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Research Article

Some Properties for an Integral Operator Defined by Generalized Hypergeometric Function

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We define a new general integral operator for meromorphic functions involving the generalized hypergeometric function. Furthermore, we study the characterization and other properties for this operator.

1. Introduction

Let Ω denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the punctured open unit disk $\mathbb{U}^* = \{z : z \in \mathbb{C}, 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$.

We say that a function $f \in \Omega$ is meromorphic starlike of order η $(0 \le \eta < 1)$ and belongs to the class $\Omega^*(\eta)$, if it satisfies the following inequality:

$$-\Re\mathrm{e}\left(\frac{zf'(z)}{f(z)}\right) > \eta. \tag{2}$$

A function $f \in \Omega$ is a meromorphic convex function of order η $(0 \le \eta < 1)$, if f satisfies the following inequality:

$$-\Re\mathrm{e}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \eta, \tag{3}$$

and we denote this class by $\Omega_k(\eta)$.

Also, let $\Omega_N(\xi)$ $(\xi > 1,\ z \in \mathbb{U})$ be the subclass of Ω, consisting of the functions f, which satisfy the inequality

$$-\Re\mathrm{e}\left(1 + \frac{zf''(z)}{f'(z)}\right) < \xi \quad (\xi > 1,\ z \in \mathbb{U}), \tag{4}$$

where the subclass $\Omega_N(\xi)$ was introduced and studied by Wang et al. [1] and Nehari and Netanyahu [2].

A q-hypergeometric function is a power series in one complex variable z with power series coefficients which depend (apart from q) on r complex upper parameters α_i, β_j, $(i = 1, \ldots, r,\ j = 1, \ldots, s, \alpha_i \in \mathbb{C}, \beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\})$ as follows:

$$\begin{aligned}&\mathbb{V}(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_s, q, z)\\&\quad= \sum_{k=0}^{\infty} \frac{(\alpha_1, q)_k \cdots (\alpha_r, q)_k}{(q, q)_k (\beta_1, q)_k \cdots (\beta_s, q)_k} \left[(-1)^k q^{\binom{k}{2}}\right]^{1+s-r} z^k,\end{aligned} \tag{5}$$

with $\binom{k}{2} = k(k-1)/2$, where $q \neq 0$ when $r > s+1$, $(r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$, and $(\alpha, q)_k$ is the Pochhammer symbol $(\alpha)_k$ defined by

$$(\alpha, q)_k = \begin{cases} 1, & k = 0; \\ (1-\alpha)(1-\alpha q)\left(1-\alpha q^2\right)\cdots\left(1-\alpha q^{k-1}\right), & k \in \mathbb{N}. \end{cases} \tag{6}$$

Tracing back the history of basic hypergeometric series defined by (5) has brought us to Heine dated in 1846. Therefore it is sometimes called Heine's series. For brief survey on q-hypergeometric functions, one may refer to [3–5] (see also [6–8]).

For $z \in \mathbb{U}$, $|q| < 1$, and $r = s+1$, the basic hypergeometric function defined in (5) takes the form

$$ {}_r\Lambda_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z) = \sum_{k=0}^{\infty}\frac{(\alpha_1;q)_k\cdots(\alpha_r;q)_k}{(q;q)_k(\beta_1;q)_k\cdots(\beta_s;q)_k}z^k, \quad (7)$$

which converges absolutely in the open unit disk $\mathbb{U}$.

Corresponding to the function ${}_r\Lambda_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z)$, let

$$ r\Psi_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z) = \frac{1}{z}\,{}_r\Lambda_s(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_s,q,z) = \frac{1}{z}+\sum_{k=1}^{\infty}\frac{(\alpha_1,q)_{k+1}\cdots(\alpha_r,q)_{k+1}}{(q,q)_{k+1}(\beta_1,q)_{k+1}\cdots(\beta_s,q)_{k+1}}z^k. \quad (8)$$

