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Research Article

Global Exponential Stability of Antiperiodic Solution for Impulsive High-Order Hopfield Neural Networks

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This paper is concerned with antiperiodic solutions for impulsive high-order Hopfield neural networks with leakage delays and continuously distributed delays. By employing a novel proof, some sufficient criteria are established to ensure the existence and global exponential stability of the antiperiodic solution, which are new and complement of previously known results. Moreover, an example and numerical simulations are given to support the theoretical result.

1. Introduction

To describe mathematically a real evolution process with a short-term perturbation, the impulsive differential equations were proposed in many fields such as control theory, physics, chemistry, population dynamics, biotechnologies, industrial robotics, and economics [1–3]. In particular, high-order neural networks with impulses have been studied extensively, and there has been a great deal of the literatures focusing on the existence and stability of equilibrium points, periodic solutions, almost periodic solutions, and antiperiodic solutions [4–13]. Reference [14] has introduced and studied the existence and exponential stability of antiperiodic solutions for the following Hopfield neural networks with time-varying and distributed delays:

$$\begin{aligned} x_i'(t) = & -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t) f_j\left(x_j\left(t-\tau_{ij}(t)\right)\right) \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\int_0^\infty K_{ij}(u) g_j\left(x_j(t-u)\right)du \\ & \times \int_0^\infty K_{il}(u) g_l\left(x_l(t-u)\right)du + I_i(t), \\ & \qquad i=1,2,\ldots,n, \quad t>0. \end{aligned} \tag{1}$$

Recently, great attention has been paid to neural networks with time delay in the leakage (or forgetting) term (see [15–19]). Specifically, Wang [20] considered the antiperiodic solution of the following impulsive high-order Hopfield neural networks with leakage delays:

$$\begin{aligned} x_i'(t) = & -c_i(t)x_i\left(t-\eta_i(t)\right) \\ & + \sum_{j=1}^{n} a_{ij}(t) g_j\left(x_j\left(t-\tau_{ij}(t)\right)\right) \\ & + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t) g_j\left(x_j\left(t-\sigma_{ijl}(t)\right)\right) \\ & \times g_l\left(x_l\left(t-\nu_{ijl}(t)\right)\right) + I_i(t), \\ & \qquad t>0, \quad t\neq t_k, \\ \Delta x_i(t_k) = & \, d_{ik}x_i(t_k), \\ x_i(t) = & \, \varphi_i(t), \quad t\in\left[-\tau_i, 0\right], \; k=1,2,\ldots. \end{aligned} \tag{2}$$

Under some reasonable conditions on coefficients of (2) and the following additional conditions:

$$t-\eta_i(t)>0, \quad \forall t>0,\ i=1,2,\ldots,n, \tag{3}$$

Wang [20] deduced the criteria on the existence and exponential stability of the antiperiodic solution for (2).

However, to the best of our knowledge, few authors have investigated the existence and exponential stability of the antiperiodic solution for impulsive high-order Hopfield neural networks with leakage delays and continuously distributed delays. Motivated by the above arguments, we consider the antiperiodic solution for the impulsive high-order Hopfield neural networks (IHHNNs) with leakage delays and continuously distributed delays as follows:

$$\begin{aligned} x_i'(t) = & -c_i(t)\,x_i(t-\eta_i(t)) \\ & +\sum_{j=1}^{n} a_{ij}(t)\, f_j\left(x_j\left(t-\tau_{ij}(t)\right)\right) \\ & +\sum_{j=1}^{n}\sum_{l=1}^{n} b_{ijl}(t)\int_0^{\infty} K_{ij}(u)\, g_j\left(x_j(t-u)\right) du \\ & \times \int_0^{\infty} K_{il}(u)\, g_l\left(x_l(t-u)\right) du + I_i(t), \\ & \qquad t>0, \quad t\neq t_k, \\ & \Delta x_i(t_k) = d_{ik} x_i(t_k), \\ & x_i(t)=\varphi_i(t), \quad t\in(-\infty,0],\ k=1,2,\ldots, \end{aligned} \tag{4}$$

where $i\in\mathcal{N}:=\{1,2,\ldots,n\}$ and n is the number of units in a neural network, $x_i(t)$ corresponds to the ith unit of the state vector at the time t, $c_i(t)>0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}(t)$ and $b_{ijl}(t)$ are the first- and second-order connection weights of the neural network, respectively, $\eta_i(t)$, $\tau_{ij}(t)$, and $K_{ij}(t)$ correspond to the leakage delay, the transmission delays, and the transmission delay kernels, respectively, g_j and f_j are the activation functions of signal transmission, and $I_i(t)$ denotes the external input at time t. For $i,j,l\in\mathcal{N}$, we always assume that $c_i, I_i, a_{ij}, b_{ijl} : R\to R$ and $\eta_i, \tau_{ij} : R\to[0,+\infty)$ are bounded continuous functions, and c_i is bounded above and below by positive constants. Consider the following : $\Delta x_i(t_k)=x_i(t_k^+)-x_i(t_k)$, $x_i(t_k^+)=\lim_{\Delta t\to 0^+} x_i(t_k+\Delta t)$, $x_i(t_k)=\lim_{\Delta t\to 0^-} x_i(t_k+\Delta t)$, $i\in\mathcal{N}$, $k=1,2,\ldots$. $t_k>0$ are impulsive moments satisfying $t_k<t_{k+1}$ and $\lim_{k\to+\infty} t_k=+\infty$. $\varphi(t)=(\varphi_1,\varphi_2,\ldots,\varphi_n)^T$ is the initial condition and $\varphi_i(\cdot)$ denotes a real-valued continuous function defined on $(-\infty,0]$, $i\in\mathcal{N}$. It is easy to see that the system (1) is a special case of system (4) with $\eta_i(t)=0$.

The purpose of this paper is to discuss the existence and exponential stability of antiperiodic solutions for system (4) without the additional condition (3) since it is unduly restrictive and unreasonable. The outline of the paper is as follows. In Section 2, we establish some preliminaries and basic results, which are useful to derive sufficient conditions on the existence and exponential stability of antiperiodic solutions for system (4) in Section 3. In Section 4, we give an example with numerical simulations to illustrate our results.

2. Preliminaries and Basic Results

Throughout this paper, we assume that the following conditions hold.

(H_1) For $i,j,l\in\mathcal{N}$ and $k\in Z^+$, where Z^+ denotes the set of all positive integers, there exists constant $\omega>0$ such that

$$\begin{gathered} c_i(t+\omega)=c_i(t), \qquad \eta_i(t+\omega)=\eta_i(t), \\ I_i(t+\omega)=-I_i(t), \qquad \tau_{ij}(t+\omega)=\tau_{ij}(t), \\ a_{ij}(t+\omega)\, f_j(v) = -a_{ij}(t)\, f_j(-v), \\ b_{ijl}(t+\omega)\int_0^{\infty} K_{ij}(u)\, g_j\left(v_j(t-u)\right) du \\ \times\int_0^{\infty} K_{il}(u)\, g_l\left(v_l(t-u)\right) du \\ = -b_{ijl}(t)\int_0^{\infty} K_{ij}(u)\, g_j\left(-v_j(t-u)\right) du \\ \times\int_0^{\infty} K_{il}(u)\, g_l\left(-v_l(t-u)\right) du, \end{gathered} \tag{5}$$

where $t,v\in R$ and v_j and v_l are real-valued bounded continuous functions defined on R.

