Research Article

Precise Asymptotics on Second-Order Complete Moment Convergence of Uniform Empirical Process

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Let $\{\xi_i, 1 \le i \le n\}$ be a sequence of iid $U[0,1]$-distributed random variables, and define the uniform empirical process $F_n(t) = n^{-1/2} \sum_{i=1}^{n} (I_{\{\xi_i \le t\}} - t), 0 \le t \le 1, \|F_n\| = \sup_{0 \le t \le 1} |F_n(t)|$. When the nonnegative function $g(x)$ satisfies some regular monotone conditions, it proves that $\lim_{\epsilon \searrow 0} (1/ - \log \epsilon) \sum_{n=1}^{\infty} (g'(n)/g(n)) \mathbb{E}\{\|F_n\|^2 I_{\{\|F_n\| \ge \epsilon \sqrt{g(n)}\}}\} = \pi^2/6$.

1. Introduction and Main Result

Let $\{X, X_n, n \ge 1\}$ be a sequence of iid random variables and $S_n = \sum_{i=1}^{n} X_i$. Hsu and Robbins [1] introduced the concept of complete convergence and obtained that $\sum_{n=1}^{\infty} P(|S_n| \ge \epsilon n) < \infty, \epsilon > 0$, whenever $\mathbb{E}X = 0$ and $\mathbb{E}X^2 < \infty$. The result is extended by Baum and Katz [2], who obtained that, for $0 < p < 2, r \ge p, \sum_{n=1}^{\infty} n^{(r/p)-2} P(|S_n| \ge \epsilon n^{1/p}) < \infty, \epsilon > 0$, if and only if $\mathbb{E}|X|^r < \infty$, and when $r \ge 1, \mathbb{E}X = 0$. Since then, some researchers concern the convergence of the series

$$\sum_{n=1}^{\infty} \varphi(n) P(|S_n| \ge \epsilon f(n)), \quad \epsilon > 0, \tag{1}$$

where $\varphi(n)$ and $f(n)$ are all positive functions defined on $[1, \infty)$, and $\sum_{n=1}^{\infty} \varphi(n) = \infty$. Because of the fact that the series tends to infinity when $\epsilon \searrow 0$, one of the interesting problems is to examine the rate when it occurs; then we need to find a suitable normalizing rate function $\psi(\epsilon)$ such that it, multiplied by the series, has a nontrivial limit. The research on this topic is usually called "precise asymptotics." Heyde [3] first proved that

$$\lim_{\epsilon \searrow 0} \epsilon^2 \sum_{n=1}^{\infty} P(|S_n| \ge \epsilon n) = \mathbb{E}X^2 \tag{2}$$

whenever $\mathbb{E}X = 0$ and $\mathbb{E}X^2 < \infty$. Chow [4] studied the similar result on complete moment convergence of $\sum_{n=1}^{\infty} n^{p\alpha-2-\alpha} \mathbb{E}\{\max_{1 \le j \le n} |S_j| - \epsilon n^{\alpha}\}_+$. Later, Liu and Lin [5] extended the result to the p ($p < 2$) order complete moment convergence, which states that when $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$, and $\mathbb{E}X^2 \log^+ |X| < \infty$, then

$$\lim_{\epsilon \searrow 0} \frac{1}{-\log \epsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} \mathbb{E}|S_n|^2 I_{\{|S_n| \ge \epsilon n\}} = 2\sigma^2. \tag{3}$$

In addition to the partial sums of iid random variables, there are some corresponding precise asymptotic results on other subjects, such as uniform empirical process, self-normalized sums, order statistics, eigenvalue statistics, and random fields. For the details on this topic, one can refer to Gut and Steinebach [6].

The paper will focus on the precise asymptotic of the uniform empirical process. Let $\{\xi_i, 1 \le i \le n\}$ be a sequence of independent $U[0,1]$-distributed random variables; we can define the uniform empirical process $F_n(t) = n^{-1/2} \sum_{i=1}^{n} (I_{\{\xi_i \le t\}} - t), 0 \le t \le 1, \|F_n\| = \sup_{0 \le t \le 1} |F_n(t)|$. Consider B to be a Brownian bridge on $D[0,1]$ and write $\|B\| = \sup_{0 \le t \le 1} |B(t)|$. Zhang and Yang [7] established some precise asymptotics on the complete convergence of the uniform empirical process; one of their main results can be stated as follows.

