Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 154294, 5 pages
http://dx.doi.org/10.1155/2014/154294

Research Article

Some Surfaces with Zero Curvature in $\mathbb{H}^2 \times \mathbb{R}$

Dae Won Yoon

Department of Mathematics Education and RINS, Gyeongsang National University, Jinju 660-701, Republic of Korea

Correspondence should be addressed to Dae Won Yoon; dwyoon@gnu.ac.kr

Received 24 December 2013; Accepted 26 February 2014; Published 24 March 2014

Academic Editor: Chong Lin

We study surfaces defined as graph of the function $z = f(x, y)$ in the product space $\mathbb{H}^2 \times \mathbb{R}$. In particular, we completely classify flat or minimal surfaces given by $f(x, y) = u(x) + v(y)$, where $u(x)$ and $v(y)$ are smooth functions.

1. Introduction

Homogenous geometries have main roles in the modern theory of manifolds. Homogenous spaces are, in a sense, the nicest examples of Riemannian manifolds and have applications in physics [1]. To underline their importance from the mathematical point of view we roughly cite the famous Thurston conjecture. This conjecture asserts that every compact orientable 3-dimensional manifold has a canonical decomposition into pieces, each of which admits a canonical geometric structure from among the eight maximal simple connected homogenous Riemannian 3-dimensional geometries [2]. The Riemannian product space $\mathbb{H}^2 \times \mathbb{R}$ is one of the eight model spaces.

Constant mean curvature and constant Gaussian curvature surfaces are one of the main objects which have drawn geometers' interest for a very long time. Recently, the study of the geometry of surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is growing very rapidly, and the interest is mainly focused on minimal and constant mean curvature surfaces [3–9].

The purpose of this paper is to study surfaces defined as graph of the function $z = f(x, y)$ in the product space $\mathbb{H}^2 \times \mathbb{R}$. In Sections 4 and 5 we classify minimal and flat surfaces defined as $f(x, y) = u(x) + v(y)$, where $u(x)$ and $v(y)$ are smooth functions.

2. Preliminaries

Let $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half plane model of the hyperbolic plane endowed with the metric, of constant Gaussian curvature -1, given by

$$g_{\mathbb{H}} = \frac{\left(dx^2 + dy^2\right)}{y^2}. \tag{1}$$

The hyperbolic space $\mathbb{H}^2$, with the group structure derived by the composition of proper affine maps, is a Lie group and the metric $g_{\mathbb{H}}$ is left invariant. Therefore, the product space $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with the left invariant product metric

$$g = \frac{dx^2 + dy^2}{y^2} + dz^2. \tag{2}$$

On the other hand, an orthonormal basis of left invariant vector fields on $\mathbb{H}^2 \times \mathbb{R}$ is

$$E_1 = y\frac{\partial}{\partial x}, \qquad E_2 = y\frac{\partial}{\partial y}, \qquad E_3 = \frac{\partial}{\partial z} \tag{3}$$

with the only nontrivial commutator relation $[E_1, E_2] = -E_1$. It follows that the Levi-Civita connection $\widetilde{\nabla}$ of $\mathbb{H}^2 \times \mathbb{R}$ is expressed as

$$\begin{aligned} &\widetilde{\nabla}_{E_1} E_1 = E_2, \qquad \widetilde{\nabla}_{E_1} E_2 = -E_1, \qquad \widetilde{\nabla}_{E_1} E_3 = 0, \\ &\widetilde{\nabla}_{E_2} E_1 = 0, \qquad \widetilde{\nabla}_{E_2} E_2 = 0, \qquad \widetilde{\nabla}_{E_2} E_3 = 0, \\ &\widetilde{\nabla}_{E_3} E_1 = 0, \qquad \widetilde{\nabla}_{E_3} E_2 = 0, \qquad \widetilde{\nabla}_{E_3} E_3 = 0. \end{aligned} \tag{4}$$