(H_2) For $i,j,l\in\mathcal{N}$, there exist constants c_i^+, η_i^+, I_i^+, a_{ij}^+, τ_{ij}^+, and b_{ijl}^+ such that

$$\begin{gathered} c_i^+ = \max_{t\in[0,\omega]} c_i(t), \qquad \eta_i^+ = \max_{t\in[0,\omega]} \eta_i(t), \\ a_{ij}^+ = \max_{t\in[0,\omega]} \left|a_{ij}(t)\right|, \qquad \tau_{ij}^+ = \max_{t\in[0,\omega]} \tau_{ij}(t), \\ b_{ijl}^+ = \max_{t\in[0,\omega]} \left|b_{ijl}(t)\right|, \qquad I_i^+ = \max_{t\in[0,\omega]} \left|I_i(t)\right|. \end{gathered} \tag{6}$$

(H_3) Consider $-2\le d_{ik}\le 0$ for $i\in\mathcal{N}$ and $k\in Z^+$.

(H_4) There exists a $q\in Z^+$ such that

$$d_{i(k+q)} = d_{ik}, \qquad t_{k+q}=t_k+\omega. \tag{7}$$

(H_5) For each $j\in\mathcal{N}$, the activation functions $f_j, g_j : R\to R$ are continuous and there exist nonnegative constants L_j^f, L_j^g, and M_j such that, for all $u,v\in R$,

$$\begin{gathered} \left|f_j(u)-f_j(v)\right| \le L_j^f |u-v|, \\ \left|g_j(u)-g_j(v)\right| \le L_j^g |u-v|, \qquad \left|g_j(u)\right|\le M_j. \end{gathered} \tag{8}$$

(H_6) For all $i,j\in\mathcal{N}$, the delay kernels $K_{ij} : [0,\infty)\to R$ are continuous, and $|K_{ij}(t)|e^{\kappa t}$ are integrable on $[0,\infty)$ for a certain positive constant κ.

(H_7) For all $t > 0$ and $i \in \mathcal{N}$, there exist positive constants ξ_i and η such that $c_i^+ \eta_i^+ < 1$ and

$$-\eta > -\left[c_i(t)\left(1 - 2c_i^+\eta_i^+\right) - \left|c_i(t) - \left(1 - \eta_i'(t)\right)c_i\left(t - \eta_i(t)\right)\right|\right]\frac{\xi_i}{1 - c_i^+\eta_i^+} + \sum_{j=1}^{n}\left|a_{ij}(t)\right|L_j^f\frac{\xi_j}{1 - c_j^+\eta_j^+} + \sum_{j=1}^{n}\sum_{l=1}^{n}\left|b_{ijl}(t)\right| \times \left(M_j\int_0^\infty\left|K_{ij}(u)\right|du\int_0^\infty\left|K_{il}(u)\right|duL_l^g\frac{\xi_l}{1 - c_l^+\eta_l^+} + \int_0^\infty\left|K_{ij}(u)\right|duL_j^g\frac{\xi_j}{1 - c_j^+\eta_j^+}\int_0^\infty\left|K_{il}(u)\right|duM_l\right). \tag{9}$$

For ease of notations, let R^n be the set of all real vectors and denote $x = (x_1, x_2, \ldots, x_n)^T \in R^n$ as a column vector, in which the symbol (T) represents the transpose of a vector. As a general rule of the theory of impulsive differential equations, we assume that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T = (x_1(t-0), x_2(t-0), \ldots, x_n(t-0))^T$ at the points of discontinuity t_k of the solution $x(t)$. From system (4), it is easy to see that the derivative $x_i'(t_k)$ does not exist in general. On the other hand, according to system (4), there exists the limit $x_i'(t_k \mp 0)$. On account of the above convention, we suppose that $x_i'(t_k) \equiv x_i'(t_k - 0)$.

Definition 1. A solution $x(t)$ of (4) is said to be ω-antiperiodic, if

$$x(t+\omega) = -x(t), \quad t \neq t_k, \qquad x\left(\left(t_k+\omega\right)^+\right) = -x\left(t_k^+\right), \quad k \in Z^+, \tag{10}$$

where the smallest positive number ω is called the antiperiod of function $x(t)$.

In the sequel, we prove some lemmas which will be used to prove our main results in Section 3.

Lemma 2. *Suppose that (H_2)–(H_7) hold. If $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is a solution of system (4) with initial conditions*

$$\left|\varphi_i(t) - \int_{t-\eta_i(t)}^{t}c_i(s)\varphi_i(s)\,ds\right| < \xi_i\frac{\gamma}{\eta}, \quad t \in (-\infty, 0], \ i \in \mathcal{N}, \tag{11}$$

then, for t in the interval of existence and $i \in \mathcal{N}$,

$$\left|x_i(t) - \int_{t-\eta_i(t)}^{t}c_i(s)x_i(s)\,ds\right| < \xi_i\frac{\gamma}{\eta}, \tag{12}$$

$$\left|x_i(t)\right| < \frac{\xi_i(\gamma/\eta)}{1 - c_i^+\eta_i^+}, \tag{13}$$

where

$$\gamma = \max_{i\in\mathcal{N}}\left\{\sum_{j=1}^{n}a_{ij}^+\left|f_j(0)\right| + \sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}^+\int_0^\infty\left|K_{ij}(u)\right|du\left|g_j(0)\right| \times \int_0^\infty\left|K_{il}(u)\right|duM_l + I_i^+\right\}. \tag{14}$$

Proof. For t in the interval of existence and $i \in \mathcal{N}$, let

$$X_i(t) = x_i(t) - \int_{t-\eta_i(t)}^{t}c_i(s)x_i(s)\,ds. \tag{15}$$

On the one hand, suppose that (12) holds. Then, for a given $\tilde{t} > 0$ in the interval of existence and $i \in \mathcal{N}$, we acquire

$$\left|x_i(t)\right| \le \left|X_i(t)\right| + \left|\int_{t-\eta_i(t)}^{t}c_i(s)x_i(s)\,ds\right| \le \xi_i\frac{\gamma}{\eta} + c_i^+\eta_i^+\sup_{s\in(-\infty,\tilde{t}]}\left|x_i(s)\right| \quad \text{for } t \in (-\infty, \tilde{t}], \tag{16}$$

which combined with (H_7) implies that (13) holds.

On the other hand, in view of (H_3), we have

$$\left|x_i\left(t_k^+\right)\right| = \left|1 + d_{ik}\right|\left|x_i\left(t_k\right)\right| \le \left|x_i\left(t_k\right)\right|. \tag{17}$$

So, if $|x_i(t_k^+)| > \xi_i(\gamma/\eta)/(1 - c_i^+\eta_i^+)$, then $|x_i(t_k)| > \xi_i(\gamma/\eta)/(1 - c_i^+\eta_i^+)$.

Thus, considering the above two cases, it is sufficient to prove (12). We proceed this by contradiction. Suppose that (12) does not hold; then there exist $\hat{i} \in \mathcal{N}$ and $t_* \in (t_k, t_{k+1})$ such that

$$\left|X_{\hat{i}}\left(t_*\right)\right| = \xi_{\hat{i}}\frac{\gamma}{\eta}, \quad (12) \text{ holds} \quad \forall t \in \left(-\infty, t_*\right), \ j \in \mathcal{N}. \tag{18}$$