Analogous to the differential operator defined in [9] which involves the q-hypergeometric functions on the normalized analytic functions, we define the following differential operator $\mathcal{N}^n_{r,s}[\alpha_i,\beta_j;q]:\Omega\to\Omega$ on the space of meromorphic functions in the class Ω by

$$\begin{aligned}
\mathcal{N}^0_{r,s}\left[a_i,b_j;q\right]f(z) &= f(z) * {}_r\Psi_s\left(a_i,b_j;q;z\right)\\
\mathcal{N}^1_{r,s}\left[a_i,b_j;q\right]f(z) &= (1-\lambda)f(z) * {}_r\Psi_s\left(a_i,b_j;q;z\right)\\
&\quad + \lambda z D_q\left(f(z) * {}_r\Psi_s\left(a_i,b_j;q;z\right)\right)\\
&\vdots\\
\mathcal{N}^n_{r,s}\left[a_i,b_j;q\right]f(z) &= \mathcal{N}^1_{r,s}\left(\mathcal{N}^{n-1}_{r,s}\left(f(z)\right)\right).
\end{aligned} \quad (9)$$

If $f \in \Omega$, then from (9) we may deduce that

$$\mathcal{N}^n_{r,s}\left[a_i,b_j;q\right]f(z) = \frac{1}{z}+\sum_{k=1}^{\infty}[1+(k-1)\lambda]^n\Upsilon_k a_k z^k, \quad (10)$$

where

$$\Upsilon_k = \frac{(\alpha_1,q)_{k+1}\cdots(\alpha_r,q)_{k+1}}{(q,q)_{k+1}(\beta_1,q)_{k+1}\cdots(\beta_s,q)_{k+1}}, \quad (11)$$

$\lambda \geq 0$, and $n \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

It should be remarked that the operator given by (10) is a generalization of many other operators considered earlier; for example, consider the following.

(i) For $\lambda = 1$, $r = 1$, $s = 0$, $\alpha_1 = q$, $q \to 1$ the operator $\mathcal{N}_{1,0}[q,q]f(z) = I^n f(z)$ was defined by Frasin and Darus [10] and studied by El-Ashwah and Aouf [11].

(ii) For $n = 0$, $\alpha_i = q^{\alpha_i}$, $\beta_j = q^{\beta_j}$, $\alpha_i > 0$, $\beta_j > 0$, $(i = 1,\ldots,r;\ j = 1,\ldots,s, r = s+1)$, $q \to 1$ the operator $\mathcal{N}_{r,s}[q^{\alpha_i},q^{\beta_j},q]f(z) = \mathscr{H}_{r,s}[\alpha_i,\beta_j]f(z)$ was imposed by Liu and Srivastava [12].

(iii) For $n = 0$, $r = 2$, $s = 1$, $\alpha_2 = q$, $q \to 1$, the operator $\mathcal{N}_{2,1}[\alpha_1,q,\beta_1,q]f(z) = \mathcal{N}[\alpha_1;\beta_1]f(z)$ was introduced and studied by Liu and Srivastava [13]. Further, we note in passing that this operator $\mathcal{N}[\alpha_1;\beta_1]f(z)$ is closely related to the Carlson-Shaffer operator $\mathcal{N}[\alpha_1;\beta_1]f(z)$ defined on the space of analytic univalent functions in $\mathbb{U}$.

(iv) For $n = 0$, $r = 1$, $s = 0$, $\alpha_1 = \lambda+1$, $q \to 1$, the operator $\mathcal{N}_{1,0}[\lambda+1,q]f(z) = \mathscr{D}^\lambda f(z) = (1/z(1-z)^{\lambda+1}) * f(z)(\lambda > -1)$, where $\mathscr{D}^\lambda$ is the differential operator, was introduced by Ganigi and Uralegaddi [14], and then it was generalized by Yang [15].

Now, making use of the differential operator $\mathcal{N}^n_{r,s}[\alpha_i,\beta_j;q]$ given by (10), we introduce the following integral operator in the class Ω.