For any vectors $X = x_1E_1 + y_1E_2 + z_1E_3$ and $Y = x_2E_1 + y_2E_2 + z_2E_3$ in $\mathbb{H}^2 \times \mathbb{R}$ the cross-product $\times$ is defined by

$$\begin{aligned} X \times Y = &\left(y_1z_2 - y_2z_1\right)E_1 + \left(x_2z_1 - x_1z_2\right)E_2 \\ &+ \left(x_1y_2 - x_2y_1\right)E_3. \end{aligned} \tag{5}$$

3. Graphs in $\mathbb{H}^2 \times \mathbb{R}$

Let us consider a surface Σ parametrized by

$$\phi(x,y) = (x, y, f(x,y)), \quad (x,y) \in \Omega, \tag{6}$$

where Ω is a domain in $\mathbb{H}^2$ and $f : \Omega \to \mathbb{R}$ is a smooth function. Then Σ is a surface defined as graph of the function f defined on $\Omega \subset \mathbb{H}^2$. In this case, we have

$$\begin{aligned} e_1 &:= \phi_x = (1, 0, f_x) = \frac{1}{y}E_1 + f_x E_3, \\ e_2 &:= \phi_y = (0, 1, f_y) = \frac{1}{y}E_2 + f_y E_3. \end{aligned} \tag{7}$$

It follows that the coefficients of the first fundamental form of Σ are given by

$$\begin{aligned} E &= g(\phi_x, \phi_x) = f_x^2 + \frac{1}{y^2}, \\ F &= g(\phi_x, \phi_y) = f_x f_y, \\ G &= g(\phi_y, \phi_y) = f_y^2 + \frac{1}{y^2}. \end{aligned} \tag{8}$$

Also, the unit normal vector field U to Σ is given by

$$U(x,y) = -\frac{f_x}{\omega y}E_1 - \frac{f_y}{\omega y}E_2 + \frac{1}{\omega y^2}E_3, \tag{9}$$

where

$$\omega = \frac{1}{y^2}\sqrt{y^2\left(f_x^2 + f_y^2\right) + 1}. \tag{10}$$

By a straightforward calculation, we obtain

$$\begin{aligned} \widetilde{\nabla}_{e_1} e_1 &= \frac{1}{y^2}E_2 + f_{xx}E_3, \\ \widetilde{\nabla}_{e_1} e_2 &= -\frac{1}{y^2}E_1 + f_{xx}E_3, \\ \widetilde{\nabla}_{e_2} e_2 &= -\frac{1}{y^2}E_2 + f_{yy}E_3, \end{aligned} \tag{11}$$

which imply that the coefficients of the second fundamental form of Σ are

$$\begin{aligned} L &= g\left(\widetilde{\nabla}_{e_1} e_1, U\right) = \frac{y f_{xx} - f_y}{\omega y^3}, \\ M &= g\left(\widetilde{\nabla}_{e_1} e_2, U\right) = \frac{y f_{xy} + f_x}{\omega y^3}, \\ N &= g\left(\widetilde{\nabla}_{e_2} e_2, U\right) = \frac{y f_{yy} + f_y}{\omega y^3}. \end{aligned} \tag{12}$$

Thus, from (8) and (12) the Gaussian curvature K and the mean curvature H are, respectively,

$$\begin{aligned} K &= \frac{1}{\omega^4 y^6}\left(\left(y f_{xx} - f_y\right)\left(y f_{yy} + f_y\right) - \left(y f_{xy} + f_x\right)^2\right), \\ H &= \frac{1}{2\omega^3 y^4}\left(\left(1 + y^2 f_y^2\right) f_{xx} - y\left(f_x^2 + f_y^2\right) f_y \right. \\ &\qquad \left. - 2y^2 f_x f_y f_{xy} + \left(1 + y^2 f_x^2\right) f_{yy}\right). \end{aligned} \tag{13}$$

Proposition 1. *Let Σ be a surface defined as graph of the function $f : \Omega \subset \mathbb{H}^2 \to \mathbb{R}$. Then Σ is a minimal surface if and only if*

$$\begin{aligned} &\left(1 + y^2 f_y^2\right) f_{xx} - y\left(f_x^2 + f_y^2\right) f_y - 2y^2 f_x f_y f_{xy} \\ &\quad + \left(1 + y^2 f_x^2\right) f_{yy} = 0. \end{aligned} \tag{14}$$