It follows that (13) holds for all $t \in (-\infty, t_*)$ and $i \in \mathcal{N}$. By virtue of (4), we obtain

$$\begin{aligned} X_{\hat{i}}'(t) &= x_{\hat{i}}'(t) - c_{\hat{i}}(t)x_{\hat{i}}(t) + \left(1 - \eta_{\hat{i}}'(t)\right)c_{\hat{i}}\left(t - \eta_{\hat{i}}(t)\right)x_{\hat{i}}\left(t - \eta_{\hat{i}}(t)\right) \\ &= -c_{\hat{i}}(t)x_{\hat{i}}(t) + \left(1 - \eta_{\hat{i}}'(t)\right)c_{\hat{i}}\left(t - \eta_{\hat{i}}(t)\right) \times x_{\hat{i}}\left(t - \eta_{\hat{i}}(t)\right) - c_{\hat{i}}(t)x_{\hat{i}}\left(t - \eta_{\hat{i}}(t)\right) \\ &\quad + \sum_{j=1}^{n}a_{\hat{i}j}(t)f_j\left(x_j\left(t - \tau_{\hat{i}j}(t)\right)\right) \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n}b_{\hat{i}jl}(t)\int_0^\infty K_{\hat{i}j}(u)g_j\left(x_j(t-u)\right)du \times \int_0^\infty K_{\hat{i}l}(u)g_l\left(x_l(t-u)\right)du + I_{\hat{i}}(t) \end{aligned}$$

$$
\begin{aligned}
&= -c_{\hat{i}}(t)\, x_{\hat{i}}(t) - \left[c_{\hat{i}}(t) - \left(1-\eta_{\hat{i}}'(t)\right) c_{\hat{i}}(t-\eta_{\hat{i}}(t))\right] \\
&\quad \times x_{\hat{i}}(t-\eta_{\hat{i}}(t)) + \sum_{j=1}^{n} a_{\hat{i}j}(t)\, f_j\left(x_j\left(t-\tau_{\hat{i}j}(t)\right)\right) \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{\hat{i}jl}(t) \int_0^{\infty} K_{\hat{i}j}(u)\, g_j\left(x_j(t-u)\right) du \\
&\quad \times \int_0^{\infty} K_{\hat{i}l}(u)\, g_l\left(x_l(t-u)\right) du + I_{\hat{i}}(t) \\
&= -c_{\hat{i}}(t)\, X_{\hat{i}}(t) - c_{\hat{i}}(t) \int_{t-\eta_{\hat{i}}(t)}^{t} c_{\hat{i}}(s)\, x_{\hat{i}}(s)\, ds \\
&\quad - \left[c_{\hat{i}}(t) - \left(1-\eta_{\hat{i}}'(t)\right) c_{\hat{i}}(t-\eta_{\hat{i}}(t))\right] x_{\hat{i}}(t-\eta_{\hat{i}}(t)) \\
&\quad + \sum_{j=1}^{n} a_{\hat{i}j}(t)\, f_j\left(x_j\left(t-\tau_{\hat{i}j}(t)\right)\right) \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{\hat{i}jl}(t) \int_0^{\infty} K_{\hat{i}j}(u)\, g_j\left(x_j(t-u)\right) du \\
&\quad \times \int_0^{\infty} K_{\hat{i}l}(u)\, g_l\left(x_l(t-u)\right) du + I_{\hat{i}}(t).
\end{aligned} \tag{19}
$$

This, together with (H_2), (H_5)–(H_7), (18), and the fact that (13) holds for all $t \in (-\infty, t_*)$ and $i \in \mathcal{N}$, we obtain

$$
\begin{aligned}
D^- \left|X_{\hat{i}}(t_*)\right| &\le -c_{\hat{i}}(t_*) \left|X_{\hat{i}}(t_*)\right| \\
&\quad + c_{\hat{i}}(t_*) \int_{t_*-\eta_{\hat{i}}(t_*)}^{t} c_{\hat{i}}(s) \left|x_{\hat{i}}(s)\right| ds \\
&\quad + \left|c_{\hat{i}}(t_*) - \left(1-\eta_{\hat{i}}'(t_*)\right) c_{\hat{i}}(t_*-\eta_{\hat{i}}(t_*))\right| \\
&\quad \times \left|x_{\hat{i}}(t_*-\eta_{\hat{i}}(t_*))\right| \\
&\quad + \left|\sum_{j=1}^{n} a_{\hat{i}j}(t_*)\, f_j\left(x_j\left(t_*-\tau_{\hat{i}j}(t_*)\right)\right)\right| \\
&\quad + \Bigg|\sum_{j=1}^{n}\sum_{l=1}^{n} b_{\hat{i}jl}(t_*) \\
&\qquad \times \int_0^{\infty} K_{\hat{i}j}(u)\, g_j\left(x_j(t_*-u)\right) du \\
&\qquad \times \int_0^{\infty} K_{\hat{i}l}(u)\, g_l\left(x_l(t_*-u)\right) du\Bigg| + \left|I_{\hat{i}}(t_*)\right| \\
&\le -c_{\hat{i}}(t_*) \left|X_{\hat{i}}(t_*)\right| + c_{\hat{i}}(t_*) \\
&\quad \times \int_{t_*-\eta_{\hat{i}}(t_*)}^{t} c_{\hat{i}}^{+} \left|x_{\hat{i}}(s)\right| ds \\
&\quad + \left|c_{\hat{i}}(t_*) - \left(1-\eta_{\hat{i}}'(t_*)\right) c_{\hat{i}}(t_*-\eta_{\hat{i}}(t_*))\right| \\
&\quad \times \left|x_{\hat{i}}(t_*-\eta_{\hat{i}}(t_*))\right| \\
&\quad + \sum_{j=1}^{n} \left|a_{\hat{i}j}(t_*)\right| \Big(\left|f_j\left(x_j\left(t_*-\tau_{\hat{i}j}(t_*)\right)\right) \\
&\qquad\qquad - f_j(0)\right| + \left|f_j(0)\right|\Big) \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \left|b_{\hat{i}jl}(t_*)\right| \\
&\quad \times \int_0^{\infty} \left|K_{\hat{i}j}(u)\right| \Big(\left|g_j\left(x_j(t_*-u)\right) - g_j(0)\right| \\
&\qquad\qquad + \left|g_j(0)\right|\Big) du \\
&\quad \times \int_0^{\infty} \left|K_{\hat{i}l}(u)\right| M_l du + \left|I_{\hat{i}}(t_*)\right| \\
&\le -c_{\hat{i}}(t_*)\, \xi_{\hat{i}} \frac{\gamma}{\eta} + c_{\hat{i}}(t_*)\, c_{\hat{i}}^{+} \eta_{\hat{i}}^{+} \frac{\xi_{\hat{i}}(\gamma/\eta)}{1-c_{\hat{i}}^{+}\eta_{\hat{i}}^{+}} \\
&\quad + \left|c_{\hat{i}}(t_*) - \left(1-\eta_{\hat{i}}'(t_*)\right) c_{\hat{i}}(t_*-\eta_{\hat{i}}(t_*))\right| \\
&\quad \times \frac{\xi_{\hat{i}}(\gamma/\eta)}{1-c_{\hat{i}}^{+}\eta_{\hat{i}}^{+}} \\
&\quad + \sum_{j=1}^{n} \left|a_{\hat{i}j}(t_*)\right| L_j^f \frac{\xi_j(\gamma/\eta)}{1-c_j^{+}\eta_j^{+}} + \sum_{j=1}^{n}\sum_{l=1}^{n} \left|b_{\hat{i}jl}(t_*)\right| \\
&\quad \times \int_0^{\infty} \left|K_{\hat{i}j}(u)\right| du L_j^g \frac{\xi_j(\gamma/\eta)}{1-c_j^{+}\eta_j^{+}} \\
&\quad \times \int_0^{\infty} \left|K_{\hat{i}l}(u)\right| du M_l \\
&\quad + \Bigg[\sum_{j=1}^{n} a_{\hat{i}j}^{+} \left|f_j(0)\right| + \sum_{j=1}^{n}\sum_{l=1}^{n} b_{\hat{i}jl}^{+} \\
&\qquad \times \int_0^{\infty} \left|K_{\hat{i}j}(u)\right| du \left|g_j(0)\right| \\
&\qquad \times \int_0^{\infty} \left|K_{\hat{i}l}(u)\right| du M_l + I_{\hat{i}}^{+}\Bigg] \\
&\le \Bigg\{-\Big[c_{\hat{i}}(t_*)\left(1-2c_{\hat{i}}^{+}\eta_{\hat{i}}^{+}\right) \\
&\qquad - \Big|c_{\hat{i}}(t_*) - \left(1-\eta_{\hat{i}}'(t_*)\right) \\
&\qquad\quad \times c_{\hat{i}}(t_*-\eta_{\hat{i}}(t_*))\Big|\Big] \\
&\quad \times \frac{\xi_{\hat{i}}}{1-c_{\hat{i}}^{+}\eta_{\hat{i}}^{+}} + \sum_{j=1}^{n} \left|a_{\hat{i}j}(t_*)\right| L_j^f \frac{\xi_j}{1-c_j^{+}\eta_j^{+}} \\
&\quad + \sum_{j=1}^{n}\sum_{l=1}^{n} \left|b_{\hat{i}jl}(t_*)\right| \\
&\quad \times \int_0^{\infty} \left|K_{\hat{i}j}(u)\right| du L_j^g \frac{\xi_j}{1-c_j^{+}\eta_j^{+}} \\
&\quad \times \int_0^{\infty} \left|K_{\hat{i}l}(u)\right| du M_l \Bigg\} \frac{\gamma}{\eta} + \gamma \\
&< -\eta \frac{\gamma}{\eta} + \gamma = 0.
\end{aligned} \tag{20}
$$