Lemma 1. *For* $1 \leq p < 2$, $r > p$, *then*

$$\lim_{\epsilon \searrow 0} \epsilon^{2(r-p)/(2-p)} \sum_{n=1}^{\infty} n^{(r/p)-2} P\left(\|F_n\| \geq \epsilon n^{(1/p)-(1/2)}\right) \tag{4}$$

$$= \frac{p}{r-p} \mathbb{E}\|B\|^{2(r-p)/(2-p)}.$$

Zang and Huang [8] obtained some results on the first-order complete moment convergence of $F_n(t)$. If the nonnegative function $g(x)$ satisfies some regular monotone conditions, they proved the following.

Lemma 2. *For any* $s > 0$, *one has*

$$\lim_{\epsilon \searrow 0} \frac{1}{-\log \epsilon} \sum_{n=1}^{\infty} \frac{g'(n)}{g(n)} \mathbb{E}\{\|F_n\| - \epsilon g^s(n)\} = \frac{1}{s} \mathbb{E}\|B\|. \tag{5}$$

Chen and Zhang [9] further got some precise asymptotic result on the second-order complete moment convergence of it. A typical result in their work can be listed as follows.

Lemma 3. *For any* $0 < \beta \leq 2$, $\delta > (2/\beta) - 1$, *one has*

$$\lim_{\epsilon \searrow 0} \epsilon^{\beta(\delta+1)-2} \sum_{n=1}^{\infty} \frac{(\log n)^{\delta-(2/\beta)}}{n} \mathbb{E}\left\{\|F_n\|^2 I_{\{\|F_n\| \geq \epsilon(\log n)^{1/\beta}\}}\right\} \tag{6}$$

$$= \frac{\beta \mathbb{E}\|B\|^{\beta(\delta+1)}}{\beta(\delta+1)-2}.$$

Based on the existing results above, we will add a general precise asymptotic result on the second-order complete moment convergence of $F_n(t)$.

Theorem 4. *Assume that the real-valued function* $g(x)$ *satisfies the following conditions.*

(A1) $g(x)$ is differentiable on the interval $[1, +\infty)$, which is nonnegative and strictly increasing to ∞.

(A2) The differentiable function $g'(x)$ is nonnegative and the function $g'(x)/g(x)$ is monotone. If $g'(x)/g(x)$ is monotone nondecreasing, we assume that $\lim_{x \to \infty}(g'(x+1)g(x)/g(x+1)g'(x)) = 1$.

One has

$$\lim_{\epsilon \searrow 0} \frac{1}{-\log \epsilon} \sum_{n=1}^{\infty} \frac{g'(n)}{g(n)} \mathbb{E}\left\{\|F_n\|^2 I_{\{\|F_n\| \geq \epsilon\sqrt{g(n)}\}}\right\} = \frac{\pi^2}{6}. \tag{7}$$

Remark 5. The assumptions on $g(x)$ are rather mild; in fact, there are lots of functions satisfying them, such as $g(x) = x^\alpha$, $g(x) = (\log x)^\beta$, and $g(x) = (\log\log x)^\gamma$ with suitable parameters $\alpha > 0, \beta > 0, \gamma > 0$.

The main proofs are presented in the next section. Throughout the paper, C denotes an absolutely positive constant whose value can be different from one place to another.

2. The Proof

We first give some propositions, which will play a key role in the proof of Theorem 4.

Proposition 6. *Under the assumptions of Theorem 4, one has*

$$\lim_{\epsilon \searrow 0} \epsilon^2 \sum_{n=1}^{\infty} g'(n) P\left\{\|B\| \geq \epsilon\sqrt{g(n)}\right\} = \mathbb{E}\|B\|^2. \tag{8}$$

Proof. If $g'(x)$ is monotone nonincreasing, by the assumptions of Theorem 4, we can see that $g'(x)P\{\|B\| \geq \epsilon\sqrt{g(n)}\}$ is also nonincreasing; thus

$$\int_1^{\infty} g'(y) P\left\{\|B\| \geq \epsilon\sqrt{g(y)}\right\} dy$$

$$\leq \sum_{n=1}^{\infty} g'(n) P\left\{\|B\| \geq \epsilon\sqrt{g(y)}\right\} \tag{9}$$

$$\leq \int_0^{\infty} g'(y) P\left\{\|B\| \geq \epsilon\sqrt{g(y)}\right\} dy.$$