Definition 1. Let $n \in \mathbb{N}$, $i \in \{1,2,\ldots,n\}$, $\delta_i > 0$. We define the integral operator $l_{\delta_1,\ldots,\delta_n}(z):\Omega^n \to \Omega$ by

$$l_{\delta_1,\ldots,\delta_n}(z) = \frac{1}{z^2}\int_0^z\left(-u^2\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_1\right)'(u)\right)^{\delta_1}\cdots\left(-u^2\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_n\right)'(u)\right)^{\delta_n}du. \quad (12)$$

Remark 2. Note that for $n = 0$, $r = 1$, $s = 0$, $\alpha_1 = q$, we obtain the integral operator defined by Mohammed and Darus [16]; see also [17–19].

With the aid of differential operator $\mathcal{N}^n_{r,s}[\alpha_i,\beta_j;q]f(z)$ given by (10), we define a new subclass of functions in Ω as follows.

Definition 3. Let a function $f \in \Omega$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Omega_k(\alpha_i,\beta_j;q,\eta)$ if, and only if, f satisfies

$$-\Re\left\{1+\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f\right)'(z)}\right\} > \eta, \quad (13)$$

where $\mathcal{N}^n_{r,s}[\alpha_i,\beta_j;q]f$ is defined in (10) and $0 \leq \eta < 1$.

Definition 4. Let a function $f \in \Omega$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Omega_{r,s}\mathscr{K}(\alpha_i,\beta_j,q,\eta,b)$ if, and only if, f satisfies

$$\Re\left\{1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f\right)'(z)}+2\right)\right\} > \eta, \quad (14)$$

where $\mathcal{N}^n_{r,s}[\alpha_i,\beta_j;q]f$ is defined in (10) and $b \in \mathbb{C}\setminus\{0\}$, $0 \leq \eta < 1$.

Definition 5. Let a function $f \in \Omega$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Omega_{r,s}\mathcal{KU}(\alpha_i, \beta_j; q, \alpha, \eta, b)$ if, and only if, f satisfies

$$\Re e \left\{ 1 - \frac{1}{b} \left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f\right)'(z)} + 2 \right) \right\} > \alpha \left| \frac{1}{b} \left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f\right)'(z)} + 2 \right) \right| + \eta, \tag{15}$$

where $\mathcal{N}^n_{r,s}[\alpha_i, \beta_j; q] f$ is defined in (10) and $\alpha \geq 0, -1 \leq \eta < 1, \alpha + \eta \geq 0, b \in \mathbb{C} \setminus \{0\}$.

Definition 6. Let a function $f \in \Omega$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Omega_{r,s}\mathcal{KUH}(\alpha_i, \beta_j, q, \alpha, b)$ if, and only if, f satisfies

$$\left| 1 - \frac{1}{b} \left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f\right)'(z)} + 2 \right) - 2\alpha\left(\sqrt{2} - 1\right) \right| < \Re e \left\{ \sqrt{2} \left(1 - \frac{1}{b} \left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f\right)'(z)} + 2 \right) \right) \right\} + 2\alpha\left(\sqrt{2} - 1\right), \tag{16}$$

where $\mathcal{N}^n_{r,s}[\alpha_i, \beta_j; q] f$ is defined in (10) and $\alpha > 0, b \in \mathbb{C} \setminus \{0\}$.

Remark 7. Putting $n = 0$, $r = 1$, $s = 0$ and $\alpha_1 = q$ in Definition 3, we obtain the result corresponding to the class mentioned in (3).

For $n = 0, r = 1, s = 0$ and $\alpha_1 = q$ in Definitions 4, 5, and 6, we obtain $\Omega\mathcal{K}_b(\eta), \Omega\mathcal{KU}(\alpha, \eta, b)$ and $\Omega\mathcal{KUH}(\alpha, b)$ the classes of meromorphic functions, introduced and studied by Mohammed and Darus [20].

2. Preliminary Definitions

We begin by recalling each of the following definitions of subclasses of meromorphic functions $\Omega\mathcal{G}_1(\eta, b), \Omega\mathcal{G}_2(\alpha, \eta, b)$ and $\Omega\mathcal{G}_3(\alpha, b)$ which will be required in our investigation.

