Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2014, Article ID 128787, 13 pages
http://dx.doi.org/10.1155/2014/128787

Research Article

Solutions of k-Hypergeometric Differential Equations

Shahid Mubeen, Mammona Naz, Abdur Rehman, and Gauhar Rahman

Department of Mathematics, University of Sargodha, Sargodha 40100, Pakistan

Correspondence should be addressed to Shahid Mubeen; smjhanda@gmail.com

Received 26 September 2013; Accepted 8 April 2014; Published 30 April 2014

Academic Editor: Bo-Qing Dong

We solve the second-order linear differential equation called the k-hypergeometric differential equation by using Frobenius method around all its regular singularities. At each singularity, we find 8 solutions corresponding to the different cases for parameters and modified our solutions accordingly.

1. Introduction

In 1769, Euler [1] formed the hypergeometric differential equation of the form

$$z(1-z)w'' + [c-(a+b+1)z]w' - abw = 0 \tag{1}$$

which has three regular singular points: 0, 1, and ∞. The hypergeometric differential equation is a prototype: every ordinary differential equation of second-order with at most three regular singular points can be brought to the hypergeometric differential equation by means of a suitable change of variables.

The solutions of hypergeometric differential equation include many of the most interesting special functions of mathematical physics. Solutions to the hypergeometric differential equation are built out of the hypergeometric series. The solution of Euler's hypergeometric differential equation is called hypergeometric function or Gaussian function ${}_2F_1$ introduced by Gauss [2].

The equation has two linearly independent solutions at each of the three regular singular points 0, 1, and ∞. Kummer [3] derived a set of 6 distinct solutions of hypergeometric differential equation. These include the hypergeometric function of Gauss and all of them could be expressed in terms of Gauss's function. For more details on Kummer's 24 solutions, see [4].

Recently, Díaz et al. [5–7] have introduced Pochhammer k-symbol. They have introduced k-gamma and k-beta functions and proved a number of their properties. They have also studied k-zeta functions and k-hypergeometric functions based on Pochhammer k-symbols for factorial functions. In 2010, Kokologiannaki [8] and Krasniqi [9] followed the work of Diaz et al. and obtained some important results for the k-beta, k-gamma, k-hypergeometric, and k-zeta functions. Also, Krasniqi [10] gave a limit for the k-gamma and k-beta functions. In 2012, Mubeen and Habibullah [11, 12] introduced a variant of fractional integrals to be called k-fractional integral which was based on k-gamma function and gave its application. They also introduced an integral representation of some generalized confluent k-hypergeometric functions and k-hypergeometric functions by using the properties of Pochhammer k-symbols, k-gamma, and k-beta functions. Furthermore, in 2013, Mubeen [13] defined a second-order linear k-hypergeometric differential equation

$$kz(1-kz)\omega'' + [c-(a+b+k)kz]\omega' - ab\omega = 0 \tag{2}$$

having one solution in the form of k-hypergeometric function ${}_2F_{1,k}$.

2. Basic Concepts

Special functions are particular mathematical functions which have more or less established names and notations due to their importance in mathematical analysis, functional analysis, physics, or other applications. The solutions of hypergeometric differential equation include many of the most interesting special functions of mathematical physics. Solutions to the hypergeometric differential equation are built out of the hypergeometric series.

Definition 1. The Pochhammer k-symbol $(a)_{n,k}$ is defined as

$$(a)_{n,k} = a(a+k)(a+2k)\cdots(a+(n-1)k) \tag{3}$$

and, for $a \neq 0$, $(a)_{0,k} = 1$, where $k > 0$.

Definition 2. The k-hypergeometric functions with three parameters a, b, c, two parameters a, b in the numerator and one parameter c in the denominator, are defined by

$${}_2F_{1,k}((a,k),(b,k);(c,k);z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}} \frac{z^n}{n!} \tag{4}$$

for all a, b, c, $c \neq 0, -1, -2, -3, \ldots, |z| < 1$, where $(a)_{n,k} = (a)(a+k)(a+2k)\cdots(a+(n-1)k)$; $(a)_{0,k} = 1$ and $k > 0$.

