

Research Article

On Best Proximity Point Theorems without Ordering

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Recently, Basha (2013) addressed a problem that amalgamates approximation and optimization in the setting of a partially ordered set that is endowed with a metric. He assumed that if A and B are nonvoid subsets of a partially ordered set that is equipped with a metric and S is a non-self-mapping from A to B , then the mapping S has an optimal approximate solution, called a best proximity point of the mapping S , to the operator equation $Sx = x$, when S is a continuous, proximally monotone, ordered proximal contraction. In this note, we are going to obtain his results by omitting ordering, proximal monotonicity, and ordered proximal contraction on S .

1. Introduction

Let S be a non-self-mapping from A to B , where A and B are nonempty subsets of a metric space X . Clearly, the set of fixed points of S can be empty. In this case, one often attempts to find an element x that is in some sense closest to $S(x)$. Best approximation theory and best proximity point analysis are applicable for solving such problems. The well-known best approximation theorem, due to Fan [1], asserts that if A is a nonempty, compact, and convex subset of a normed linear space X and S is a continuous function from A to X , then there exists a point x in A such that the distance of x to $S(x)$ is equal to the distance of $S(x)$ to A . Such a point x is called a best approximation point of S in A . A point x in A is said to be a best proximity point for S , if the distance of x to $S(x)$ is equal to the distance of A to B . The aim of best proximity point theory is to provide sufficient conditions that assure the existence of best proximity points. Investigation of several variants of contractions for the existence of a best proximity point can be found in [1–15]. In most of the papers on the best proximity, the ordering, proximal monotonicity, and ordered proximal contraction on the mapping S play a key role. A natural question arises that it is possible that we can have other ways that may not require the ordering as well as proximal monotonicity and ordered proximal contraction on the mapping S . Very recently, Basha [5] addressed a problem

that amalgamates approximation and optimization in the setting of a partially ordered set that is endowed with a metric. He assumed that if A and B are nonvoid subsets of a partially ordered set that is equipped with a metric and S is a non-self-mapping from A to B , then the mapping S has an optimal approximate solution, called a best proximity point of the mapping S , to the operator equation $Sx = x$, when S is a continuous, proximally monotone, ordered proximal contraction. In this note, we are going to obtain his results by omitting ordering, proximal monotonicity, and ordered proximal contraction on S .

2. Preliminary Results

Let X be a nonempty set and let d be a metric on X . Unless otherwise specified, it is assumed throughout this paper that A and B are nonempty subsets of X . We recollect the following notations and preliminary results:

$$d(A, B) := \inf \{d(x, y) \mid x \in A, y \in B\},$$

$$A_0 := \{x \in A \mid d(x, y) = d(A, B) \text{ for some } y \in B\}, \quad (1)$$

$$B_0 := \{y \in B \mid d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

Proposition 1. *Let A and B be two compact subsets of a metric space (X, d) . Then both A_0 and B_0 are nonempty sets.*

Proof. Suppose that both A and B are two compact subsets of a metric space (X, d) . Let $\{(a_n, b_n)\} \subseteq A \times B$ such that $d(a_n, b_n) \rightarrow d(A, B)$ as $n \rightarrow \infty$. Since A and B are compact, $A \times B$ is also compact. There exists $\{(a_{n_k}, b_{n_k})\} \subseteq \{(a_n, b_n)\}$ such that

$$(a_{n_k}, b_{n_k}) \rightarrow (a, b) \in A \times B \quad \text{as } k \rightarrow \infty. \quad (2)$$

Note that (2) is equivalent to

$$a_{n_k} \rightarrow a \in A, \quad b_{n_k} \rightarrow b \in B \quad \text{as } k \rightarrow \infty. \quad (3)$$

Let us consider

$$\begin{aligned} d(A, B) &\leq d(a, b) \\ &\leq d(a, a_{n_k}) + d(a_{n_k}, b_{n_k}) + d(b_{n_k}, b) \\ &\leq d(a, a_{n_k}) + d(A, B) + d(b_{n_k}, b). \end{aligned} \quad (4)$$