Definition 8 (see [20]). Let a function $f \in \Omega$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Omega\mathcal{G}_1(\eta, b)$ if, and only if, f satisfies

$$\Re e \left\{ 1 - \frac{1}{b} \left(\frac{z\left(zf''(z) + 3f'(z)\right)}{zf'(z) + 2f(z)} + 1 \right) \right\} > \eta, \tag{17}$$

where $b \in \mathbb{C} \setminus \{0\}, 0 \leq \eta < 1$.

Definition 9 (see [20]). Let a function $f \in \Omega$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Omega\mathcal{G}_2(\alpha, \eta, b)$ if, and only if, f satisfies

$$\Re e \left\{ 1 - \frac{1}{b} \left(\frac{z\left(zf''(z) + 3f'(z)\right)}{zf'(z) + 2f(z)} + 1 \right) \right\} > \alpha \left| \frac{1}{b} \left(\frac{z\left(zf''(z) + 3f'(z)\right)}{zf'(z) + 2f(z)} + 1 \right) \right| + \eta, \tag{18}$$

where $\alpha \geq 0, -1 \leq \eta < 1, \alpha + \eta \geq 0, b \in \mathbb{C} \setminus \{0\}$.

Definition 10 (see [20]). Let a function $f \in \Omega$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Omega\mathcal{G}_3(\alpha, b)$ if, and only if, f satisfies

$$\left| 1 - \frac{1}{b} \left(\frac{z\left(zf''(z) + 3f'(z)\right)}{zf'(z) + 2f(z)} + 1 \right) - 2\alpha\left(\sqrt{2} - 1\right) \right| < \Re e \left\{ \sqrt{2} \left(1 - \frac{1}{b} \left(\frac{z\left(zf''(z) + 3f'(z)\right)}{zf'(z) + 2f(z)} + 1 \right) \right) \right\} + 2\alpha\left(\sqrt{2} - 1\right), \tag{19}$$

where $\alpha > 0, b \in \mathbb{C} \setminus \{0\}$.

3. Main Results

In this section, we study some properties for the integral operator $l_{\delta_1,\dots,\delta_n}(z)$ defined by (12) of the subclasses given by Definitions 3, 4, 5, and 6.

Theorem 11. *Let $\delta_i > 0$, for $i \in \{1, 2, \dots, n\}$ and $f_i \in \Omega$. If*

$$\sum_{i=1}^{n} \delta_i \left(1 - \eta_i\right) < 2, \quad \left(0 \leq \eta_i < 1\right) \tag{20}$$

and $f_i \in \Omega_k(\alpha_i, \beta_j, q, \eta_i)$, then $l_{\delta_1,\dots,\delta_n}(z)$ is in the class $\Omega_N(\xi)$, $\xi > 1$.

Proof. By differentiating the equality given by (12), we get

$$z^2 l'_{\delta_1,\dots,\delta_n}(z) + 2z l_{\delta_1,\dots,\delta_n}(z) = \left(-z^2\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f_1\right)'(z)\right)^{\delta_1} \cdots \left(-z^2\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f_n\right)'(z)\right)^{\delta_n}. \tag{21}$$

Again, differentiating (21), we deduce that

$$z^2 l''_{\delta_1,\dots,\delta_n}(z) + 4z l'_{\delta_1,\dots,\delta_n}(z) + 2 l_{\delta_1,\dots,\delta_n}(z) = \sum_{i=1}^{n} \delta_i \left(\frac{\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f_i\right)'(z)} + \frac{2}{z} \right), \tag{22}$$

$$\left[\left(-z^2\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f_1\right)'(z)\right)^{\delta_1} \cdots \left(-z^2\left(\mathcal{N}^n_{r,s}\left[\alpha_i, \beta_j; q\right] f_n\right)'(z)\right)^{\delta_n} \right]. \tag{23}$$

By using (21) and (23), we have

$$\begin{aligned}&\frac{z^2 l''_{\delta_1,\dots,\delta_n}(z)+4zl'_{\delta_1,\dots,\delta_n}(z)+2l_{\delta_1,\dots,\delta_n}(z)}{z^2 l'_{\delta_1,\dots,\delta_n}(z)+2zl_{\delta_1,\dots,\delta_n}(z)}\\&\quad=\sum_{i=1}^{n}\delta_i\left(\frac{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+\frac{2}{z}\right).\end{aligned}\tag{24}$$