Proposition 2. *Let Σ be a surface defined as graph of the function $f : \Omega \subset \mathbb{H}^2 \to \mathbb{R}$. Then Σ is flat if and only if*

$$\left(y f_{xx} - f_y\right)\left(y f_{yy} + f_y\right) - \left(y f_{xy} + f_x\right)^2 = 0. \tag{15}$$

Remark 3. Some examples are satisfying the ODE (14) studied in [7]. Also, examples in Lorentz product space $\mathbb{H}^2 \times \mathbb{R}_1$ can be found in [10].

4. Minimal Surfaces Defined by $f(x,y) = u(x) + v(y)$

Let Σ be a surface in $\mathbb{H}^2 \times \mathbb{R}$ parametrized by

$$\phi(x,y) = (x, y, u(x) + v(y)) \tag{16}$$

for all $y > 0$, where $u(x)$ and $v(y)$ are smooth functions. We suppose that Σ is a minimal surface. Then, from (14) we have the following minimal surface equation:

$$\begin{aligned} &\left(1 + y^2\left(v'\right)^2\right) u'' - y\left(\left(u'\right)^2 + \left(v'\right)^2\right) v' \\ &\quad + \left(1 + y^2\left(u'\right)^2\right) v'' = 0. \end{aligned} \tag{17}$$

In order to solve it, divide first by $1 + y^2(v')^2 \neq 0$; then we get

$$u'' - \frac{y\left(\left(u'\right)^2 + \left(v'\right)^2\right)}{1 + y^2\left(v'\right)^2} v' + \frac{1 + y^2\left(u'\right)^2}{1 + y^2\left(v'\right)^2} v'' = 0, \tag{18}$$

for all $x, y \in \Omega$. Differentiating with respect to x, we obtain

$$u''' + 2\left(\frac{y^2 v'' - y v'}{1 + y^2\left(v'\right)^2}\right) u' u'' = 0. \tag{19}$$

First of all, we suppose that $u'' = 0$ on an open interval; that is, $u(x) = ax + b$, $a, b \in \mathbb{R}$. In this case, from (17) we obtain

$$v'' - \frac{a^2 y}{1 + a^2 y^2} v' - \frac{y}{1 + a^2 y^2}\left(v'\right)^3 = 0. \tag{20}$$

We put $v'(y) = p(y)$. Then the last equation can be written as

$$p' - \frac{y}{1 + a^2 y^2} \left(a^2 p + p^3\right) = 0. \tag{21}$$

Its general solution is given by

$$p = \pm \frac{c_1 a \sqrt{1 + a^2 y^2}}{\sqrt{1 - c_1^2 \left(1 + a^2 y^2\right)}}. \tag{22}$$

From this, we thus have

$$v(y) = \pm \int \frac{c_1 a \sqrt{1 + a^2 y^2}}{\sqrt{1 - c_1^2 \left(1 + a^2 y^2\right)}} dy, \tag{23}$$

where $c_1 \in \mathbb{R}$.

Now, we assume that $u'' \neq 0$ on an open interval, and divide (19) by $u'u''$. It follows that

$$\frac{u'''}{u'u''} + 2\frac{y^2 v'' - y v'}{1 + y^2 (v')^2} = 0. \tag{24}$$

Hence we deduce the existence of a real number $k \in \mathbb{R}$ such that

$$u''' = 2ku'u'', \qquad y^2 v'' - y v' = -k\left(1 + y^2 (v')^2\right). \tag{25}$$

Let us distinguish the following cases according to k.

Case 1. If $k = 0$, then $u''' = 0$ and $yv'' - v' = 0$. It follows that $u(x) = a_1 x^2 + b_1 x + c_1$ ($a_1 \neq 0, b_1, c_1 \in \mathbb{R}$). If $v' = 0$, then $v(y) = a_2$ ($a_2 \in \mathbb{R}$). In this case, from (17) we obtain $a_1 = 0$; it is a contradiction. If $v' \neq 0$, then we get $v(y) = (1/2)b_2 y^2 + c_2$ ($b_2 \neq 0, c_2 \in \mathbb{R}$). In such case, (17) is polynomial equation on x and y. From the coefficients of y^4 and the constant term we have $2a_1 - b_2 = 0$ and $2a_1 + b_2 = 0$, which imply $a_1 = 0$ and $b_2 = 0$. It is a contradiction.