By employing (3) and letting $k \rightarrow \infty$ in (4), we obtain $d(a, b) = d(A, B)$. Hence $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$. This completes the proof. \square

Proposition 2. *Let A be a compact and let B be a closed subset of the Euclidian space \mathbb{R}^n with norm $\|\cdot\|$. Then both A_0 and B_0 are nonempty set.*

Proof. Suppose that A is compact and B is closed subset of the Euclidian space $(\mathbb{R}^n, \|\cdot\|)$. Let $\{(a_n, b_n)\} \subseteq A \times B$ such that $\|a_n - b_n\| \rightarrow d(A, B)$ as $n \rightarrow \infty$. Since A is compact, there exists $\{a_{n_k}\} \subseteq \{a_n\}$ such that

$$a_{n_k} \rightarrow a \in A \quad \text{as } k \rightarrow \infty. \quad (5)$$

Note that

$$\|b_{n_k}\| \leq \|b_{n_k} - a_{n_k}\| + \|a_{n_k}\| \leq d(A, B) + \|a_{n_k}\|, \quad (6)$$

for all $k \in \mathbb{N}$. This means that $\{b_{n_k}\}$ is bounded. It follows from the Bolzano-Weierstrass theorem and the closeness of B that there exists $\{b_{n_{k_j}}\} \subseteq \{b_{n_k}\}$ such that

$$b_{n_{k_j}} \rightarrow b \in B \quad \text{as } j \rightarrow \infty. \quad (7)$$

Let us consider

$$\begin{aligned} d(A, B) &\leq \|a - b\| \\ &\leq \|a - a_{n_{k_j}}\| + \|a_{n_{k_j}} - b_{n_{k_j}}\| + \|b_{n_{k_j}} - b\| \\ &\leq \|a - a_{n_{k_j}}\| + d(A, B) + \|b_{n_{k_j}} - b\|. \end{aligned} \quad (8)$$

By employing (5) and (7) and letting $j \rightarrow \infty$ in (8), we obtain $\|a - b\| = d(A, B)$. Hence $A_0 \neq \emptyset$ and $B_0 \neq \emptyset$. This completes the proof. \square

Proposition 3. *Let A and B be two nonempty subsets of a metric space (X, d) . Then the following are satisfied.*

- (i) *If B is compact and A is closed, then A_0 is a closed subset of X .*
- (ii) *If A is compact and B is closed, then B_0 is a closed subset of X .*
- (iii) *If both A and B are compact, then A_0 and B_0 are nonempty and closed.*

Proof. (i) It is trivial in the case of $A_0 = \emptyset$. Suppose that $A_0 \neq \emptyset$ and let $\{a_n\} \subseteq A_0$ such that

$$a_n \rightarrow a \in X \quad \text{as } n \rightarrow \infty. \quad (9)$$

Note that $A_0 \subseteq A$ and A is closed, so we have $a \in A$. Since $a_n \in A_0$, there is $b_n \in B$ such that $d(a_n, b_n) = d(A, B)$. It follows from the compactness of B that there exists $\{b_{n_k}\} \subseteq \{b_n\}$ such that

$$b_{n_k} \rightarrow b \in B \quad \text{as } k \rightarrow \infty. \quad (10)$$

Now, let $\{a_{n_k}\} \subseteq \{a_n\}$ and consider

$$\begin{aligned} d(A, B) &\leq d(a, b) \\ &\leq d(a, a_{n_k}) + d(a_{n_k}, b_{n_k}) + d(b_{n_k}, b) \\ &\leq d(a, a_{n_k}) + d(A, B) + d(b_{n_k}, b). \end{aligned} \quad (11)$$

By employing (9) and (10) and letting $k \rightarrow \infty$ in (11), we obtain $d(a, b) = d(A, B)$. This implies that $a \in A_0$ and, hence, A_0 is closed. This completes the proof.