Definition 3. In mathematics, the method of Frobenius [14], named after Ferdinand George Frobenius, is a method to find an infinite series solution for a second-order ordinary differential equation of the form

$$P_2(z)w'' + P_1(z)w' + P_0(z)w = 0 \tag{5}$$

about the regular singular point z_0. After dividing this equation by $P_2(z)$, we obtain a differential equation of the form

$$w'' + \frac{P_1(z)}{P_2(z)}w' + \frac{P_0(z)}{P_2(z)}w = 0 \tag{6}$$

which is not solvable with regular power series methods if either $P_1(z)/P_2(z)$ or $P_0(z)/P_2(z)$ is not analytic at $z = z_0$. The Frobenius method enables us to obtain a power series solution to such a differential equation, provided that $P_1(z)$ and $P_0(z)$ are themselves analytic at z_0 or, being analytic elsewhere, both their limits at z_0 exist.

3. The Solutions of the k-Hypergeometric Differential Equation

In this section, we solve the following ordinary linear second-order k-hypergeometric differential equation defined by Mubeen [13]

$$kz(1-kz)w'' + [c - (a+b+k)kz]w' - abw = 0 \tag{7}$$

using Frobenius method. We usually use this method for complicated ordinary differential equations. This method is used to find an infinite series solution for a second-order ordinary differential equation about regular singular points of that equation. We prove that this equation has three regular singular points, namely, at $z = 0$ and $z = 1/k$ and around ∞, and then we will be able to consider a solution in the form of a series.

As this is a second-order differential equation, we must have two linearly independent solutions. The problem however will be that our assumed solutions may or may not be independent or worse may not be defined (depending on the values of the parameters of the equation). This is why we study the different cases for parameters and modify our assumed solutions accordingly.

3.1. Solution at $z = 0$. Let

$$\begin{aligned} P_0(z) &= -ab, \\ P_1(z) &= c - (a+b+k)kz, \\ P_2(z) &= kz(1-kz). \end{aligned} \tag{8}$$

Then $P_2(0) = 0$ and $P_2(1/k) = 0$.

Hence, $z = 0$ and $z = 1/k$ are singular points.

Let us start with $z = 0$.

To see if it is regular, we study the following limits:

$$\begin{gathered} \lim_{z \to z_0} \frac{(z-z_0)P_1(z)}{P_2(z)} = \lim_{z \to 0} \frac{(z-0)(c-(a+b+k)kz)}{kz(1-kz)} = \frac{c}{k}, \\ \lim_{z \to z_0} \frac{(z-z_0)^2 P_0(z)}{P_2(z)} = \lim_{z \to 0} \frac{(z-0)^2(-ab)}{kz(1-kz)} = 0. \end{gathered} \tag{9}$$

Hence, both limits exist and $z = 0$ is a regular singular point.

Therefore, we assume the solution of the form

$$w = \sum_{n=0}^{\infty} d_n z^{n+\beta} \tag{10}$$

with $d_0 \neq 0$.

Hence

$$\begin{aligned} w' &= \sum_{n=0}^{\infty} d_n(n+\beta)z^{n+\beta-1}, \\ w'' &= \sum_{n=0}^{\infty} d_n(n+\beta)(n+\beta-1)z^{n+\beta-2}. \end{aligned} \tag{11}$$

By substituting these into the k-hypergeometric differential equation (7), we get

$$\begin{aligned} & k\sum_{n=0}^{\infty} d_n(n+\beta)(n+\beta-1)z^{n+\beta-1} \\ & \quad - k^2\sum_{n=0}^{\infty} d_n(n+\beta)(n+\beta-1)z^{n+\beta} \\ & \quad + c\sum_{n=0}^{\infty} d_n(n+\beta)z^{n+\beta-1} \\ & \quad - (a+b+k)k\sum_{n=0}^{\infty} d_n(n+\beta)z^{n+\beta} \\ & \quad - ab\sum_{n=0}^{\infty} d_n z^{n+\beta} = 0. \end{aligned} \tag{12}$$

In order to simplify this equation, we need all powers of z to be the same, equal to $n+\beta-1$, the smallest power. Hence, we switch the indices as follows:

$$\begin{aligned}
&k\sum_{n=0}^{\infty} d_n (n+\beta)(n+\beta-1) z^{n+\beta-1} \\
&\quad - k^2 \sum_{n=1}^{\infty} d_{n-1} (n+\beta-1)(n+\beta-2) z^{n+\beta-1} \\
&\quad + c \sum_{n=0}^{\infty} d_n (n+\beta) z^{n+\beta-1} \\
&\quad - (a+b+k) k \sum_{n=1}^{\infty} d_{n-1} (n+\beta-1) z^{n+\beta-1} \\
&\quad - ab \sum_{n=1}^{\infty} d_{n-1} z^{n+\beta-1} = 0.
\end{aligned} \tag{13}$$