It contradicts $D^-|X_{\hat{i}}(t_*)| \ge 0$. Hence, (12) holds. The proof is now completed. □

Remark 3. Under conditions (H_2)–(H_7), the solution of system (4) always exists (see [1, 2]). On account of the boundedness of this solution, it follows from the theory of impulsive differential equations in [1] that the solution of system (4) can be defined on $[0, +\infty)$.

Lemma 4. *Suppose that* (H_1)*–*(H_7) *are true. Let* $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ *be the solution of system (4) with initial value* $\varphi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \ldots, \varphi_n^*(t))^T$*, and let* $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ *be the solution of system (4) with initial value* $\varphi(t) = (\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t))^T$*. Then, there exists a positive constant* λ *such that*

$$x_i(t) - x_i^*(t) = O\left(e^{-\lambda t}\right), \quad i \in \mathcal{N}. \tag{21}$$

Proof. With the help of (H_7) and a similar discussion as that in the proof of (2.9) in [20], we can select $\kappa > \lambda > 0$ and $\overline{\eta} > 0$ such that $c_i(t) > \lambda$, and

$$\begin{aligned}
&-\Big[(c_i(\rho) - \lambda)(1 - 2c_i^+\eta_i^+) \\
&\quad - \left|c_i(\rho)e^{\lambda\eta_i(\rho)} - \left(1 - \eta_i'(\rho)\right)c_i(\rho - \eta_i(\rho))\right|\Big] \\
&\quad \times \frac{\xi_i}{1 - c_i^+\eta_i^+} \\
&\quad + \sum_{j=1}^n \left|a_{ij}(\rho)\right| L_j^f e^{\lambda\tau_{ij}(\rho)} \frac{\xi_j}{1 - c_j^+\eta_j^+} \\
&\quad + \sum_{j=1}^n \sum_{l=1}^n \left|b_{ijl}(\rho)\right| \\
&\qquad \times \Bigg(M_j \int_0^\infty \left|K_{ij}(u)\right| du \int_0^\infty \left|K_{il}(u)\right| e^{\lambda u} du \\
&\qquad \times L_l^g \frac{\xi_l}{1 - c_l^+\eta_l^+} + \int_0^\infty \left|K_{ij}(u)\right| e^{\lambda u} du \\
&\qquad \times L_j^g \frac{\xi_j}{1 - c_j^+\eta_j^+} \int_0^\infty \left|K_{il}(u)\right| du M_l\Bigg) \\
&< -\overline{\eta}, \quad t \geq 0, \ i \in \mathcal{N}.
\end{aligned} \tag{22}$$

Let $y(t) = x(t) - x^*(t)$ and $Y_i(t) = e^{\lambda t}y_i(t) - \int_{t-\eta_i(t)}^t c_i(s)e^{\lambda s}y_i(s)ds$, $i \in \mathcal{N}$. Then,

$$\begin{aligned}
y_i'(t) = &-c_i(t)y_i(t - \eta_i(t)) \\
&+ \sum_{j=1}^n a_{ij}(t)\left[f_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) - f_j\left(x_j^*\left(t - \tau_{ij}(t)\right)\right)\right] \\
&+ \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t)\Bigg[\int_0^\infty K_{ij}(u) g_j\left(x_j(t-u)\right) du \\
&\quad \times \int_0^\infty K_{il}(u) g_l(x_l(t-u)) du \\
&\quad - \int_0^\infty K_{ij}(u) g_j\left(x_j^*(t-u)\right) du \\
&\quad \times \int_0^\infty K_{il}(u) g_l(x_l^*(t-u)) du\Bigg], \\
&\qquad t > 0, \quad t \neq t_k, \\
y_i^+(t_k^+) = &\left(1 + d_{ik}\right) y_i(t_k), \quad k \in Z^+,
\end{aligned} \tag{23}$$

which leads to

$$\begin{aligned}
Y_i'(t) &= \lambda e^{\lambda t} y_i(t) + e^{\lambda t} y_i'(t) \\
&\quad - \left[c_i(t)e^{\lambda t}y_i(t) - \left(1 - \eta_i'(t)\right)c_i(t - \eta_i(t)) \times e^{\lambda(t-\eta_i(t))} y_i(t - \eta_i(t))\right] \\
&= \lambda e^{\lambda t} y_i(t) \\
&\quad - \left[c_i(t)e^{\lambda t}y_i(t) - \left(1 - \eta_i'(t)\right) \times c_i(t - \eta_i(t)) e^{\lambda(t-\eta_i(t))} y_i(t - \eta_i(t))\right] \\
&\quad + e^{\lambda t}\Bigg\{-c_i(t)y_i(t - \eta_i(t)) \\
&\qquad + \sum_{j=1}^n a_{ij}(t)\left[f_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) - f_j\left(x_j^*\left(t - \tau_{ij}(t)\right)\right)\right] \\
&\qquad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) \\
&\qquad \times \Bigg[\int_0^\infty K_{ij}(u) g_j\left(x_j(t-u)\right) du \\
&\qquad \times \int_0^\infty K_{il}(u) g_l(x_l(t-u)) du \\
&\qquad - \int_0^\infty K_{ij}(u) g_j\left(x_j^*(t-u)\right) du \\
&\qquad \times \int_0^\infty K_{il}(u) g_l(x_l^*(t-u)) du\Bigg]\Bigg\} \\
&= -(c_i(t) - \lambda)Y_i(t) - (c_i(t) - \lambda) \\
&\quad \times \int_{t-\eta_i(t)}^t c_i(s)e^{\lambda s}y_i(s)ds \\
&\quad - \left[c_i(t) - \left(1 - \eta_i'(t)\right)c_i(t - \eta_i(t))e^{-\lambda\eta_i(t)}\right] \\
&\quad \times e^{\lambda t}y_i(t - \eta_i(t)) \\
&\quad + e^{\lambda t}\Bigg[\sum_{j=1}^n a_{ij}(t)\left(f_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) - f_j\left(x_j^*\left(t - \tau_{ij}(t)\right)\right)\right) \\
&\quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) \\
&\quad \times \Bigg(\int_0^\infty K_{ij}(u) g_j\left(x_j(t-u)\right) du \\
&\qquad \times \int_0^\infty K_{il}(u) g_l(x_l(t-u)) du \\
&\qquad - \int_0^\infty K_{ij}(u) g_j\left(x_j^*(t-u)\right) du \\
&\qquad \times \int_0^\infty K_{il}(u) g_l(x_l^*(t-u)) du\Bigg)\Bigg], \\
&\qquad t > 0, \quad t \neq t_k,
\end{aligned} \tag{24}$$

$$\left|Y_i\left(t_k^+\right)\right| = \left|\left(1+d_{ik}\right)Y_i\left(t_k\right)\right|. \tag{25}$$