Case 2. If $k \neq 0$, then from the first equation in (25) we have

$$u'' = e^{2ku + d_1}, \tag{26}$$

where $d_1 \in \mathbb{R}$. Let

$$u = \frac{1}{2k}\left(-d_1 + \ln g\right) \tag{27}$$

be any solution of (26), where g is a smooth function. Then (26) can be rewritten as

$$gg'' - \left(g'\right)^2 = 2kg^3. \tag{28}$$

We put $p = g'$. Then, we have

$$\frac{dp}{dg} - \frac{1}{g} p = 2kg^2 p^{-1}. \tag{29}$$

We again put $t = p^2$. In this case the above equation becomes

$$\frac{dt}{dg} - \frac{2}{g} t = 4kg^2 \tag{30}$$

and its general solution is given by

$$t = g^2 \left(4kg + c_1\right). \tag{31}$$

Thus, we get

$$\frac{dg}{dx} = \pm g \sqrt{4kg + c_1}. \tag{32}$$

After an integration, we can find

$$g = \frac{c_1}{4k} \tan^2 \left(8k^2 \sqrt{c_1} \left(\pm x + c_2\right)\right) - \frac{c_1}{4k}, \tag{33}$$

where $c_2 \in \mathbb{R}$. By combining (27) and (33), we thus have

$$u(x) = \frac{1}{2k}\left[-d_1 + \ln\left(\frac{c_1}{4k} \tan^2 \left(8k^2 \sqrt{c_1} \left(\pm x + c_2\right)\right) - \frac{c_1}{4k}\right)\right]. \tag{34}$$

Now, we consider the second equation in (25). Since $y > 0$, we yield

$$v'' + \frac{k}{y^2} - \frac{1}{y} v' + k\left(v'\right)^2 = 0. \tag{35}$$

We put $p = v'$. Then, the above equation becomes

$$p' + \frac{k}{y^2} - \frac{1}{y} p + kp^2 = 0. \tag{36}$$

Since $k \neq 0$, without loss of generality we take $k = 1$ or $k = -1$.

Subcase i. Let $k = 1$. We do the change

$$p = \frac{1}{y} + \frac{1}{h(y)}, \tag{37}$$

where h is a nonzero smooth function. Then, (36) can be rewritten as the form

$$h' - \frac{1}{y} h = 1. \tag{38}$$

Thus, its general solution is

$$h(y) = y\left(\ln y + c_1\right), \tag{39}$$

where $c_1 \in \mathbb{R}$. So, $p = (1/y) + (1/y(\ln y + c_1))$ and from its integration we can obtain

$$v(y) = \ln\left(c_2 y \ln\left(y + c_1\right)\right), \tag{40}$$

where $c_2 \in \mathbb{R}$.

Subcase ii. Let $k = -1$. We put

$$p = -\frac{1}{y} + \frac{1}{h(y)}, \tag{41}$$

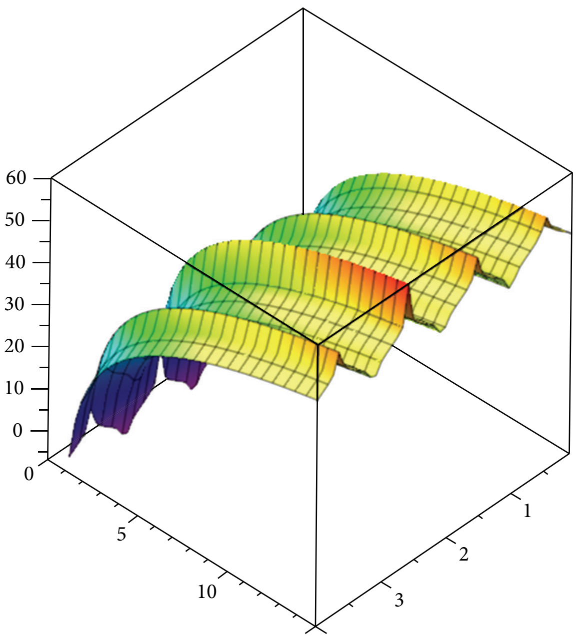

Figure 1: A minimal surface defined by (34) and (44).