If $g'(x)$ is monotone nondecreasing, by the assumption that $\lim_{n \to \infty}(g'(n+1)/g'(n)) = 1$, we can find that, for any $0 < \theta_0 < 1$, there exists a sufficient large number $N = N(\theta_0)$, such that $g'(n+1)/g'(n) \leq 1 + \theta_0$ and $g'(n+1)/g'(n) \geq 1 - \theta_0$ for all $n \geq N$. Thus we have

$$\frac{1}{1+\theta_0} \int_N^{\infty} g'(y) P\left\{\|B\| \geq \epsilon\sqrt{g(y)}\right\} dy$$

$$\leq \sum_{n=N}^{\infty} g'(n) P\left\{\|B\| \geq \epsilon\sqrt{g(n)}\right\} \tag{10}$$

$$\leq \frac{1}{1-\theta_0} \int_{N-1}^{\infty} g'(y) P\left\{\|B\| \geq \epsilon\sqrt{g(y)}\right\} dy.$$

At the same time, by making a change of variables and integration by parts, for any $\theta \geq 0$, we have

$$\lim_{\epsilon \searrow 0} \epsilon^2 \int_\theta^{\infty} g'(y) P\left\{\|B\| \geq \epsilon\sqrt{g(y)}\right\} dy$$

$$= \lim_{\epsilon \searrow 0} 2 \int_{2\epsilon\sqrt{g(\theta)}}^{\infty} y P(|\xi| \geq y) dy \tag{11}$$

$$= 2 \int_0^{\infty} y P(\|B\| \geq y) dy$$

$$= \mathbb{E}\|B\|^2.$$

By relations (9)–(11) and the fact that the result of the proposition will remain unchanged when we add or subtract some finite sums on the left hand of it, we can complete the proof by taking $\theta_0 \to 0$. □

Proposition 7. *Under the assumptions of Theorem 4, one has*

$$\lim_{\epsilon \searrow 0} \epsilon^2 \sum_{n=1}^{\infty} g'(n) \left\{P\left(\|F_n\| \geq \epsilon\sqrt{g(n)}\right) - P\left(\|B\| \geq \epsilon\sqrt{g(n)}\right)\right\}$$

$$= 0. \tag{12}$$

Proof. A well known fact in Billingsley [10] reveals that the uniform empirical process converges weakly to Brownian bridge, $F_n \Rightarrow B$. By continuous mapping theorem, we have $\|F_n\| \Rightarrow \|B\|$. Thus, as $n \to \infty$,

$$\Delta_n = \sup_{x \in \mathbb{R}} |P\{\|F_n\| \geq x\} - P(\|B\| \geq x)| \longrightarrow 0. \quad (13)$$

Let $A(\epsilon) = g^{-1}(M\epsilon^{-2})$, where $g^{-1}(x)$ is the inverse function of $g(x)$ and M is an arbitrary positive number; then there exists a positive constant C such that

$$\sum_{n \leq A(\epsilon)} g'(n) \leq C \int_1^{A(\epsilon)} g'(x)\,dx \leq Cg(A(\epsilon)) = CM\epsilon^{-2}; \quad (14)$$

thus

$$\epsilon^2 \sum_{n \leq A(\epsilon)} g'(n) \leq C < \infty. \quad (15)$$

By Toeplitz Lemma listed in Appendix, we know

$$\lim_{\epsilon \searrow 0} \epsilon^2 \sum_{n \leq A(\epsilon)} g'(n) \left\{ P\left(\|F_n\| \geq \epsilon\sqrt{g(n)}\right) \right.$$

$$\left. -P\left(\|B\| \geq \epsilon\sqrt{g(n)}\right) \right\} \quad (16)$$

$$\leq \lim_{\epsilon \searrow 0} \epsilon^2 \sum_{n \leq A(\epsilon)} g'(n) \Delta_n = 0.$$