We define a positive constant $\overline{M}$ as follows:

$$\overline{M} = \max_{i\in\mathcal{N}}\left\{\sup_{s\in(-\infty,0]}\left|Y_i(s)\right|\right\}. \tag{26}$$

There is a positive number K such that

$$\left|Y_i(t)\right| \le \overline{M} < K\xi_i, \quad \forall t\in(-\infty,0],\ i\in\mathcal{N}. \tag{27}$$

We assert that

$$\left|Y_i(t)\right| < K\xi_i, \quad \forall t>0,\ i\in\mathcal{N}. \tag{28}$$

Clearly, (28) holds for $t=0$. We first prove that (28) is true for $0<t\le t_1$. Otherwise, there exist $i\in\mathcal{N}$ and $\rho\in(0,t_1]$ such that

$$\left|Y_i(\rho)\right| = K\xi_i, \qquad \left|Y_j(t)\right| < K\xi_j, \quad \forall t\in[0,\rho),\ j\in\mathcal{N}. \tag{29}$$

It follows that, for $t\in[0,\rho)$ and $j\in\mathcal{N}$,

$$\begin{aligned} e^{\lambda t}\left|y_j(t)\right| &\le \left|e^{\lambda t}y_j(t) - \int_{t-\eta_j(t)}^{t} c_j(s)e^{\lambda s}y_j(s)\,ds\right| \\ &\quad + \left|\int_{t-\eta_j(t)}^{t} c_j(s)e^{\lambda s}y_j(s)\,ds\right| \\ &\le K\xi_j + c_j^+\eta_j^+ \sup_{s\in(-\infty,\rho]} e^{\lambda s}\left|y_j(s)\right|, \end{aligned} \tag{30}$$

and thus

$$e^{\lambda t}\left|y_j(t)\right| \le \sup_{s\in(-\infty,\rho]} e^{\lambda s}\left|y_j(s)\right| \le \frac{K\xi_j}{1-c_j^+\eta_j^+}. \tag{31}$$

Calculating the upper left derivative of $|Y_i(t)|$, together with (22), (24), (29), (31), (H_2), and (H_5)–(H_7), we get

$$\begin{aligned} 0 &\le D^-\left|Y_i(\rho)\right| \\ &\le -\left(c_i(\rho)-\lambda\right)\left|Y_i(\rho)\right| \\ &\quad + \left|-\left(c_i(\rho)-\lambda\right)\times\int_{\rho-\eta_i(\rho)}^{\rho} c_i(s)e^{\lambda s}y_i(s)\,ds\right| \\ &\quad + \Bigg|-\left[c_i(\rho)-\left(1-\eta_i'(\rho)\right)c_i\left(\rho-\eta_i(\rho)\right)e^{-\lambda\eta_i(\rho)}\right] \\ &\qquad \times e^{\lambda\rho}y_i\left(\rho-\eta_i(\rho)\right) + e^{\lambda\rho} \\ &\qquad \times\Bigg[\sum_{j=1}^{n}a_{ij}(\rho)\Big(f_j\left(x_j\left(\rho-\tau_{ij}(\rho)\right)\right) \\ &\qquad\qquad - f_j\left(x_j^*\left(\rho-\tau_{ij}(\rho)\right)\right)\Big) \\ &\qquad + \sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(\rho) \\ &\qquad\quad \times\left(\int_0^\infty K_{ij}(u)g_j\left(x_j(\rho-u)\right)du\right. \\ &\qquad\quad \times\int_0^\infty K_{il}(u)g_l\left(x_l(\rho-u)\right)du \\ &\qquad\quad - \int_0^\infty K_{ij}(u)g_j\left(x_j^*(\rho-u)\right)du \\ &\qquad\quad \left.\times\int_0^\infty K_{il}(u)g_l\left(x_l^*(\rho-u)\right)du\right)\Bigg]\Bigg| \\ &\le -\left(c_i(\rho)-\lambda\right)\left|Y_i(\rho)\right| + \left(c_i(\rho)-\lambda\right)c_i^+\eta_i^+\frac{K\xi_i}{1-c_i^+\eta_i^+} \\ &\quad + \left|c_i(\rho)-\left(1-\eta_i'(\rho)\right)c_i\left(\rho-\eta_i(\rho)\right)e^{-\lambda\eta_i(\rho)}\right| \\ &\quad \times e^{\lambda\eta_i(\rho)}e^{\lambda(\rho-\eta_i(\rho))}y_i\left(\rho-\eta_i(\rho)\right) \\ &\quad + \sum_{j=1}^{n}\left|a_{ij}(\rho)\right|L_j^f e^{\lambda\tau_{ij}(\rho)}e^{\lambda(\rho-\tau_{ij}(\rho))} \\ &\qquad \times\left|y_j\left(\rho-\tau_{ij}(\rho)\right)\right| \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n}\left|b_{ijl}(\rho)\right|e^{\lambda\rho} \\ &\quad \times\left(\left|\int_0^\infty K_{ij}(u)g_j\left(x_j(\rho-u)\right)du\right.\right. \\ &\qquad \times\int_0^\infty K_{il}(u)g_l\left(x_l(\rho-u)\right)du \\ &\qquad - \int_0^\infty K_{ij}(u)g_j\left(x_j(\rho-u)\right)du \\ &\qquad \left.\times\int_0^\infty K_{il}(u)g_l\left(x_l^*(\rho-u)\right)du\right| \\ &\qquad + \left|\int_0^\infty K_{ij}(u)g_j\left(x_j(\rho-u)\right)du\right. \\ &\qquad \times\int_0^\infty K_{il}(u)g_l\left(x_l^*(\rho-u)\right)du \\ &\qquad - \int_0^\infty K_{ij}(u)g_j\left(x_j^*(\rho-u)\right)du \\ &\qquad \left.\left.\times\int_0^\infty K_{il}(u)g_l\left(x_l^*(\rho-u)\right)du\right|\right) \\ &\le -\left(c_i(\rho)-\lambda\right)K\xi_i + \left(c_i(\rho)-\lambda\right)c_i^+\eta_i^+\frac{K\xi_i}{1-c_i^+\eta_i^+} \\ &\quad + \left|c_i(\rho)e^{\lambda\eta_i(\rho)}-\left(1-\eta_i'(\rho)\right)c_i\left(\rho-\eta_i(\rho)\right)\right|\frac{K\xi_i}{1-c_i^+\eta_i^+} \\ &\quad + \sum_{j=1}^{n}\left|a_{ij}(\rho)\right|L_j^f e^{\lambda\tau_{ij}(\rho)}\frac{K\xi_j}{1-c_j^+\eta_j^+} \\ &\quad + \sum_{j=1}^{n}\sum_{l=1}^{n}\left|b_{ijl}(\rho)\right|\times\left(\int_0^\infty\left|K_{ij}(u)\right|\left|g_j\left(x_j(\rho-u)\right)\right|du\right. \\ &\qquad \times\int_0^\infty\left|K_{il}(u)\right|L_l^g e^{\lambda u}e^{\lambda(\rho-u)} \\ &\qquad\quad \times\left|y_l(\rho-u)\right|du \\ &\qquad + \int_0^\infty\left|K_{ij}(u)\right|L_j^g e^{\lambda u}e^{\lambda(\rho-u)} \\ &\qquad\quad \times\left|y_j(\rho-u)\right|du \\ &\qquad \left.\times\int_0^\infty\left|K_{il}(u)\right|\left|g_l\left(x_l^*(\rho-u)\right)\right|du\right) \\ &\le -\left(c_i(\rho)-\lambda\right)K\xi_i + \left(c_i(\rho)-\lambda\right)c_i^+\eta_i^+\frac{K\xi_i}{1-c_i^+\eta_i^+} \end{aligned}$$