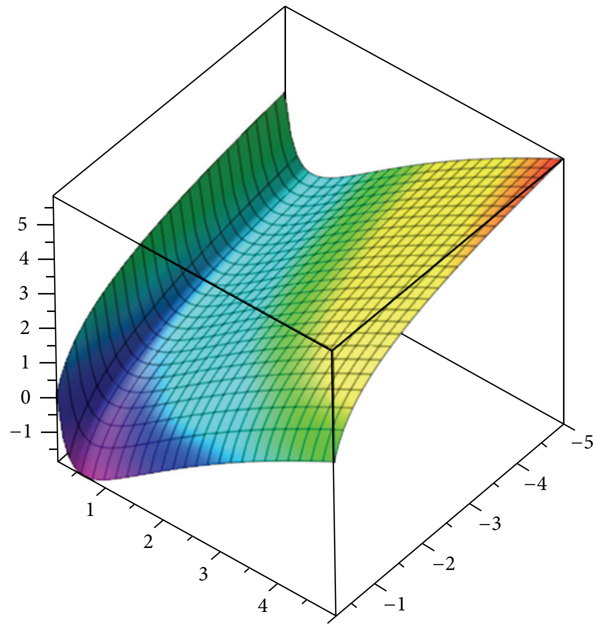

Figure 2: A flat surface defined by (52) and (55).

where h is a nonzero smooth function. Then, (36) becomes

$$h' - \frac{1}{y}h = -1 \tag{42}$$

and its general solution is given by

$$h(y) = -y(\ln y + c_1), \tag{43}$$

where $c_1 \in \mathbb{R}$. Thus, we have

$$v(y) = -\ln(c_2 y \ln(y + c_1)), \tag{44}$$

where $c_2 \in \mathbb{R}$. The surface given by (34) and (44) is shown in Figure 1.

Consequently, we have the following.

Theorem 4. *Let Σ be a surface defined as graph of the function $f(x,y) = u(x) + v(y)$. If Σ is a minimal surface, then Σ is parametrized as*

$$\phi(x,y) = (x, y, u(x) + v(y)), \tag{45}$$

where

(1) $u(x) = ax + b$ *and* $v(y) = \pm\int (c_1 a\sqrt{1+a^2y^2}/\sqrt{1-c_1^2(1+a^2y^2)})dy$ *with* $a, b, c_1 \in \mathbb{R}$*, or*

(2) $u(x) = (1/2k)[-c_3 + \ln((c_1/4k)\tan^2(8k^2\sqrt{c_1}(\pm x + c_2)) - (c_1/4k))]$ *and* $v(y) = \pm\ln(d_1 y \ln(y + d_2))$ *with* $k \neq 0, c_1, c_2, c_3, d_1, d_2 \in \mathbb{R}$.

5. Flat Surfaces Defined by $f(x,y) = u(x) + v(y)$

Let Σ be a surface defined by (16). Assume that Σ is a flat surface. Then, from (15) we have the following flat surface equation:

$$y(yv'' + v')u'' - (yv'' + v')v' - (u')^2 = 0. \tag{46}$$

In order to solve it, differentiating with respect to x, we have

$$y(yv'' + v')\frac{d}{dx}(u'') - \frac{d}{dx}((u')^2) = 0. \tag{47}$$

Thus, there exists a nonzero real number k such that

$$\frac{d}{dx}(u'') = k\frac{d}{dx}((u')^2), \qquad y(yv'' + v') = \frac{1}{k}. \tag{48}$$