Multiplying (24) by z yields that

$$\begin{aligned}&\frac{z^2 l''_{\delta_1,\dots,\delta_n}(z)+4zl'_{\delta_1,\dots,\delta_n}(z)+2l_{\delta_1,\dots,\delta_n}(z)}{zl'_{\delta_1,\dots,\delta_n}(z)+2l_{\delta_1,\dots,\delta_n}(z)}\\&\quad=\sum_{i=1}^{n}\delta_i\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right),\end{aligned}\tag{25}$$

or, equivalently, that

$$\begin{aligned}&\frac{z\left(zl''_{\delta_1,\dots,\delta_n}(z)+3l'_{\delta_1,\dots,\delta_n}(z)\right)}{zl'_{\delta_1,\dots,\delta_n}(z)+2l_{\delta_1,\dots,\delta_n}(z)}+1\\&\quad=\sum_{i=1}^{n}\delta_i\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right).\end{aligned}\tag{26}$$

Thus, we get

$$\begin{aligned}&-\left(\frac{zl''_{\delta_1,\dots,\delta_n}(z)}{l'_{\delta_1,\dots,\delta_n}(z)}+1\right)-2\\&\quad=\left(-\sum_{i=1}^{n}\delta_i\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)+1\right)\\&\qquad\times\left(1+\frac{2l_{\delta_1,\dots,\delta_n}(z)}{zl'_{\delta_1,\dots,\delta_n}(z)}\right).\end{aligned}\tag{27}$$

That is,

$$\begin{aligned}&-\left(\frac{zl''_{\delta_1,\dots,\delta_n}(z)}{l'_{\delta_1,\dots,\delta_n}(z)}+1\right)\\&\quad=\left(\frac{2l_{\delta_1,\dots,\delta_n}(z)}{zl'_{\delta_1,\dots,\delta_n}(z)}\right)\\&\qquad\times\left(-\sum_{i=1}^{n}\delta_i\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)+1\right)\\&\qquad+\sum_{i=1}^{n}\delta_i\left(-\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+1\right)\right)\\&\qquad+3-\sum_{i=1}^{n}\delta_i.\end{aligned}\tag{28}$$

Taking the real parts of both terms of the last expression and using the fact that $\operatorname{Re}(z)\le|z|$ for all z, we impose

$$\begin{aligned}&-\mathfrak{Re}\left(\frac{zl''_{\delta_1,\dots,\delta_n}(z)}{l'_{\delta_1,\dots,\delta_n}(z)}+1\right)\\&\quad=\mathfrak{Re}\left\{\left(\frac{2l_{\delta_1,\dots,\delta_n}(z)}{zl'_{\delta_1,\dots,\delta_n}(z)}\right)\right.\\&\qquad\left.\times\left(-\sum_{i=1}^{n}\delta_i\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)+1\right)\right\}\\&\qquad+\sum_{i=1}^{n}\delta_i\mathfrak{Re}\left\{-\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+1\right)\right\}\\&\qquad+3-\sum_{i=1}^{n}\delta_i\\&\quad\le\left|\frac{2l_{\delta_1,\dots,\delta_n}(z)}{zl'_{\delta_1,\dots,\delta_n}(z)}\right.\\&\qquad\left.\times\left(-\sum_{i=1}^{n}\delta_i\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)+1\right)\right|\\&\qquad+\sum_{i=1}^{n}\delta_i\mathfrak{Re}\left\{-\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j,q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+1\right)\right\}\\&\qquad+3-\sum_{i=1}^{n}\delta_i.\end{aligned}\tag{29}$$