$$
\begin{aligned}
&+\left|c_i(\rho)e^{\lambda\eta_i(\rho)}-\left(1-\eta_i'(\rho)\right)c_i\left(\rho-\eta_i(\rho)\right)\right|\frac{K\xi_i}{1-c_i^+\eta_i^+}\\
&+\sum_{j=1}^{n}\left|a_{ij}(\rho)\right|L_j^f e^{\lambda\tau_{ij}(\rho)}\frac{K\xi_j}{1-c_j^+\eta_j^+}\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\left|b_{ijl}(\rho)\right|\times\left(M_j\int_0^\infty\left|K_{ij}(u)\right|du\right.\\
&\qquad\times\int_0^\infty\left|K_{il}(u)\right|e^{\lambda u}duL_l^g\frac{K\xi_l}{1-c_l^+\eta_l^+}\\
&\qquad+\int_0^\infty\left|K_{ij}(u)\right|e^{\lambda u}duL_j^g\frac{K\xi_j}{1-c_j^+\eta_j^+}\\
&\qquad\left.\times\int_0^\infty\left|K_{il}(u)\right|duM_l\right)\\
=&\left\{-\left[\left(c_i(\rho)-\lambda\right)\left(1-2c_i^+\eta_i^+\right)\right.\right.\\
&\quad-\left|c_i(\rho)e^{\lambda\eta_i(\rho)}-\left(1-\eta_i'(\rho)\right)\right.\\
&\quad\left.\left.\times c_i\left(\rho-\eta_i(\rho)\right)\right|\right]\frac{\xi_i}{1-c_i^+\eta_i^+}\\
&+\sum_{j=1}^{n}\left|a_{ij}(\rho)\right|L_j^f e^{\lambda\tau_{ij}(\rho)}\frac{\xi_j}{1-c_j^+\eta_j^+}\\
&+\sum_{j=1}^{n}\sum_{l=1}^{n}\left|b_{ijl}(\rho)\right|\times\left(M_j\int_0^\infty\left|K_{ij}(u)\right|du\right.\\
&\qquad\times\int_0^\infty\left|K_{il}(u)\right|e^{\lambda u}du\\
&\qquad\times L_l^g\frac{\xi_l}{1-c_l^+\eta_l^+}\\
&\qquad+\int_0^\infty\left|K_{ij}(u)\right|e^{\lambda u}du\\
&\qquad\times L_j^g\frac{\xi_j}{1-c_j^+\eta_j^+}\\
&\qquad\left.\left.\times\int_0^\infty\left|K_{il}(u)\right|duM_l\right)\right\}K\\
<&-\overline{\eta}K<0,
\end{aligned}
\tag{32}
$$

which is a contradiction. Therefore, (28) holds for $t\in[0,t_1]$. From (25), (28), (31), and (H_3), we know that

$$
\begin{aligned}
\left|y_i(t_1)\right|e^{\lambda t_1}&<\frac{K\xi_i}{1-c_i^+\eta_i^+},\quad i\in\mathcal{N},\\
\left|y_i(t_1^+)\right|e^{\lambda t_1^+}&=\left|1+d_{i1}\right|\left|y_i(t_1)\right|e^{\lambda t_1}\\
&\le\left|y_i(t_1)\right|e^{\lambda t_1}<\frac{K\xi_i}{1-c_i^+\eta_i^+},\quad i\in\mathcal{N}.
\end{aligned}
\tag{33}
$$

Thus, using the same argument as the above procedure, we can obtain

$$
\left|y_i(t)\right|e^{\lambda t}<\frac{K\xi_i}{1-c_i^+\eta_i^+},\quad\forall t\in\left[t_1,t_2\right],\ i\in\mathcal{N}.\tag{34}
$$

Further, we have

$$
\left|y_i(t)\right|e^{\lambda t}<\frac{K\xi_i}{1-c_i^+\eta_i^+},\quad\forall t>0,\ i\in\mathcal{N}.\tag{35}
$$

That is,

$$
\left|x_i(t)-x_i^*(t)\right|\le\frac{K\xi_i}{1-c_i^+\eta_i^+}e^{-\lambda t},\quad\forall t>0,\ i\in\mathcal{N}.\tag{36}
$$

□

Remark 5. If $x^*(t)=(x_1^*(t),x_2^*(t),\ldots,x_n^*(t))^T$ is an ω-antiperiodic solution of system (4), it follows from Lemma 4 that $x^*(t)$ is globally exponentially stable.

3. Main Results

In this section, we will study the existence and global exponential stability of the antiperiodic solution for system (4).

Theorem 6. *Suppose that all conditions in Lemma 4 are satisfied. Then, system (4) has exactly one ω-antiperiodic solution $x^*(t)$. Moreover, $x^*(t)$ is globally exponentially stable.*

Proof. Let $x(t)=(x_1(t),x_2(t),\ldots,x_n(t))^T$ be a solution of system (4). By Remark 3, the solution $x(t)$ can be defined for all $t\in[0,+\infty)$. By hypotheses (H_1), we have, for any natural number h and $i\in\mathcal{N}$,

$$
\begin{aligned}
&\left((-1)^{h+1}x_i(t+(h+1)\omega)\right)'\\
&\quad=(-1)^{h+1}\left(x_i(t+(h+1)\omega)\right)'=(-1)^{h+1}\\
&\qquad\times\left\{-c_i(t+(h+1)\omega)\right.\\
&\qquad\quad\times x_i\left(t+(h+1)\omega-\eta_i(t+(h+1)\omega)\right)\\
&\qquad\quad+\sum_{j=1}^{n}a_{ij}(t+(h+1)\omega)\\
&\qquad\quad\times f_j\left(x_j\left(t+(h+1)\omega-\tau_{ij}(t+(h+1)\omega)\right)\right)\\
&\qquad\quad+\sum_{j=1}^{n}\sum_{l=1}^{n}b_{ijl}(t+(h+1)\omega)\\
&\qquad\quad\times\int_0^\infty K_{ij}(u)g_j\left(x_j(t+(h+1)\omega-u)\right)du\\
&\qquad\quad\times\int_0^\infty K_{il}(u)g_l\left(x_l(t+(h+1)\omega-u)\right)du\\
&\qquad\quad\left.+I_i(t+(h+1)\omega)\right\}\\
&\quad=(-1)^{h+1}\left\{-c_i(t)x_i\left(t+(h+1)\omega-\eta_i(t)\right)\right.\\
&\qquad\quad+\sum_{j=1}^{n}a_{ij}(t)(-1)^{h+1}f_j\\
&\qquad\quad\times\left((-1)^{h+1}x_j\left(t+(h+1)\omega-\tau_{ij}(t)\right)\right)\\
&\qquad\quad+\sum_{j=1}^{n}\sum_{l=1}^{n}(-1)^{h+1}b_{ijl}(t)
\end{aligned}
$$

$$\times \int_0^\infty K_{ij}(u)\, g_j \times \left((-1)^{h+1} x_j(t+(h+1)\omega-u)\right) du \times \int_0^\infty K_{il}(u)\, g_l \times \left((-1)^{h+1} x_l(t+(h+1)\omega-u)\right) du + (-1)^{h+1} I_i(t) \Bigg\}$$

$$\begin{aligned} &= -c_i(t)(-1)^{h+1} x_i(t+(h+1)\omega-\eta_i(t)) \\ &\quad + \sum_{j=1}^n a_{ij}(t) f_j\left((-1)^{h+1} x_j\left(t+(h+1)\omega-\tau_{ij}(t)\right)\right) \\ &\quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) \\ &\quad \times \int_0^\infty K_{ij}(u)\, g_j\left((-1)^{h+1} x_j(t+(h+1)\omega-u)\right) du \\ &\quad \times \int_0^\infty K_{il}(u)\, g_l\left((-1)^{h+1} x_l(t+(h+1)\omega-u)\right) du \\ &\quad + I_i(t), \quad t \neq t_k. \end{aligned} \tag{37}$$