Thus isolating the first terms of the sums starting from 0, we get

$$\begin{aligned}
&d_0 (k\beta(\beta-1) + c\beta) z^{\beta-1} \\
&\quad + k\sum_{n=1}^{\infty} d_n (n+\beta)(n+\beta-1) z^{n+\beta-1} \\
&\quad - k^2 \sum_{n=1}^{\infty} d_{n-1} (n+\beta-1)(n+\beta-2) z^{n+\beta-1} \\
&\quad + c \sum_{n=1}^{\infty} d_n (n+\beta) z^{n+\beta-1} \\
&\quad - (a+b+k) k \sum_{n=1}^{\infty} d_{n-1} (n+\beta-1) z^{n+\beta-1} \\
&\quad - ab \sum_{n=1}^{\infty} d_{n-1} z^{n+\beta-1} = 0.
\end{aligned} \tag{14}$$

Now, from the linear independence of all powers of z, that is, of the functions 1, z, z^2, and so forth, the coefficients of z^r vanish for all r. Hence, from the first term, we have

$$d_0 (k\beta(\beta-1) + c\beta) = 0 \tag{15}$$

which is the indicial equation.

Since $d_0 \neq 0$, we have

$$k\beta(\beta-1) + c\beta = 0. \tag{16}$$

Hence, the solutions of the above indicial equation are given below:

$$\beta_1 = 0, \qquad \beta_2 = 1 - \frac{c}{k}. \tag{17}$$

Also, from the rest of the terms, we have

$$\begin{aligned}
&(n+\beta)[k(n+\beta-1)+c] d_n \\
&\quad = \left[k^2 (n+\beta-1)(n+\beta-2) \right. \\
&\qquad \left. + (a+b+k) k (n+\beta-1) + ab\right] d_{n-1}.
\end{aligned} \tag{18}$$

Hence, we get the following recurrence relation:

$$d_n = \frac{(a+k(n+\beta-1))(b+k(n+\beta-1))}{(n+\beta)(c+k(n+\beta-1))} d_{n-1} \tag{19}$$

for $n \geq 1$.

Let us now simplify this relation by giving d_n in terms of d_0 instead of d_{n-1}.

From the recurrence relation, we have the following:

$$d_n = \frac{(a+\beta k)_{n,k} (b+\beta k)_{n,k}}{(c+\beta k)_{n,k} (\beta+1)_n} d_0 \tag{20}$$

for $n \geq 1$.

Hence, our assumed solution takes the form

$$w = d_0 \sum_{n=0}^{\infty} \frac{(a+\beta k)_{n,k} (b+\beta k)_{n,k}}{(c+\beta k)_{n,k}} \frac{z^{n+\beta}}{(\beta+1)_n}. \tag{21}$$

3.2. Analysis of the Solutions in terms of the Difference "$(c/k) - 1$" of the Two Roots. Now, we study the solutions corresponding to the different cases for the expression $\beta_1 - \beta_2 = (c/k) - 1$ (this reduces to studying the nature of the parameter c/k whether it is an integer or not).

Case 1 ("c/k" not an integer). Let c/k be not an integer. Then

$$\begin{aligned}
w_1 &= w|_{\beta=0}, \\
w_2 &= w|_{\beta=1-(c/k)}.
\end{aligned} \tag{22}$$

Since

$$w = d_0 \sum_{n=0}^{\infty} \frac{(a+\beta k)_{n,k} (b+\beta k)_{n,k}}{(c+\beta k)_{n,k}} \frac{z^{n+\beta}}{(\beta+1)_n}, \tag{23}$$

therefore, we have

$$\begin{aligned}
w_1 &= d_0 \sum_{n=0}^{\infty} \frac{(a)_{n,k} (b)_{n,k}}{(c)_{n,k}} \frac{z^n}{(1)_n} \\
&= d_0 \, {}_2F_{1,k}((a,k),(b,k);(c,k);z), \\
w_2 &= d_0 \sum_{n=0}^{\infty} \frac{(a+(1-(c/k))k)_{n,k} (b+(1-(c/k))k)_{n,k}}{(c+(1-(c/k))k)_{n,k}} \\
&\qquad \times \frac{z^{n+1-(c/k)}}{(2-(c/k))_n} \\
&= d_0 z^{1-(c/k)} \, {}_2F_{1,k} \\
&\qquad \times ((a+k-c,k),(b+k-c,k);(2k-c,k);z).
\end{aligned} \tag{24}$$