The proof of (ii) is obvious from (i) and also the proof (iii) follows from Proposition 1 and (i) and (ii). \square

The next result extends Proposition 3.1 of [10] from normed linear spaces to metrizable topological vector spaces.

Proposition 4. *Let X be real topological vector space whose topology is induced by translation invariant metric d with the property*

$$d(tx, x) = (1 - t)d(x, 0_X), \quad \forall (0 < t < 1, x \in X), \quad (*)$$

where 0_X denotes the zero vector of X . Let A and B be two closed subsets of X such that $d(A, B) > 0$. Then

$$A_0 \subseteq bd(A), \quad B_0 \subseteq bd(B), \quad (12)$$

where $bd(A)$ and $bd(B)$ are denoted by the boundary of A and B , respectively.

Proof. Let $a \in A_0$. Then there exists $b \in B$ such that $d(a, b) = d(A, B) > 0$. It is obvious that $A = \text{int } A \cup bd(A)$. Let on the contrary $a \in \text{int } A$. Then, there is closed neighborhood of the 0_X (the zero vector) and especially positive number ε such that $a + t(b - a) \in A$, for all $t \in [0, \varepsilon]$. Let

$$t_0 = \max \{t \in [0, 1] : a + t(b - a) \in A\}. \quad (13)$$

It is clear from $A \cap B = \emptyset$ and the closeness of A and B with $b \in B$ that $0 < t_0 < 1$ and $a + t_0(b - a) \in A$. Hence, it follows from the translation invariant property and (*) that

$$\begin{aligned}
 0 < d(a, b) &= d(A, B) \\
 &\leq d(a + t_0(b - a), b) \\
 &= d(a + t_0(b - a) - a, b - a) \\
 &= d(t_0(b - a), b - a) = (1 - t_0)d((b - a), 0_X) \\
 &= (1 - t_0)d((b - a + a), 0_X + a) \\
 &= (1 - t_0)d(a, b) < d(a, b),
 \end{aligned}
 \tag{14}$$

which is a contradiction. This completes the proof. \square

The following example shows that there are metrizable topological vector spaces with the properties cited in the previous proposition which are not normable.

Example 5. Let X be a real vector space and $\mathbf{A} = \{P_n\}_{n \in \mathbb{N}}$ a countable family of seminorms on X which separates the nonzero points of X from 0_X (the zero vector of X). For each $y \in X$ and each index $n \in \mathbb{N}$, define $g_{y,n}(x) = P_n(x - y)$. Let τ be the topology on X generated by the family $\{g\}_{(y,n) \in X \times \mathbb{N}}$. One can see that (X, τ) is a topological vector space (even locally convex space). One can verify the topology τ induced by the translation invariant meter $d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \min(P_n(x - y), 1)$. Moreover, for each positive enough small number t , we have $d(tx, x) = (1 - t)d(x, 0_X)$ for each $x \in X$. However, X is not normable.

3. Main Results

In this section, we provide an existence result for the best proximity point of the mapping S on the metric space X by omitting ordering, proximal monotonicity, and ordered proximal contraction on S .

We begin with an example which shows that it is possible in the finite dimensional Euclidean space that the proximity points set for even a linear mapping (here projection) be empty.

Example 6. Let $X = \mathbb{R}^2$, $A = \{(x, 1/x) \mid x > 0\}$, and $B = \{(x, 0) \mid x \geq 0\}$. Define function $S : A \rightarrow B$ by

$$S\left(x, \frac{1}{x}\right) = (x, 0), \quad \forall \left(x, \frac{1}{x}\right) \in A. \tag{15}$$

It is clear that S is continuous (since it is projection). It is not hard to verify that

- (i) both A and B are closed subset of X ;
- (ii) $d(A, B) = 0$;
- (iii) there is no $x^* \in X$ such that $d(x^*, Sx^*) = d(A, B)$ (i.e., there is no best proximity point).