On the other hand, by the similar argument in (11), we have

$$\epsilon^2 \sum_{n > A(\epsilon)} g'(n) P\left\{\|F_n\| \geq \epsilon\sqrt{g(n)}\right\}$$

$$\leq C\epsilon^2 \int_{A(\epsilon)}^\infty g'(x) P\left\{\|F_n\| \geq \epsilon\sqrt{g(x)}\right\}\,dx \quad (17)$$

$$\leq C \int_{\sqrt{M}}^\infty yP\left\{\|F_n\| \geq y\right\}\,dy.$$

By the result of Kiefer and Wolfowitz [11], there exists $c_0 > 0$, such that

$$P\{\|F_n\| \geq y\} \leq Ce^{-c_0 y^2}. \quad (18)$$

Then, by letting $\epsilon \searrow 0$ and then $M \to \infty$, we can get

$$\lim_{M \to \infty} \limsup_{\epsilon \searrow 0} \epsilon^2 \sum_{n > A(\epsilon)} g'(n) P\left\{\|F_n\| \geq \epsilon\sqrt{g(n)}\right\}$$

$$\leq \lim_{M \to \infty} C \int_{\sqrt{M}}^\infty ye^{-c_0 y^2}\,dy = 0. \quad (19)$$

By Lemma 2.1 in Zhang and Yang [7], for any $x \in \mathbb{R}$,

$$P\{\|B\| \geq x\} \leq 2e^{-2x^2}. \quad (20)$$

Then, a similar argument in (19) can deduce that

$$\lim_{M \to \infty} \limsup_{\epsilon \searrow 0} \epsilon^2 \sum_{n > A(\epsilon)} g'(n) P\left\{\|B\| \geq \epsilon\sqrt{g(n)}\right\} = 0. \quad (21)$$

By combining (16), (19), and (21) and using the triangular inequality, we can complete the proof. □

Proposition 8. *Under the assumptions of Theorem 4, one has*

$$\lim_{\epsilon \searrow 0} \frac{1}{-\log \epsilon} \sum_{n=1}^\infty \frac{g'(n)}{g(n)} \int_{\epsilon\sqrt{g(n)}}^\infty xP(\|B\| \geq x)\,dx = \mathbb{E}\|B\|^2. \quad (22)$$

Proof. Similar to the argument in Proposition 6, no matter whether the function $g'(x)/g(x)$ is monotone nonincreasing or monotone nondecreasing, we can deduce the following relations by applying the change of variables and the L'Hôpital's rule:

$$\lim_{\epsilon \searrow 0} \frac{1}{-\log \epsilon} \sum_{n=1}^\infty \frac{g'(n)}{g(n)} \int_{\epsilon\sqrt{g(n)}}^\infty xP(\|B\| \geq x)\,dx$$

$$= \lim_{\epsilon \searrow 0} \frac{1}{-\log \epsilon} \int_1^\infty \frac{g'(x)}{g(x)} \int_{\epsilon\sqrt{g(x)}}^\infty tP(\|B\| \geq t)\,dt\,dx$$

$$= \lim_{\epsilon \searrow 0} \frac{2}{-\log \epsilon} \int_{\epsilon\sqrt{g(1)}}^\infty \frac{1}{y} \int_y^\infty tP(\|B\| \geq t)\,dt\,dy \quad (23)$$

$$= \int_0^\infty 2tP(\|B\| \geq t)\,dt$$

$$= \mathbb{E}\|B\|^2.$$

Thus, the proof is completed. □

Proposition 9. *Under the assumptions of Theorem 4, one has*

$$\lim_{\epsilon \searrow 0} \frac{1}{-\log \epsilon} \sum_{n=1}^\infty \frac{g'(n)}{g(n)}$$

$$\times \int_{\epsilon\sqrt{g(n)}}^\infty x\left[P(\|B\| \geq x) - P(\|F_n\| \geq x)\right]\,dx = 0. \quad (24)$$