Hence

$$w = A' w_1 + B' w_2. \tag{25}$$

Let

$$\begin{aligned}
A' d_0 &= A, \\
B' d_0 &= B.
\end{aligned} \tag{26}$$

Then

$$w = A\,{}_2F_{1,k}\left((a,k),(b,k);(c,k);z\right) + Bz^{1-(c/k)}\,{}_2F_{1,k} \times \left((a+k-c,k),(b+k-c,k);(2k-c,k);z\right). \tag{27}$$

Case 2 ("$c/k=1$" (i.e., $c=k$)). Let $c=k$. Then

$$w_1 = w|_{\beta=0}. \tag{28}$$

Since we have

$$w = d_0\sum_{n=0}^{\infty}\frac{(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{(c+\beta k)_{n,k}}\frac{z^{n+\beta}}{(\beta+1)_n}, \tag{29}$$

using $c=k$, we get

$$w = d_0\sum_{n=0}^{\infty}\frac{(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{((\beta+1)k)_{n,k}}\frac{z^{n+\beta}}{(\beta+1)_n}. \tag{30}$$

Hence,

$$\begin{aligned} w_1 &= d_0\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(k)_{n,k}}\frac{z^n}{(1)_n} \\ &= d_0\,{}_2F_{1,k}\left((a,k),(b,k);(k,k);z\right), \end{aligned} \tag{31}$$

$$w_2 = \left.\frac{\partial w}{\partial\beta}\right|_{\beta=0}. \tag{32}$$

To calculate this derivative, let

$$\begin{aligned} M_n &= \frac{(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{((\beta+1)k)_{n,k}(\beta+1)_n} \\ &= \frac{(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{k^n(\beta+1)_n^2}. \end{aligned} \tag{33}$$

Then

$$\begin{aligned} \ln(M_n) &= \ln\frac{(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{k^n(\beta+1)_n^2} \\ &= \ln(a+\beta k)_{n,k} + \ln(b+\beta k)_{n,k} - 2k^n\ln(\beta+1)_n. \end{aligned} \tag{34}$$

Since

$$\begin{aligned} &\ln(a+\beta k)_{n,k} \\ &\quad= \ln\left[(a+\beta k)(a+(\beta+1)k)\cdots(a+(\beta+n-1)k)\right] \\ &\quad= \sum_{j=0}^{n-1}\ln(a+(\beta+j)k), \end{aligned} \tag{35}$$

therefore

$$\ln(M_n) = \sum_{j=0}^{n-1}\left[\ln(a+(\beta+j)k) + \ln(b+(\beta+j)k) - 2k^n\ln(1+\beta+j)\right]. \tag{36}$$

Differentiating both sides of the equation with respect to β, we get

$$\frac{\partial M_n}{\partial\beta} = \frac{(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{k^n(\beta+1)_n^2} \times\sum_{j=0}^{n-1}\left[\frac{k}{a+(\beta+j)k} + \frac{k}{b+(\beta+j)k} - \frac{2k^n}{1+\beta+j}\right]. \tag{37}$$

Since

$$w = d_0 z^{\beta}\sum_{n=0}^{\infty}M_n z^n, \tag{38}$$

therefore

$$\frac{\partial w}{\partial\beta} = d_0 z^{\beta}\sum_{n=0}^{\infty}\frac{(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{k^n(\beta+1)_n^2} \times\left[\ln z + \sum_{j=0}^{n-1}\left(\frac{k}{a+(\beta+j)k} + \frac{k}{b+(\beta+j)k} - \frac{2k^n}{1+\beta+j}\right)\right]z^n. \tag{39}$$

For $\beta=0$, we get

$$w_2 = d_0\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(k)_{n,k}} \times\left[\ln z + \sum_{j=0}^{n-1}\left(\frac{k}{a+jk} + \frac{k}{b+jk} - \frac{2k^n}{1+j}\right)\right]\frac{z^n}{(1)_n}. \tag{40}$$