Further, by hypothesis of (H_4), we obtain

$$\begin{aligned} &(-1)^{h+1} x_i\left((t_k+(h+1)\omega)^+\right) \\ &\quad = (-1)^{h+1} x_i\left(t^+_{k+(h+1)q}\right) \\ &\quad = (-1)^{h+1}\left(1+d_{i(k+(h+1)q)}\right) x_i\left(t_{k+(h+1)q}\right) \\ &\quad = \left(1+d_{ik}\right)(-1)^{h+1} x_i\left(t_k+(h+1)\omega\right), \\ &\qquad\qquad k = 1, 2, \ldots. \end{aligned} \tag{38}$$

Hence, for any natural number h, we obtain that $(-1)^{h+1}x(t+(h+1)\omega)$ is a solution of system (4) for all $t+(h+1)\omega \geq 0$. Hence, $-x(t+\omega)$ is also a solution of (4) with initial values:

$$-x_i(s+\omega), \quad s \in (-\infty, 0], \ i \in \mathcal{N}. \tag{39}$$

Then, by the proof of Lemma 4, for $i \in \mathcal{N}$, there exists a constant $K > 0$ such that, for any natural number h,

$$\begin{aligned} &\left|(-1)^{h+1} x_i(t+(h+1)\omega) - (-1)^h x_i(t+h\omega)\right| \\ &\quad = \left|x_i(t+h\omega) - (-x_i(t+h\omega+\omega))\right| \\ &\quad \leq \frac{K\xi_i}{1-c_i^+\eta_i^+} e^{-\lambda(t+h\omega)} \\ &\quad = \frac{K\xi_i}{1-c_i^+\eta_i^+} e^{-\lambda t}\left(\frac{1}{e^{\lambda\omega}}\right)^h, \quad t+h\omega \geq 0, \ t \neq t_k, \\ &\left|(-1)^{h+1} x_i\left((t_k+(h+1)\omega)^+\right) - (-1)^h x_i\left((t_k+h\omega)^+\right)\right| \\ &\quad = \left|1+d_{ik}\right|\left|x_i(t_k+h\omega) - (-x_i(t_k+h\omega+\omega))\right| \\ &\quad \leq \frac{K\xi_i}{1-c_i^+\eta_i^+} e^{-\lambda(t_k+h\omega)} \\ &\quad = \frac{K\xi_i}{1-c_i^+\eta_i^+} e^{-\lambda t_k}\left(\frac{1}{e^{\lambda\omega}}\right)^h, \quad k \in Z^+. \end{aligned} \tag{40}$$

Furthermore, for any natural number m and $i \in \mathcal{N}$, we can obtain

$$\begin{aligned} &(-1)^{m+1} x_i(t+(m+1)\omega) \\ &\quad = x_i(t) + \sum_{h=0}^m \left[(-1)^{h+1} x_i(t+(h+1)\omega) - (-1)^h x_i(t+h\omega)\right], \quad t+h\omega \geq 0, \ t \neq t_k, \\ &(-1)^{m+1} x_i\left((t_k+(m+1)\omega)^+\right) \\ &\quad = x_i(t_k^+) + \sum_{h=0}^m \left[(-1)^{h+1} x_i\left((t_k+(h+1)\omega)^+\right) - (-1)^h x_i\left((t_k+h\omega)^+\right)\right], \quad k \in Z^+. \end{aligned} \tag{41}$$

Due to (40)–(41), we know that $(-1)^m x(t+m\omega)$ converges uniformly to a piecewise continuous function $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ on any compact set of R.

Next, we show that $x^*(t)$ is an ω-antiperiodic solution of system (4). It is easy to see that $x^*(t)$ is ω-antiperiodic, since

$$\begin{aligned} x_i^*(t+\omega) &= \lim_{m\to+\infty} (-1)^m x_i(t+\omega+m\omega) \\ &= -\lim_{m+1\to+\infty} (-1)^{m+1} x_i(t+(m+1)\omega) \\ &= -x_i^*(t), \quad t \neq t_k, \\ x_i^*\left((t_k+\omega)^+\right) &= -\lim_{m+1\to+\infty} (-1)^{m+1} x_i\left((t_k+(m+1)\omega)^+\right) \\ &= -x_i^*(t_k^+), \quad k \in Z^+, \end{aligned} \tag{42}$$

where $i \in \mathcal{N}$. Observing that the right side of (4) is piecewise continuous, together with (37) and (38), we find that $\{(-1)^{m+1} x_i'(t+(m+1)\omega)\}$ converges uniformly to a piecewise continuous function on any compact set of $R \setminus \{t_1, t_2, \ldots\}$. Therefore, letting $m \to +\infty$ on both sides of (37) and (38), we get

$$\begin{aligned} x_i^{*\prime}(t) &= -c_i(t)\, x_i^*(t-\eta_i(t)) \\ &\quad + \sum_{j=1}^n a_{ij}(t) f_j\left(x_j^*\left(t-\tau_{ij}(t)\right)\right) \\ &\quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) \int_0^\infty K_{ij}(u)\, g_j\left(x_j^*(t-u)\right) du \\ &\quad \times \int_0^\infty K_{il}(u)\, g_l\left(x_l^*(t-u)\right) du \\ &\quad + I_i(t), \quad t > 0, \ t \neq t_k, \\ x_i^*(t_k^+) &= \left(1+d_{ik}\right) x_i^*(t_k), \quad k \in Z^+, \\ &\qquad i \in \mathcal{N}. \end{aligned} \tag{43}$$

Thus, $x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T$ is an ω-antiperiodic solution of system (4).

Finally, adopting the same procedure as in the proof of Lemma 4, we can show that $x^*(t)$ is globally exponentially stable. This completes the proof. □

4. An Example

In this section, we provide an example to demonstrate the results obtained in the previous sections.

Example 1. Consider the following IHHNNs consisting of two neurons with leakage delays and continuously distributed delays:

