0}s. \tag{11}$$

For $t \in J_1$, q_1-integrating both sides of (1), we obtain

$$u(t) = u(t_1^+) + \int_{t_1}^t f(s,u(s))\, d_{q_1}s. \tag{12}$$

In view of $\Delta u(t_1) = u(t_1^+) - u(t_1^-) = I_1(u(t_1))$, it follows that

$$\begin{aligned} u(t) &= u(0) + \int_{t_1}^t f(s,u(s))\, d_{q_1}s \\ &\quad + \int_0^{t_1} f(s,u(s))\, d_{q_0}s + I_1(u(t_1)), \quad \forall t \in J_1. \end{aligned} \tag{13}$$

Similarly, we get

$$\begin{aligned} u(t) &= u(0) + \int_{t_k}^t f(s,u(s))\, d_{q_k}s \\ &\quad + \sum_{0<t_k<t}\left[\int_{t_{k-1}}^{t_k} f(s,u(s))\, d_{q_{k-1}}s + I_k(u(t_k))\right], \\ & \qquad t \in J_k. \end{aligned} \tag{14}$$

Using the antiperiodic boundary value condition $u(0) = -u(T)$, we obtain (9). Conversely, assume that u is a solution of the impulsive q_k-integral equation (9); then by a direct computation, it follows that the solution given by (9) satisfies q_k-difference equation (1). This completes the proof. □

3. Main Results

Define an operator $\mathcal{T} : \mathrm{PC}(J, \mathbb{R}) \to \mathrm{PC}(J, \mathbb{R})$ as

$$\begin{aligned}\mathcal{T}u(t) = {} & \int_{t_k}^{t} f(s, u(s))\, d_{q_k}s \\ & + \sum_{0<t_k<t} \left[\int_{t_{k-1}}^{t_k} f(s, u(s))\, d_{q_{k-1}}s + I_k(u(t_k)) \right] \\ & - \frac{1}{2}\left[\sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} f(s, u(s))\, d_{q_i}s + \sum_{i=1}^{m} I_i(u(t_i)) \right].\end{aligned} \tag{15}$$

Obviously, problem (1) is equivalent to a fixed point problem $u = \mathcal{T}u$. In consequence, problem (1) has a solution if and only if the operator $\mathcal{T}$ has a fixed point.

Theorem 2. *Assume that there exist continuous functions $a(t), b(t)$ and a nonnegative constant L such that*

$$\begin{aligned}\left| f(t, u(t)) \right| &\le a(t) + b(t)\left| u(t) \right|, \\ \left| I_k(u) \right| &\le L, \quad k = 1, 2, \ldots, m.\end{aligned} \tag{16}$$

Then, problem (1) *has at least one solution.*

Proof. Let us denote $\sup_{t\in J}|a(t)| = A$ and $\sup_{t\in J}|b(t)| = B$. Take $R \ge 3(AT + mL)/(2 - 3BT) > 0$ and define $B_R = \{u \in \mathrm{PC}(J, \mathbb{R}) \mid \|u\| \le R\}$. It is easy to verify that B_R is a bounded, closed, and convex subset of $\mathrm{PC}(J, \mathbb{R})$.

In order to show that there exists a solution for problem (1), we have to establish that the operator $\mathcal{T}$ has a fixed point in B_R. The proof consists of two steps:

(i) $\mathcal{T} : B_R \to B_R$.

For any $u \in B_R$, we have

$$\begin{aligned}|\mathcal{T}u(t)| \le {} & \int_{t_k}^{t} \left| f(s, u(s)) \right| d_{q_k}s \\ & + \sum_{0<t_k<t} \left[\int_{t_{k-1}}^{t_k} \left| f(s, u(s)) \right| d_{q_{k-1}}s + \left| I_k(u(t_k)) \right| \right] \\ & + \frac{1}{2}\left[\sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} \left| f(s, u(s)) \right| d_{q_i}s + \sum_{i=1}^{m} \left| I_i(u(t_i)) \right| \right] \\ \le {} & \int_{t_k}^{t} [a(s) + b(s)|u(s)|]\, d_{q_k}s \\ & + \sum_{0<t_k<t} \left[\int_{t_{k-1}}^{t_k} [a(s) + b(s)|u(s)|]\, d_{q_{k-1}}s + L \right] \\ & + \frac{1}{2}\left[\sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} [a(s) + b(s)|u(s)|]\, d_{q_i}s + \sum_{i=1}^{m} L \right] \\ \le {} & [A + B\|u\|](t - t_k) \\ & + [A + B\|u\|]\, t_k + mL + \frac{1}{2}[(A + B\|u\|)T + mL] \\ \le {} & \frac{3(AT + mL)}{2} + \frac{3BT\|u\|}{2} \le R,\end{aligned} \tag{17}$$

which means $\|\mathcal{T}u\| \le R$. So, $\mathcal{T}$ is $B_R \to B_R$. Consider the following:

(ii) the operator $\mathcal{T}$ is relatively compact.