To achieve understanding in Example 6, let us see Figure 1.

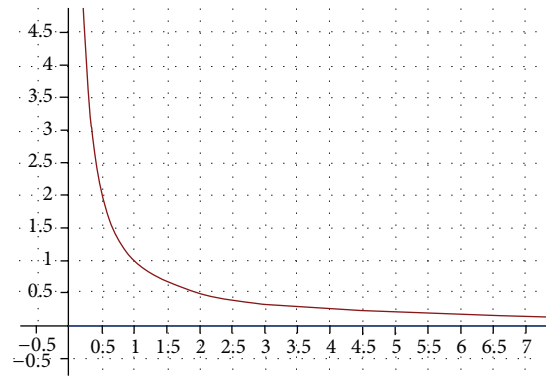


FIGURE 1

Proposition 7. Let A be a compact subset and let B be a nonvoidsubset of a metric space (X, d) . Let $S : A \rightarrow B$ be continuous with the property that there exists $\bar{B} \in \mathfrak{B}$ such that $\bar{B} \subseteq S(A)$, where

$$\mathfrak{B} = \{\{b_n\} \subseteq B \mid \exists \{a_n\} \subseteq A \text{ such that } d(a_n, b_n) \downarrow d(A, B)\}. \tag{16}$$

Then, there exists an element x^* in A such that

$$d(x^*, Sx^*) = d(A, B). \tag{17}$$

Proof. Pick $\{b_n\} = \bar{B} \in \mathfrak{B}$ such that $\{b_n\} = \bar{B} \subseteq S(A)$. Then there exists $\{a_n\} \subseteq A$ such that $d(a_n, b_n) \downarrow d(A, B)$ and $b_n = Sa_n$. Since A is compact, there exists $\{a_{n_i}\} \subseteq \{a_n\}$ such that

$$a_{n_i} \rightarrow x^* \quad \text{as } i \rightarrow \infty. \tag{18}$$

By using the continuity of S , we can conclude that

$$\begin{aligned}
 d(x^*, Sx^*) &= \lim_{i \rightarrow \infty} d(a_{n_i}, Sa_{n_i}) \\
 &= \lim_{i \rightarrow \infty} d(a_{n_i}, b_{n_i}) = d(A, B).
 \end{aligned}
 \tag{19}$$

This completes the proof. \square

The following result establishes an existence result in order to be nonempty best proximity point set for the mapping S without assuming any ordering, proximal monotonicity, and ordered proximal contraction on the S . It is worth noting that it is only an existence result without applying any iteration method (see Theorem 3.1 of [5]).

Theorem 8. Let A and B be nonvoidclosed subsets of a complete metric space (X, d) such that A_0 is nonvoidand A is totally bounded. Let $S : A \rightarrow B$ and $g : A \rightarrow A$ satisfy the following conditions:

- (a) S and g are continuous;
- (b) $S(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$.

Then, there exists an element x^* in A_0 such that

$$d(gx^*, Sx^*) = d(A, B). \tag{20}$$

Proof. Suppose that $A_0 \neq \emptyset$. Let $x_0 \in A_0$ and note that $S(A_0) \subseteq B_0$. Then, we have

$$\begin{aligned} Sx_0 \in S(A_0) \subseteq B_0 \\ = \{y \in B \mid d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned} \quad (21)$$