Setting

$$\begin{aligned}\xi&=\left|\frac{2l_{\delta_1,\dots,\delta_n}(z)}{zl'_{\delta_1,\dots,\delta_n}(z)}\right.\\&\qquad\left.\times\left(-\sum_{i=1}^{n}\delta_i\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)+1\right)\right|\\&\quad+\sum_{i=1}^{n}\delta_i\mathfrak{Re}\left(-\left(\frac{z\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}^n_{r,s}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+1\right)\right)\\&\quad+3-\sum_{i=1}^{n}\delta_i,\end{aligned}\tag{30}$$

and since

$$\left|\frac{2l_{\delta_1,\ldots,\delta_n}(z)}{zl'_{\delta_1,\ldots,\delta_n}(z)} \times\left(-\sum_{i=1}^{n}\delta_i\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)+1\right)\right| > 0,\quad f_i\in\Omega_k\left(\alpha_i,\beta_j,q,\eta_i\right), \tag{31}$$

we conclude that

$$\xi > \sum_{i=1}^{n}\delta_i\left(\eta_i-1\right)+3. \tag{32}$$

Hence, by (20), we have $\xi > 1$, so that $l_{\delta_1,\ldots,\delta_n}(z)\in\Omega_N(\xi)$, which evidently completes the proof of Theorem 11. □

Theorem 12. *Let $\delta_i > 0$, for $i\in\{1,2,\ldots,n\}$, $f_i\in\Omega_{r,s}\mathcal{K}(\alpha_i,\beta_j,q,\eta_i,b)$ $(0\le\eta_i<1)$ and $b\in\mathbb{C}\setminus\{0\}$. If*

$$0<\sum_{i=1}^{n}\delta_i\left(1-\eta_i\right)\le 1, \tag{33}$$

then $l_{\delta_1,\ldots,\delta_n}(z)$ is in the class $\Sigma\mathcal{G}_1(\gamma,b)$, where $\gamma = 1-\sum_{i=1}^{n}\delta_i(1-\eta_i)$.

Proof. We know from the proof of Theorem 11 that

$$\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1 =\sum_{i=1}^{n}\delta_i\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right), \tag{34}$$

which is equivalent to

$$1-\frac{1}{b}\left\{\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l(z)}+1\right\} =1-\sum_{i=1}^{n}\delta_i +\sum_{i=1}^{n}\delta_i\left\{1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right\}. \tag{35}$$

Taking the real part of both terms of (35), we get

$$\mathfrak{Re}\left\{1-\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right\} =\sum_{i=1}^{n}\delta_i\mathfrak{Re}\left\{1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right\} +1-\sum_{i=1}^{n}\delta_i. \tag{36}$$

Now, since $f_i\in\Omega_{r,s}\mathcal{K}(\alpha_i,\beta_j,q,\eta_i,b)$, we have

$$\mathfrak{Re}\left\{1-\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right\} >\sum_{i=1}^{n}\delta_i\eta_i+1-\sum_{i=1}^{n}\delta_i=1-\sum_{i=1}^{n}\delta_i\left(1-\eta_i\right). \tag{37}$$

Letting

$$\gamma=1-\sum_{i=1}^{n}\delta_i\left(1-\eta_i\right), \tag{38}$$

and in view of (33), it follows that $0\le\gamma<1$, so that $l_{\delta_1,\ldots,\delta_n}(z)\in\Omega\mathcal{G}_1(\gamma,b)$. This completes the proof of Theorem 12. □

Theorem 13. *Let $\delta_i > 0$, for $i\in\{1,2,\ldots,n\}$, $f_i\in\Omega_{r,s}\mathcal{K}\mathcal{U}(\alpha_i,\beta_j,q,\alpha,\eta,b)$ $(\alpha\ge 0,-1\le\eta<1,\alpha+\eta\ge 0)$ and $b\in\mathbb{C}\setminus\{0\}$. If*

$$\sum_{i=1}^{n}\delta_i-1\le 0, \tag{39}$$

then $l_{\delta_1,\ldots,\delta_n}(z)$ is in the class $\Omega\mathcal{G}_2(\alpha,\eta,b)$.