Let $\sup_{(t,u)\in J\times B_R}|f(t, u)| = \overline{f}$. For any $t', t'' \in J_k$ $(k = 0, 1, 2, \ldots, m)$ with $t' < t''$, we have

$$\begin{aligned}&\left| \mathcal{T}u(t'') - \mathcal{T}u(t') \right| \\ &\quad \le \left| \int_{t_k}^{t''} f(s, u(s))\, d_{q_k}s - \int_{t_k}^{t'} f(s, u(s))\, d_{q_k}s \right| \\ &\quad \le \int_{t'}^{t''} \left| f(s, u(s)) \right| d_{q_k}s \\ &\quad \le \overline{f}\left| t'' - t' \right|\end{aligned} \tag{18}$$

which is independent of u and tends to zero as $t'' - t' \to 0$. Thus, $\mathcal{T}$ is equicontinuous. Hence, $\mathcal{T}B_R$ is relatively compact as $\mathcal{T}B_R \subset B_R$ is uniformly bounded. Further, it is obvious that the operator $\mathcal{T}$ is continuous in view of continuity of f and I_k. Therefore, the operator $\mathcal{T} : \mathrm{PC}(J, \mathbb{R}) \to \mathrm{PC}(J, \mathbb{R})$ is completely continuous on B_R. By the application of Schauder fixed point theorem, we conclude that the operator $\mathcal{T}$ has at least one fixed point in B_R. This, in turn, implies that problem (1) has at least one solution. □

Theorem 3. *Assume that there exist a function $M(t) \in C(J, \mathbb{R}^+)$ and a positive constant N such that $3(MT + mN) < 2$ and*

$$\begin{aligned}\left| f(t, u) - f(t, v) \right| &\le M(t)|u - v|, \\ \left| I_k(u) - I_k(v) \right| &\le N|u - v|,\end{aligned} \tag{19}$$

for $t \in J$, $u, v \in \mathbb{R}$ and $k = 1, 2, \ldots, m$, and $M = \sup_{t\in J}|M(t)|$. Then, problem (1) *has a unique solution.*

Proof. For $\forall u, v \in \mathrm{PC}(J, \mathbb{R})$, we have

$$
\begin{aligned}
&|(\mathcal{T}u)(t) - (\mathcal{T}v)(t)| \\
&\leq \int_{t_k}^{t} |f(s,u(s)) - f(s,v(s))|\, d_{q_k}s \\
&\quad + \sum_{0<t_k<t} \left[\int_{t_{k-1}}^{t_k} |f(s,u(s)) - f(s,v(s))|\, d_{q_{k-1}}s + |I_k(u(t_k)) - I_k(v(t_k))| \right] \\
&\quad + \frac{1}{2}\left[\sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} |f(s,u(s)) - f(s,v(s))|\, d_{q_i}s + \sum_{i=1}^{m} |I_i(u(t_i)) - I_i(v(t_i))| \right] \\
&\leq \int_{t_k}^{t} M(s)\, |(u-v)(s)|\, d_{q_k}s \\
&\quad + \sum_{0<t_k<t} \left[\int_{t_{k-1}}^{t_k} M(s)\, |(u-v)(s)|\, d_{q_{k-1}}s + N\, |(u-v)(t_k)| \right] \\
&\quad + \frac{1}{2}\left[\sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} M(s)\, |(u-v)(s)|\, d_{q_i}s + \sum_{i=1}^{m} N\, |(u-v)(t_i)| \right] \\
&\leq \left\{ \int_{t_k}^{t} M(s)\, d_{q_k}s + \sum_{0<t_k<t} \left[\int_{t_{k-1}}^{t_k} M(s)\, d_{q_{k-1}}s + N \right] + \frac{1}{2}\left[\sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} M(s)\, d_{q_i}s + mN \right] \right\} \|u-v\| \\
&\leq \left\{ [MT + mN] + \frac{1}{2}[MT + mN] \right\} \|u-v\| \\
&\leq \frac{3(MT+mN)}{2} \|u-v\|.
\end{aligned} \tag{20}
$$

By the given condition, $3(MT + mN) < 2$, it follows that $\|\mathcal{T}u - \mathcal{T}v\| < \|u - v\|$. Therefore, $\mathcal{T}$ is a contraction. By the contraction mapping principle, problem (1) has a unique solution. □

4. Example

Example 1. Consider impulsive antiperiodic boundary value problem of nonlinear q_k-difference equation:

$$
\begin{gathered}
D_{1/(5+k)}u(t) = \frac{t^2}{10}\sin u(t) + 2 + e^t, \quad t \in [0,1],\ t \neq \frac{k}{3+k}, \\
\Delta u\left(\frac{k}{3+k}\right) = e^{-u^2(k/(3+k))}, \quad k = 1, 2, \ldots, 5, \\
u(0) = -u(1),
\end{gathered} \tag{21}
$$

where $q_k = 1/(5+k)$ $(k = 0, 1, 2, \ldots, 5)$, $t_k = k/(3+k)$ $(k = 1, 2, \ldots, 5)$, $f(t,u) = (t^2/10)\sin u + 2 + e^t$, and $I_k(u) = e^{-u^2}$. With $a(t) = 2 + e^t$, $b(t) = t^2/10$, and $L = 1$, it is easy to verify that all conditions of Theorem 2 hold. Thus, by Theorem 2, problem (21) has at least one solution.

Example 2. Consider impulsive antiperiodic boundary value problem of nonlinear q_k-difference equation:

$$
\begin{gathered}
D_{1/(4+k)}u(t) = \frac{t^3}{15}\arctan u(t) + \ln(2+t), \\
t \in [0,1],\ t \neq \frac{2+k}{3+k}, \\
\Delta u\left(\frac{2+k}{3+k}\right) = \frac{e^t}{45}\sin u\left(\frac{2+k}{3+k}\right), \quad k = 1, 2, 3, \\
u(0) = -u(1),
\end{gathered} \tag{22}
$$

where $q_k = 1/(4+k)$ $(k = 0, 1, 2, 3)$, $t_k = (2+k)/(3+k)$ $(k = 1, 2, 3)$, $f(t,u) = (t^3/15)\arctan u + \ln(2+t)$, and $I_k(u) = (e^t/45)\sin u$. With $M(t) = t^3/15$, $M = 1/15$, and $N = e/45$, combining with $T = 1$ and $m = 3$, it is easy to verify that all conditions of Theorem 3 hold. Thus, by Theorem 3, problem (22) has a unique solution.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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