So, we have $Sx_0 \in B$ and there exists

$$x'_0 \in A \text{ such that } d(x'_0, Sx_0) = d(A, B). \quad (22)$$

Equation (22) indicates that $x'_0 \in A_0$. Since $A_0 \subseteq g(A_0)$, there exists $x_1 \in A_0$ such that $x'_0 = gx_1$. Thus

$$d(gx_1, Sx_0) = d(A, B). \quad (23)$$

In the next step, since $x_1 \in A_0$, we obtain $Sx_1 \in S(A_0) \subseteq B_0$. Then, we have $Sx_1 \in B$ and there exists

$$x'_1 \in A \text{ such that } d(x'_1, Sx_1) = d(A, B). \quad (24)$$

Equation (24) indicates that $x'_1 \in A_0$. Since $A_0 \subseteq g(A_0)$, there exists $x_2 \in A_0$ such that $x'_1 = gx_2$. Thus

$$d(gx_2, Sx_1) = d(A, B). \quad (25)$$

Following by this way, we can produce the sequence $\{x_n\} \subseteq A_0$ such that

$$d(gx_{n+1}, Sx_n) = d(A, B), \quad \forall n \in \mathbb{N}. \quad (26)$$

Since $A_0 \subseteq A$ and A is totally bounded, there exists a subsequence $\{x_{n_i}\} \subseteq \{x_n\}$ such that $\{x_{n_i}\}$ is a Cauchy sequence. By using the completeness of X , we have

$$x_{n_i} \longrightarrow x^* \text{ as } i \longrightarrow \infty. \quad (27)$$

Applying the continuity of S and g , we obtain

$$d(gx^*, Sx^*) = \lim_{i \rightarrow \infty} d(gx_{n_i+1}, Sx_{n_i}) = d(A, B). \quad (28)$$

This completes the proof. \square

If $g = I$ (the identity mapping), then Theorem 8 reduces to the following corollary.

Corollary 9. *Let A and B be nonvoidclosed subsets of a complete metric space (X, d) such that A_0 is nonvoid and A is totally bounded. Let $S : A \rightarrow B$ be a continuous function such that $S(A_0) \subseteq B_0$. Then, there exists an element x^* in A_0 such that*

$$d(x^*, Sx^*) = d(A, B). \quad (29)$$

If $A = B$, then $A = A_0 = B_0 = B$. Then, by Corollary 9, we obtain the following corollary which says that the fixed points set of the mapping S is nonempty.

Corollary 10. *Let A be a nonvoidclosed and totally bounded subset of a complete metric space (X, d) . Let $S : A \rightarrow A$ be continuous. Then, $F(S) \neq \emptyset$, where $F(S)$ denotes the set of all fixed points of S .*

In the following result, we are going to relax the continuity of the mappings S and g (see conditions (a) and (c) of Theorem 3.1 in [5]).

Theorem 11. *Let A be a nonvoidcompact subset and let B be a nonvoidsubset of a complete metric space (X, d) . Let $S : A \rightarrow B$ and $g : A \rightarrow A$ and define $g \otimes S : A \rightarrow A \times B$ by*

$$(g \otimes S)(x) = (gx, Sx), \quad \forall x \in A. \quad (30)$$

Suppose that $(g \otimes S)(A) = A \times B$ and $d \circ (g \otimes S)$ is lower semicontinuous where d is the distance function of the metric space (X, d) . Then, there exists an element x^ in A such that*

$$d(gx^*, Sx^*) = d(A, B). \quad (31)$$

Proof. By the assumption, we notice that

$$\begin{aligned} \{(a, b) \mid (a, b) \in A \times B\} &= A \times B \\ &= (g \otimes S)(A) \\ &= \{(g \otimes S)(\bar{a}) \mid \bar{a} \in A\}. \end{aligned} \quad (32)$$

Then, we have

$$\begin{aligned} d(A, B) &= \inf \{d(a, b) \mid (a, b) \in A \times B\} \\ &= \inf \{d((g \otimes S)(\bar{a})) \mid \bar{a} \in A\} \\ &= \inf \{(d \circ (g \otimes S))(\bar{a}) \mid \bar{a} \in A\}. \end{aligned} \quad (33)$$