Proof. We want to show that

$$\mathfrak{Re}\left\{1-\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right\} -\alpha\left|\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right|-\eta > 0. \tag{40}$$

We find from (26) and (36) that

$$\begin{aligned}
&\mathfrak{Re}\left\{1-\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right\}\\
&\quad-\alpha\left|\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right|-\eta\\
&\quad=1-\sum_{i=1}^{n}\delta_i\\
&\qquad+\sum_{i=1}^{n}\delta_i\mathfrak{Re}\left\{1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right\}\\
&\qquad-\alpha\left|\sum_{i=1}^{n}\delta_i\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right|-\eta.
\end{aligned}\tag{41}$$

Since $f_i \in \Omega_{r,s}\mathscr{KU}(\alpha_i,\beta_j,q,\alpha,\delta,b)$, it follows from Definition 5 that

$$\begin{aligned}
&\mathfrak{Re}\left\{1-\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right\}\\
&\quad-\alpha\left|\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right|-\eta\\
&\quad>1-\sum_{i=1}^{n}\delta_i\\
&\qquad+\sum_{i=1}^{n}\delta_i\left\{\alpha\left|\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right|+\eta\right\}\\
&\qquad-\alpha\sum_{i=1}^{n}\delta_i\left|\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right|-\eta\\
&\quad=1-\sum_{i=1}^{n}\delta_i+\sum_{i=1}^{n}\delta_i\eta-\eta.
\end{aligned}\tag{42}$$

From hypothesis (39), we note that

$$(1-\eta)\left(1-\sum_{i=1}^{n}\delta_i\right)\geq 0.\tag{43}$$

Hence $l_{\delta_1,\ldots,\delta_n} \in \Omega\mathscr{G}_2(\alpha,\eta,b)$. Therefore, we complete the proof of Theorem 13. □

Theorem 14. *Let $\delta_i > 0$ for $i \in \{1,2,\ldots,n\}$, $f_i \in \Omega_{r,s}\mathscr{KUH}(\alpha_i,\beta_j,q;\alpha,b)$ ($\alpha>0$ and $b\in\mathbb{C}\setminus\{0\}$). If*

$$\sum_{i=1}^{n}\delta_i-1\leq 0,\tag{44}$$

then $l_{\delta_1,\ldots,\delta_n}(z)$ is in the class $\Omega\mathscr{G}_3(\alpha,b)$.

Proof. Our aim is to prove that

$$\begin{aligned}
&\mathfrak{R}\left\{\sqrt{2}\left(1-\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right)\right\}\\
&\quad+2\alpha\left(\sqrt{2}-1\right)\\
&\quad-\left|1-\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right.\\
&\qquad\left.-2\alpha\left(\sqrt{2}-1\right)\right|\\
&\quad>0.
\end{aligned}\tag{45}$$

With the aid of (26), (36) can be written as follows:

$$\begin{aligned}
&\mathfrak{R}\left\{\sqrt{2}\left(1-\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right)\right\}\\
&\quad+2\alpha\left(\sqrt{2}-1\right)\\
&\quad-\left|1-\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right.\\
&\qquad\left.-2\alpha\left(\sqrt{2}-1\right)\right|\\
&=\mathfrak{R}\left\{\sqrt{2}\left[1-\sum_{i=1}^{n}\delta_i\frac{1}{b}\right.\right.\\
&\qquad\left.\left.\times\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right]\right\}\\
&\quad+2\alpha\left(\sqrt{2}-1\right)\\
&\quad-\left|1-\sum_{i=1}^{n}\delta_i\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right.\\
&\qquad\left.-2\alpha\left(\sqrt{2}-1\right)\right|\\
&=\sqrt{2}-\sqrt{2}\sum_{i=1}^{n}\delta_i\mathfrak{R}\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\\
&\quad+2\alpha\left(\sqrt{2}-1\right)
\end{aligned}$$

$$-\left|1-\sum_{i=1}^{n}\delta_i\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right.$$

$$\left.-2\alpha\left(\sqrt{2}-1\right)\right|$$

$$=\sqrt{2}+\sqrt{2}\sum_{i=1}^{n}\delta_i\Re$$

$$\times\left\{1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right\}$$

$$-\sqrt{2}\sum_{i=1}^{n}\delta_i+2\alpha\left(\sqrt{2}-1\right)$$

$$-\left|1+\sum_{i=1}^{n}\delta_i\left[1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right.\right.$$