Proof. If we denote $\widetilde{A}(\epsilon) = g^{-1}(\epsilon^{-2})$, where $g^{-1}(x)$ is the inverse function of $g(x)$, then we can write

$$\sum_{n=1}^\infty \frac{g'(n)}{g(n)} \int_{\epsilon\sqrt{g(n)}}^\infty x\left[P(\|B\| \geq x) - P(\|F_n\| \geq x)\right]\,dx$$

$$\leq \sum_{n \leq \widetilde{A}(\epsilon)} \frac{g'(n)}{g(n)} \int_{\epsilon\sqrt{g(n)}}^\infty x\left|P(\|B\| \geq x) - P(\|F_n\| \geq x)\right|\,dx$$

$$+ \sum_{n > \widetilde{A}(\epsilon)} \frac{g'(n)}{g(n)} \int_{\epsilon\sqrt{g(n)}}^\infty xP(\|B\| \geq x)\,dx$$

$$+ \sum_{n > \widetilde{A}(\epsilon)} \frac{g'(n)}{g(n)} \int_{\epsilon\sqrt{g(n)}}^\infty xP(\|F_n\| \geq x)\,dx$$

$$:= J_1 + J_2 + J_3. \quad (25)$$

For the term J_1, via the change of variable $x = y + \epsilon \sqrt{g(n)}$, we have

$$\int_{\epsilon\sqrt{g(n)}}^{\infty} x \left| P\left(\|B\| \geq x\right) - P\left(\|F_n\| \geq x\right)\right| dx$$

$$= \int_{0}^{\infty} \left(y + \epsilon\sqrt{g(n)}\right) \left| P\left(\|B\| \geq y + \epsilon\sqrt{g(n)}\right) \right.$$

$$\left. - P\left(\|F_n\| \geq y + \epsilon\sqrt{g(n)}\right)\right| dy$$

$$\leq \left(\Delta_{n1} + \Delta_{n2} + \Delta_{n3}\right), \tag{26}$$

where

$$\Delta_{n1}$$

$$= \int_{0}^{\Delta_n^{-1/4}} \left(y + \epsilon\sqrt{g(n)}\right) \left| P\left(\|B\| \geq y + \epsilon\sqrt{g(n)}\right) \right.$$

$$\left. - P\left(\|F_n\| \geq y + \epsilon\sqrt{g(n)}\right)\right| dy,$$

$$\Delta_{n2} = \int_{\Delta_n^{-1/4}}^{\infty} \left(y + \epsilon\sqrt{g(n)}\right) P\left(\|F_n\| \geq y + \epsilon\sqrt{g(n)}\right) dy,$$

$$\Delta_{n3} = \int_{\Delta_n^{-1/4}}^{\infty} \left(y + \epsilon\sqrt{g(n)}\right) P\left(\|B\| \geq y + \epsilon\sqrt{g(n)}\right) dy,$$

$$\tag{27}$$

and Δ_n is defined by (13).

Since $n \leq \widetilde{A}(\epsilon)$ implies $\epsilon\sqrt{g(n)} \leq 1$, then we can see that

$$\Delta_{n1} \leq \Delta_n \left(\Delta_n^{-1/4} + \epsilon\sqrt{g(n)}\right) \Delta_n^{-1/4}$$

$$= \Delta_n^{1/2} + \Delta_n^{3/4} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{28}$$

By relation (18), when $n \to \infty$, we have

$$\Delta_{n2} \leq C \int_{\Delta_n^{-1/4}}^{\infty} y e^{-c_0 y^2} dy \longrightarrow 0. \tag{29}$$

By using relation (20), the similar argument can prove that

$$\Delta_{n3} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{30}$$

Note that $\sum_{n=1}^{\widetilde{A}(\epsilon)} (g'(n)/g(n)) \leq C \int_0^{\widetilde{A}(\epsilon)} (g'(x)/g(x) dx \leq -C\log\epsilon$. A combination of (28)–(30) and Toeplitz Lemma can lead to that

$$\lim_{\epsilon\searrow 0} \frac{1}{-\log\epsilon} \sum_{n\leq\widetilde{A}(\epsilon)} \frac{g'(n)}{g(n)} \left(\Delta_{n1} + \Delta_{n2} + \Delta_{n3}\right) = 0, \tag{31}$$

which indicates that

$$\lim_{\epsilon\searrow 0} \frac{1}{-\log\epsilon} J_1 = 0. \tag{32}$$

For the term J_2, using relation (18) again, the same argument in Proposition 8 can deduce that

$$\sum_{n>\widetilde{A}(\epsilon)} \frac{g'(n)}{g(n)} \int_{\epsilon\sqrt{g(n)}}^{\infty} xP\left(\|F_n\| \geq x\right) dx$$