By using the lower semicontinuity of $(d \circ (g \otimes S))$, we have that there exists $x^* \in A$ such that

$$\begin{aligned} d(gx^*, Sx^*) &= (d \circ (g \otimes S))(x^*) \\ &= \min \{(d \circ (g \otimes S))(\bar{a}) \mid \bar{a} \in A\} \\ &= \inf \{(d \circ (g \otimes S))(\bar{a}) \mid \bar{a} \in A\} \\ &= d(A, B). \end{aligned} \quad (34)$$

This completes the proof. \square

Corollary 12. *Let A be a nonvoidcompact subset and let B be a nonvoidsubset of a complete metric space (X, d) . Let $S : A \rightarrow B$ and $g : A \rightarrow A$ be continuous and surjective. Define $g \otimes S : A \rightarrow A \times B$ by*

$$(g \otimes S)(x) = (gx, Sx), \quad \forall x \in A. \quad (35)$$

Then, there exists an element x^ in A such that*

$$d(gx^*, Sx^*) = d(A, B). \quad (36)$$

Proof. It is obvious that the continuity and surjectivity of S and g imply the lower semicontinuity of $d \circ (g \otimes S)$, where d is the distance function of the metric space (X, d) and $(g \otimes S)(A) = A \times B$, respectively. Applying Theorem 11, we have the desired result. \square

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

- [1] K. Fan, "Extensions of two fixed point theorems of F. E. Browder," *Mathematische Zeitschrift*, vol. 112, pp. 234–240, 1969.
- [2] C. di Bari, T. Suzuki, and C. Vetro, "Best proximity points for cyclic Meir-Keeler contractions," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 11, pp. 3790–3794, 2008.
- [3] A. A. Eldred and P. Veeramani, "Existence and convergence of best proximity points," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1001–1006, 2006.
- [4] S. Karpagam and S. Agrawal, "Best proximity point theorems for p -cyclic Meir-Keeler contractions," *Fixed Point Theory and Applications*, vol. 2009, Article ID 197308, 2009.
- [5] S. S. Basha, "Best proximity point theorems on partially ordered sets," *Optimization Letters*, vol. 7, no. 5, pp. 1035–1043, 2013.
- [6] S. Sadiq Basha, "Extensions of Banach's contraction principle," *Numerical Functional Analysis and Optimization*, vol. 31, no. 4–6, pp. 569–576, 2010.
- [7] S. Sadiq Basha, "Best proximity points: global optimal approximate solutions," *Journal of Global Optimization*, vol. 49, no. 1, pp. 15–21, 2011.
- [8] V. Sankar Raj and P. Veeramani, "Best proximity pair theorems for relatively nonexpansive mappings," *Applied General Topology*, vol. 10, no. 1, pp. 21–28, 2009.
- [9] N. Shahzad, S. Sadiq Basha, and R. Jeyaraj, "Common best proximity points: global optimal solutions," *Journal of Optimization Theory and Applications*, vol. 148, no. 1, pp. 69–78, 2011.
- [10] S. Sadiq Basha and P. Veeramani, "Best proximity pair theorems for multifunctions with open fibres," *Journal of Approximation Theory*, vol. 103, no. 1, pp. 119–129, 2000.
- [11] S. S. Basha, "Common best proximity points: global minimal solutions," *TOP*, vol. 21, no. 1, pp. 182–188, 2013.
- [12] S. Sadiq Basha, "Global optimal approximate solutions," *Optimization Letters*, vol. 5, no. 4, pp. 639–645, 2011.
- [13] S. Sadiq Basha, "Best proximity point theorems generalizing the contraction principle," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 17, pp. 5844–5850, 2011.
- [14] K. Włodarczyk, R. Plebaniak, and C. Obczyński, "Convergence theorems, best approximation and best proximity for set-valued dynamic systems of relatively quasi-asymptotic contractions in cone uniform spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 72, no. 2, pp. 794–805, 2010.
- [15] K. Włodarczyk, R. Plebaniak, and A. Banach, "Best proximity points for cyclic and noncyclic set-valued relatively quasi-asymptotic contractions in uniform spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 9, pp. 3332–3341, 2009.