$$
\begin{aligned}
x_1'(t) = & -1.5x_1\left(t-\frac{|\cos\pi t|}{1000}\right) \\
& +\frac{|\sin\pi t|}{32}f_1\left(x_1\left(t-2|\sin\pi t|\right)\right) \\
& +\frac{|\cos\pi t|}{32}f_2\left(x_2\left(t-2|\cos\pi t|\right)\right)+\frac{\cos\pi t}{32} \\
& \times\left[\int_0^\infty \sin ue^{-u}g_1\left(x_1(t-u)\right)du\right. \\
& \quad\times\int_0^\infty \cos ue^{-u}g_1\left(x_1(t-u)\right)du \\
& \quad+\int_0^\infty \sin ue^{-u}g_1\left(x_1(t-u)\right)du \\
& \quad\times\int_0^\infty \sin ue^{-u}g_2\left(x_2(t-u)\right)du \\
& \quad+\int_0^\infty \cos 2ue^{-u}g_2\left(x_2(t-u)\right)du \\
& \quad\times\int_0^\infty \sin 2ue^{-u}g_1\left(x_1(t-u)\right)du \\
& \quad+\int_0^\infty \cos 2ue^{-u}g_2\left(x_2(t-u)\right)du \\
& \quad\left.\times\int_0^\infty \cos 2ue^{-u}g_2\left(x_2(t-u)\right)du\right] \\
& +10\sin\pi t, \\
x_2'(t) = & -1.5x_2\left(t-\frac{|\cos\pi t|}{1000}\right) \\
& +\frac{|\cos\pi t|}{32}f_1\left(x_1\left(t-2|\cos\pi t|\right)\right) \\
& +\frac{|\sin\pi t|}{32}f_2\left(x_2\left(t-2|\sin\pi t|\right)\right)+\frac{\sin\pi t}{32} \\
& \times\left[\int_0^\infty \sin ue^{-u}g_1\left(x_1(t-u)\right)du\right. \\
& \quad\times\int_0^\infty \cos ue^{-u}g_1\left(x_1(t-u)\right)du \\
& \quad+\int_0^\infty \sin ue^{-u}g_1\left(x_1(t-u)\right)du \\
& \quad\times\int_0^\infty \sin ue^{-u}g_2\left(x_2(t-u)\right)du \\
& \quad+\int_0^\infty \cos 2ue^{-u}g_2\left(x_2(t-u)\right)du \\
& \quad\times\int_0^\infty \sin 2ue^{-u}g_1\left(x_1(t-u)\right)du \\
& \quad+\int_0^\infty \cos 2ue^{-u}g_2\left(x_2(t-u)\right)du \\
& \quad\left.\times\int_0^\infty \cos 2ue^{-u}g_2\left(x_2(t-u)\right)du\right]+10\cos\pi t, \\
& \qquad t\neq k-0.5, \\
& x_i\left(t_k^+\right)=\left(1+d_{ik}\right)x_i\left(t_k\right), \\
& d_{i(2s)}=-2,\qquad d_{i(2s-1)}=-1, \\
& t_k=k-0.5,\quad i=1,2, \\
& k,s=1,2,\ldots.
\end{aligned}
\tag{44}
$$

Here, it is assumed that the activation functions

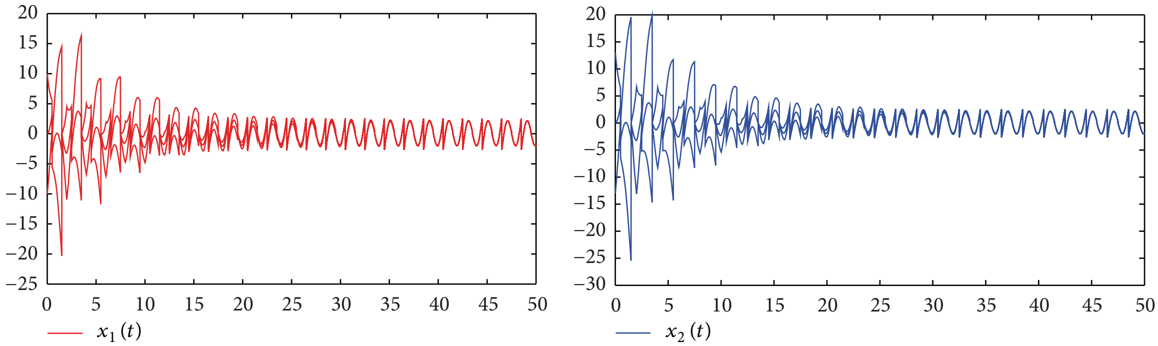
$$
\begin{aligned}
f_1(x)&=f_2(x)=2x, \\
g_1(x)&=g_2(x)=|x+1|-|x-1|.
\end{aligned}
\tag{45}
$$

Note that

$$
\begin{gathered}
L_1^f=L_2^f=L_1^g=L_2^g=2,\qquad M_2=M_2=2, \\
c_1(t)=c_2(t)=1.5,\qquad \int_0^\infty\left|K_{ij}(u)\right|du\le 1, \\
a_{11}(t)=\frac{|\sin\pi t|}{32},\qquad a_{12}(t)=\frac{|\cos\pi t|}{32}, \\
a_{21}(t)=\frac{|\cos\pi t|}{32},\qquad a_{22}(t)=\frac{|\sin\pi t|}{32}, \\
b_{111}(t)=b_{112}(t)=b_{121}(t)=b_{122}(t)=\frac{\cos\pi t}{32}, \\
b_{211}(t)=b_{212}(t)=b_{221}(t)=b_{222}(t)=\frac{\sin\pi t}{32}, \\
\eta_1(t)=\eta_2(t)=\frac{|\cos\pi t|}{1000}, \\
I_1(t)=10\sin\pi t,\qquad I_2(t)=10\cos\pi t, \\
\tau_{11}(t)=2|\sin\pi t|,\qquad \tau_{12}(t)=2|\cos\pi t|, \\
\tau_{21}(t)=2|\cos\pi t|,\qquad \tau_{22}(t)=2|\sin\pi t|.
\end{gathered}
\tag{46}
$$

Then, we obtain

$$
\begin{aligned}
& -\left[c_i(t)\left(1-2c_i^+\eta_i^+\right)-\left|c_i(t)-\left(1-\eta_i'(t)\right)c_i\left(t-\eta_i(t)\right)\right|\right] \\
& \quad\times\frac{\xi_i}{1-c_i^+\eta_i^+}+\sum_{j=1}^n\left|a_{ij}(t)\right|L_j^f\frac{\xi_j}{1-c_j^+\eta_j^+}+\sum_{j=1}^n\sum_{l=1}^n\left|b_{ijl}(t)\right| \\
& \quad\times\left(M_j\int_0^\infty\left|K_{ij}(u)\right|du\int_0^\infty\left|K_{il}(u)\right|duL_l^g\frac{\xi_l}{1-c_l^+\eta_l^+}\right. \\
& \quad\quad\left.+\int_0^\infty\left|K_{ij}(u)\right|duL_j^g\frac{\xi_j}{1-c_j^+\eta_j^+}\int_0^\infty\left|K_{il}(u)\right|duM_l\right) \\
< & -\left[1.5\left(1-2\times\frac{1.5}{1000}\right)-1.5\times\frac{\pi}{1000}\right]\frac{1}{1-1.5/1000} \\
& +\frac{1}{32}\times 2\times\frac{1}{1-1.5/1000}\times 2 \\
& +\frac{1}{32}\left(2\times 2\times\frac{1}{1-1.5/1000}+2\times\frac{1}{1-1.5/1000}\times 2\right) \\
& \times 4\approx -0.3619<-0.3,\qquad \xi_i=1,\ i=1,2.
\end{aligned}
\tag{47}
$$



(a) (b)

FIGURE 1: Numerical solutions $x(t) = (x_1(t), x_2(t))^T$ of system (1) for initial value $\varphi(t) \equiv (10, -13)^T, (-10, 13)^T, (0, 0)^T, t \in (-\infty, 0]$.

It follows that system (1) satisfies all the conditions in Theorem 6. Therefore, system (1) has exactly one 1-antiperiodic solution. Moreover, the 1-antiperiodic solution is globally exponentially stable. The fact is verified by the numerical simulations in Figure 1.

Remark 7. Since [4–13] mainly obtained the stability results on the impulsive Hopfield neural networks without leakage delays and [15–19] dealt with nonimpulsive neural networks, it can be observed that all the results in the literature and the references therein cannot be applicable to prove the existence and exponential stability of 1-antiperiodic solution for IHHNNs (1). Furthermore,

$$t - \eta_i(t) = t - \frac{|\cos \pi t|}{1000} < 0 \tag{48}$$

is possible for some $t > 0, i = 1, 2$; one can find that the results in [20] are invalid. This implies that the results of this paper are essentially new.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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