$$\leq C \int_{\widetilde{A}(\epsilon)}^{\infty} \frac{g'(y)}{g(y)} \int_{\epsilon\sqrt{g(y)}}^{\infty} xP\left(\|F_n\| \geq x\right) dx\, dy$$

$$\leq C \int_1^{\infty} \frac{1}{t} \int_t^{\infty} xP\left(\|F_n\| \geq x\right) dx\, dt$$

$$\leq C \int_1^{\infty} x \log x\, e^{-c_0 x^2} dx < \infty, \tag{33}$$

which implies that

$$\lim_{\epsilon\searrow 0} \frac{1}{-\log\epsilon} J_2 = 0. \tag{34}$$

At last, by using relation (20) again, the similar argument as in the discussion on J_2 can yield that

$$\lim_{\epsilon\searrow 0} \frac{1}{-\log\epsilon} J_3 = 0. \tag{35}$$

Combining (25) and (32)–(35), we can complete the proof. □

Proof of Theorem 4. According to the fact that, for any random variable ζ and $a \in \mathbb{R}$,

$$\mathbb{E}\zeta I_{\{\zeta\geq a\}} = aP\left(\zeta \geq a\right) + \int_a^{\infty} P\left(\zeta \geq x\right) dx, \tag{36}$$

we have

$$\mathbb{E}\left\{\|F_n\|^2 I_{\{\|F_n\|\geq\epsilon\sqrt{g(n)}\}}\right\}$$

$$= \epsilon^2 g(n) P\left(\|F_n\| \geq \epsilon\sqrt{g(n)}\right) \tag{37}$$

$$+ \int_{\epsilon\sqrt{g(n)}}^{\infty} 2xP\left(\|F_n\| \geq x\right) dx.$$

By Propositions 6 and 7, we know

$$\lim_{\epsilon\searrow 0} \epsilon^2 \sum_{n=1}^{\infty} g'(n) P\left(\|F_n\| \geq \epsilon\sqrt{g(n)}\right) = \mathbb{E}\|B\|^2, \tag{38}$$

which shows that

$$\lim_{\epsilon\searrow 0} \frac{1}{-\log\epsilon} \epsilon^2 \sum_{n=1}^{\infty} \frac{g'(n)}{g(n)} g(n) P\left(\|F_n\| \geq \epsilon\sqrt{g(n)}\right) = 0. \tag{39}$$

By Propositions 8 and 9, we can get that

$$\lim_{\epsilon\searrow 0} \frac{1}{-\log\epsilon} \sum_{n=1}^{\infty} \frac{g'(n)}{g(n)} \int_{\epsilon\sqrt{g(n)}}^{\infty} xP\left(\|F_n\| \geq x\right) dx = \mathbb{E}\|B\|^2. \tag{40}$$

A final combination of the above two relations can reveal that

$$\lim_{\epsilon \searrow 0} \frac{1}{-\log \epsilon} \sum_{n=1}^{\infty} \frac{g'(n)}{g(n)} E\left\{\|F_n\|^2 I_{\{\|F_n\| \geq \epsilon \sqrt{g(n)}\}}\right\} = 2\mathbb{E}\|B\|^2. \quad (41)$$

From Csörgö and Révèsz [12], we know $P(\|B\| \geq x) = 2\sum_{k=1}^{\infty}(-1)^{k+1}e^{-2k^2x^2}$, $x > 0$, which yields that

$$\mathbb{E}\|B\|^2 = 2\int_0^{\infty} xP\left(\|B\| \geq x\right)dx = \sum_{k=1}^{\infty}(-1)^{k+1}k^{-2} = \frac{\pi^2}{12}. \quad (42)$$

This concludes the proof of Theorem 4. $\qquad\square$

Appendix

Lemma A.1 (Toeplitz Lemma in Loève [13]). *Let $a_{nk}, k = 1, 2, \ldots, k_n$, be numbers such that, for every fixed k, $a_{nk} \to 0$, and for all n, $\sum_k |a_{nk}| \leq c < \infty$. If $x_n \to 0$, then*

$$x_n' = \sum_k a_{nk}x_k \longrightarrow 0. \quad (A.1)$$

In particular, if $b_n = \sum_{k=1}^n a_k \uparrow \infty$ and $x_n \to 0$, then

$$\frac{1}{b_n}\sum_{k=1}^n a_k x_k \longrightarrow x. \quad (A.2)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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