$$\left.-2\alpha\left(\sqrt{2}-1\right)\right]$$

$$\left.-\sum_{i=1}^{n}\delta_i+2\alpha\left(\sqrt{2}-1\right)\sum_{i=1}^{n}\delta_i-2\alpha\left(\sqrt{2}-1\right)\right|$$

$$=\sqrt{2}\left(1-\sum_{i=1}^{n}\delta_i\right)+2\alpha\left(\sqrt{2}-1\right)$$

$$+\sqrt{2}\sum_{i=1}^{n}\delta_i\Re$$

$$\times\left\{1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right\}$$

$$-\left|\left[1-2\alpha\left(\sqrt{2}-1\right)\right]\left(1-\sum_{i=1}^{n}\delta_i\right)\right.$$

$$+\sum_{i=1}^{n}\delta_i\left[1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right.$$

$$\left.\left.-2\alpha\left(\sqrt{2}-1\right)\right]\right|$$

$$\geq\sqrt{2}\left(1-\sum_{i=1}^{n}\delta_i\right)+2\alpha\left(\sqrt{2}-1\right)$$

$$+\sqrt{2}\sum_{i=1}^{n}\delta_i\Re$$

$$\times\left\{1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right\}$$

$$-\sum_{i=1}^{n}\delta_i\left|1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right.$$

$$\left.-2\alpha\left(\sqrt{2}-1\right)\right|$$

$$-\left|1-2\alpha\left(\sqrt{2}-1\right)\right|\left(1-\sum_{i=1}^{n}\delta_i\right)$$

$$=\sum_{i=1}^{n}\delta_i\left\{\mathfrak{Re}\sqrt{2}\right.$$

$$\times\left[1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right]$$

$$+2\alpha\left(\sqrt{2}-1\right)$$

$$-\left|1-\frac{1}{b}\left(\frac{z\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)''(z)}{\left(\mathcal{N}_{r,s}^{n}\left[\alpha_i,\beta_j;q\right]f_i\right)'(z)}+2\right)\right.$$

$$\left.\left.-2\alpha\left(\sqrt{2}-1\right)\right|\right\}+\sqrt{2}\left(1-\sum_{i=1}^{n}\delta_i\right)$$

$$+2\alpha\left(\sqrt{2}-1\right)-2\alpha\left(\sqrt{2}-1\right)\sum_{i=1}^{n}\delta_i$$

$$-\left|1-2\alpha\left(\sqrt{2}-1\right)\right|\left(1-\sum_{i=1}^{n}\delta_i\right). \quad (46)$$

Since $f_i \in \Omega_{r,s}\mathcal{KUH}(\alpha_i,\beta_j,q,\alpha,b)$, it follows from Definition 6 and by a simple calculation that

$$\Re\left\{\sqrt{2}\left(1-\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right)\right\}$$

$$+2\alpha\left(\sqrt{2}-1\right)$$

$$-\left|1-\frac{1}{b}\left(\frac{z\left(zl''_{\delta_1,\ldots,\delta_n}(z)+3l'_{\delta_1,\ldots,\delta_n}(z)\right)}{zl'_{\delta_1,\ldots,\delta_n}(z)+2l_{\delta_1,\ldots,\delta_n}(z)}+1\right)\right.$$

$$\left.-2\alpha\left(\sqrt{2}-1\right)\right|$$

$$> \left[\sqrt{2} + 2\alpha\left(\sqrt{2} - 1\right) - \left|1 - 2\alpha\left(\sqrt{2} - 1\right)\right|\right]\left(1 - \sum_{i=1}^{n}\delta_i\right)$$
$$> \left(1 - \sum_{i=1}^{n}\delta_i\right)\min\left\{\left(\sqrt{2} - 1\right)(1 + 4\alpha), \sqrt{2} + 1\right\}$$
$$\geq 0, \tag{47}$$

which means that $l_{\delta_1,\ldots,\delta_n} \in \Omega\mathcal{G}_3(\alpha, b)$, and we obtain the proof of the theorem. □

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

Ibtisam Aldawish and Maslina Darus read and approved the final paper.

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