Hence,

$$w = C'w_1 + D'w_2. \tag{41}$$

Let

$$\begin{aligned} C'd_0 &= C, \\ D'd_0 &= D. \end{aligned} \tag{42}$$

Then

$$w = C\,{}_2F_{1,k}\left((a,k),(b,k);(k,k);z\right) + D\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(k)_{n,k}} \times\left[\ln z + \sum_{j=0}^{n-1}\left(\frac{k}{a+jk} + \frac{k}{b+jk} - \frac{2k^n}{1+j}\right)\right]\frac{z^n}{(1)_n}. \tag{43}$$

Case 3 ("c/k" an integer and "$c/k \neq 1$"). Here, we discuss the further two cases:

(i) "$(c/k) < 1$." Let $(c/k) < 1$. Then from the recurrence relation

$$d_n = \frac{(a+(n+\beta-1)k)(b+(n+\beta-1)k)}{(n+\beta)(c+(n+\beta-1)k)} d_{n-1}, \quad (44)$$

we see that, when $\beta = 0$ (the smaller root), $d_{1-(c/k)} \to \infty$. Thus, we must make the substitution

$$d_0 = g_0 (\beta - \beta_i) k, \quad (45)$$

where β_i is the root for which our solution is infinite.

Therefore, we take

$$d_0 = g_0 \beta k \quad (46)$$

and our assumed solution in equation (29) takes the new form

$$w_g = g_0 z^{\beta} \sum_{n=0}^{\infty} \frac{(\beta k)(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{(c+\beta k)_{n,k}(\beta+1)_n} z^n. \quad (47)$$

Then

$$w_1 = w_g \big|_{\beta=0}. \quad (48)$$

As we can see, all terms before

$$\frac{(\beta k)(a+\beta k)_{1-(c/k),k}(b+\beta k)_{1-(c/k),k}}{(\beta+1)_{1-(c/k)}(c+\beta k)_{1-(c/k),k}} z^{1-(c/k)} \quad (49)$$

vanish because of the βk in the numerator.

Starting from this term, however, the βk in the numerator vanishes. To see this, note that

$$(c+\beta k)_{1-(c/k),k} = (c+\beta k)(c+(\beta+1)k)\cdots(\beta k). \quad (50)$$

Hence, our assumed solution takes the form

$$w_1 = \frac{g_0}{(c)_{-(c/k),k}} \sum_{n=1-(c/k)}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(1)_n (k)_{n+(c/k)-1,k}} z^j. \quad (51)$$

Now

$$w_2 = \frac{\partial w_g}{\partial \beta} \bigg|_{\beta=1-(c/k)}. \quad (52)$$

To calculate this derivative, let

$$M_n = \frac{(\beta k)(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{(\beta+1)_n (c+\beta k)_{n,k}}. \quad (53)$$

Then following the method in the previous case $c/k = 1$, we get

$$\frac{\partial M_n}{\partial \beta} = M_n \left[\frac{1}{\beta} + \sum_{j=0}^{n-1} \left(\frac{k}{a+(\beta+j)k} + \frac{k}{b+(\beta+j)k} - \frac{1}{\beta+1+j} - \frac{k}{c+(\beta+j)k} \right) \right]. \quad (54)$$

Since

$$w_g = g_0 z^{\beta} \sum_{n=0}^{\infty} M_n z^n, \quad (55)$$

therefore

$$\frac{\partial w_g}{\partial \beta} = g_0 z^{\beta} \sum_{n=0}^{\infty} \frac{(\beta k)(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{(\beta+1)_n (c+\beta k)_{n,k}} \times \left[\ln z + \frac{1}{\beta} + \sum_{j=0}^{j=n-1} \left(\frac{k}{a+(\beta+j)k} + \frac{k}{b+(\beta+j)k} - \frac{1}{\beta+1+j} - \frac{k}{c+(\beta+j)k} \right) \right] z^n. \quad (56)$$

At $\beta = 1 - (c/k)$, we get

$$w_2 = g_0 (k-c) z^{1-(c/k)} \times \sum_{n=0}^{\infty} \frac{(a+k-c)_{n,k}(b+k-c)_{n,k}}{(2k-c)_{n,k}} \times \left[\ln z + \frac{k}{k-c} + k \sum_{j=0}^{n-1} \left(\frac{1}{(a+k-c)+jk} + \frac{1}{(b+k-c)+jk} - \frac{1}{(2k-c)+jk} - \frac{1}{(1+j)k} \right) \right] \frac{z^n}{n!}. \quad (57)$$