From the first equation in (48), we get

$$u'' = k(u')^2 + c_1, \tag{49}$$

where $c_1 \in \mathbb{R}$. We put $p = u'$, and it follows that we yield

$$\frac{dp}{du} = \frac{kp^2 + c_1}{p}. \tag{50}$$

From this, the general solution is

$$p = \pm\sqrt{\frac{1}{k}e^{2k(u+c_2)} - \frac{c_1}{k}}, \tag{51}$$

where $c_2 \in \mathbb{R}$. We can assume that $c_1 = 0$. From the last equation we can easily obtain (see Figure 2)

$$u(x) = \pm\frac{1}{k}\left(\ln\left(-\sqrt{k}(x + c_3)\right) + kc_2\right), \tag{52}$$

where $c_3 \in \mathbb{R}$.

In order to solve the second equation in (48), divide by y^2 and put $q = v'$. Then, we get

$$q' + \frac{1}{y}q = \frac{1}{ky^2} \tag{53}$$

and its general solution is given by

$$q = \frac{1}{y}\left(\frac{1}{k}\ln y + d_1\right), \tag{54}$$

where $d_1 \in \mathbb{R}$. From this, we thus obtain (see Figure 2)

$$v(y) = \frac{1}{2k}(\ln y)^2 + d_1 \ln y + d_2, \tag{55}$$

where $d_2 \in \mathbb{R}$.

As a conclusion, we have the following.

Theorem 5. *Let Σ be a surface defined as graph of the function $f(x, y) = u(x) + v(y)$. If Σ is a flat surface, then Σ is parametrized as*

$$\phi(x, y) = (x, y, u(x) + v(y)), \tag{56}$$

where $u(x) = \pm(1/k)(\ln(-\sqrt{k}(x + c_1)) + kc_2)$ and $v(y) = (1/2k)(\ln y)^2 + d_1 \ln y + d_1$ with $k \neq 0, c_1, c_2, d_1, d_2 \in \mathbb{R}$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This paper was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A2003994).

References

[1] J. Gegenberg, S. Vaidya, and J. F. Vázquez-Poritz, "Thurston geometries from eleven dimensions," *Classical and Quantum Gravity*, vol. 19, no. 23, pp. L199–L204, 2002.

[2] W. Thurston, *Three-Dimensional Geometry and Topology*, vol. 35 of *Princeton Mathematical Series*, Princeton University Press, Princeton, NJ, USA, 1997.

[3] J. M. Espinar, M. M. Rodríguez, and H. Rosenberg, "The extrinsic curvature of entire minimal graphs in $H^2 \times R$," *Indiana University Mathematics Journal*, vol. 59, no. 3, pp. 875–889, 2010.

[4] I. Fernández and P. Mira, "Harmonic maps and constant mean curvature surfaces in $H^2 \times R$," *The American Journal of Mathematics*, vol. 129, no. 4, pp. 1145–1181, 2007.

[5] Y. W. Kim, S. E. Koh, H. Shin, and S. D. Yang, "Helicoidal minimal surfaces in $H^2 \times R$," *Bulletin of the Australian Mathematical Society*, vol. 86, no. 1, pp. 135–149, 2012.

[6] S. Montaldo and I. I. Onnis, "Invariant CMC surfaces in $H^2 \times R$," *Glasgow Mathematical Journal*, vol. 46, no. 2, pp. 311–321, 2004.

[7] S. Montaldo and I. I. Onnis, "A note on surfaces in $H^2 \times R$," *Bollettino della Unione Matematica Italiana B*, vol. 10, no. 3, pp. 939–950, 2007.

[8] B. Nelli and H. Rosenberg, "Minimal surfaces in $H^2 \times R$," *Bulletin of the Brazilian Mathematical Society*, vol. 33, no. 2, pp. 263–292, 2002.

[9] D. W. Yoon, "Minimal translation surfaces in $H^2 \times R$," *Taiwanese Journal of Mathematics*, vol. 17, no. 5, pp. 1545–1556, 2013.

[10] A. L. Albujer, "New examples of entire maximal graphs in $H^2 \times R_1$," *Differential Geometry and its Applications*, vol. 26, no. 4, pp. 456–462, 2008.