Hence

$$w = E' w_1 + F' w_2. \quad (58)$$

Let

$$E' g_0 = E, \qquad F' g_0 = F. \quad (59)$$

Then

$$w = \frac{E}{(c)_{-(c/k),k}} \sum_{j=1-(c/k)}^{\infty} \frac{(a)_{j,k}(b)_{j,k}}{(k)_{j+(c/k)-1,k}} \frac{z^j}{(1)_j} + F(k-c) z^{1-(c/k)} \times \sum_{n=0}^{\infty} \frac{(a+k-c)_{n,k}(b+k-c)_{n,k}}{(2k-c)_{n,k}} \times \left[\ln z + \frac{k}{k-c} + k \sum_{j=0}^{n-1} \left(\frac{1}{(a+k-c)+jk} + \frac{1}{(b+k-c)+jk} - \frac{1}{(2k-c)+jk} - \frac{1}{(1+j)k} \right) \right] \frac{z^n}{n!}. \quad (60)$$

(ii) "$(c/k) > 1$". Let $(c/k) > 1$. Then, from the recurrence relation

$$d_n = \frac{(a+(\beta+n-1)k)(b+(\beta+n-1)k)}{(n+\beta)(c+(\beta+n-1)k)} d_{n-1}, \quad (61)$$

we see that, when $\beta = 1-(c/k)$ (the smaller root), $d_{(c/k)-1} \to \infty$. Thus, we must make the substitution

$$d_0 = g_0(\beta - \beta_i)k, \quad (62)$$

where β_i is the root for which our solution is infinite.

Hence we take

$$d_0 = g_0 \left(\beta + \frac{c}{k} - 1 \right) k \quad (63)$$

and our assumed solution takes the new form

$$w_g = g_0 z^{\beta} \sum_{n=0}^{\infty} \frac{((\beta+(c/k)-1)k)(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{(\beta+1)_n (c+\beta k)_{n,k}} z^n. \quad (64)$$

Then

$$w_1 = w_g \big|_{\beta=1-(c/k)}. \quad (65)$$

As we can see, all terms before

$$\frac{((\beta+(c/k)-1)k)(a+\beta k)_{(c/k)-1,k}(b+\beta k)_{(c/k)-1,k}}{(\beta+1)_{(c/k)-1}(c+\beta k)_{(c/k)-1,k}} z^{(c/k)-1} \quad (66)$$

vanish because of the "$\beta + (c/k) - 1$" in the numerator.

Starting from this term, however, the "$\beta + (c/k) - 1$" in the numerator vanishes. To see this, note that

$$(\beta+1)_{(c/k)-1} = (\beta+1)(\beta+2)\cdots\left(\beta + \frac{c}{k} - 1\right). \quad (67)$$

Hence, our solution takes the form

$$w_1 = \frac{g_0 z^{1-(c/k)}}{(2k-c)_{(c/k)-2,k}} \sum_{n=(c/k)-1}^{\infty} \frac{(a+k-c)_{n,k}(b+k-c)_{n,k}}{(k)_{n+1-(c/k),k}} \frac{z^n}{(1)_n}. \quad (68)$$

Now

$$w_2 = \frac{\partial w_g}{\partial \beta} \bigg|_{\beta=0}. \quad (69)$$

To calculate this derivative, let

$$M_n = \frac{((\beta+(c/k)-1)k)(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{(\beta+1)_n (c+\beta k)_{n,k}}. \quad (70)$$

Then following the method in the previous case $(c/k) < 1$, we get

$$\frac{\partial M_n}{\partial \beta} = M_n \left[\frac{1}{\beta+(c/k)-1} + \sum_{j=0}^{n-1} \left(\frac{k}{a+(\beta+j)k} + \frac{k}{b+(\beta+j)k} - \frac{1}{\beta+1+j} - \frac{k}{c+(\beta+j)k} \right) \right],$$

$$\frac{\partial w_g}{\partial \beta} = g_0 z^{\beta} \sum_{n=0}^{\infty} \frac{((\beta+(c/k)-1)k)(a+\beta k)_{n,k}(b+\beta k)_{n,k}}{(\beta+1)_n (c+\beta k)_{n,k}} \times \left[\ln z + \frac{1}{\beta+(c/k)-1} + \sum_{j=0}^{n-1} \left(\frac{k}{a+(\beta+j)k} + \frac{k}{b+(\beta+j)k} - \frac{1}{\beta+1+j} - \frac{k}{c+(\beta+j)k} \right) \right] z^n. \quad (71)$$

At $\beta = 0$, we get

$$w_2 = g_0(c-k) \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}} \times \left[\ln z + \frac{k}{c-k} + \sum_{j=0}^{n-1} \left(\frac{k}{a+jk} + \frac{k}{b+jk} - \frac{1}{1+j} - \frac{k}{c+jk} \right) \right] \frac{z^n}{n!}. \quad (72)$$

Hence

$$w = G' w_1 + H' w_2. \tag{73}$$

Let

$$\begin{aligned} G' g_0 &= G, \\ H' g_0 &= H. \end{aligned} \tag{74}$$

Then

$$\begin{aligned} w = {} & \frac{G}{(2k-c)_{(c/k)-2,k}} z^{1-(c/k)} \\ & \times \sum_{j=(c/k)-1}^{\infty} \frac{(a+k-c)_{j,k}(b+k-c)_{j,k}}{(k)_{j+1-(c/k),k}} \\ & \times \frac{z^j}{(1)_j} + H(c-k) \\ & \times \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}} \\ & \times \left[\ln z + \frac{k}{c-k} \right. \\ & + \sum_{j=0}^{n-1} \left(\frac{k}{a+jk} + \frac{k}{b+jk} \right. \\ & \left. \left. - \frac{1}{1+j} - \frac{k}{c+jk} \right) \right] \frac{z^n}{n!}. \end{aligned} \tag{75}$$

3.3. Solution at z=1/k. Let us now study the singular point $z = 1/k$.

To see if it is regular, we study the following limits:

$$\begin{aligned} \lim_{z \to z_0} \frac{(z - z_0) P_1(z)}{P_2(z)} &= \lim_{z \to 1/k} \frac{(z - (1/k))(c - (a+b+k)kz)}{kz(1-kz)} \\ &= \frac{c - (a+b+k)}{k}, \end{aligned} \tag{76}$$

$$\lim_{z \to z_0} \frac{(z - z_0)^2 P_0(z)}{P_2(z)} = \lim_{z \to 1/k} \frac{(z - (1/k))^2 (-ab)}{kz(1-kz)} = 0.$$

As both limits exist, therefore $z = 1/k$ is a regular singular point.

Now instead of assuming a solution in the form

$$w = \sum_{n=0}^{\infty} d_n \left(z - \frac{1}{k} \right)^{n+\beta}, \tag{77}$$

we will try to express the solutions of this case in terms of the solutions for the point $z = 0$. We proceed as follows.

We have the k-hypergeometric differential equation of the form

$$kz(1-kz) w'' + [c - (a+b+k)kz] w' - abw = 0. \tag{78}$$

Let $y = (1/k) - z$. Then

$$\begin{aligned} \frac{dw}{dz} &= \frac{dw}{dy}\frac{dy}{dz} = -\frac{dw}{dy} = -\dot{w}, \\ \frac{d^2w}{dz^2} &= \frac{d}{dz}\left(\frac{dw}{dz}\right) = \frac{d}{dz}\left(-\frac{dw}{dy}\right) = \frac{d}{dy}\left(-\frac{dw}{dy}\right)\frac{dy}{dz} \\ &= \frac{d^2w}{dy^2} = \ddot{w}. \end{aligned} \tag{79}$$

Hence, k-hypergeometric differential equation (7) takes the form

$$\begin{aligned} & ky(1-ky)\ddot{w} + [a+b-c+k-(a+b+k)ky]\dot{w} \\ & \quad - abw = 0. \end{aligned} \tag{80}$$

Since $y = (1/k) - z$, the solution of the k-hypergeometric differential equation at $z = 1/k$ is the same as the solution for this equation at $y = 0$. But the solution at $y = 0$ is identical to the solution we obtained for the point $z = 0$ if we replace c by $a + b - c + k$.

Hence, to get the solutions, we just make the substitution in the previous results.

Note also that, for $z = 0$,

$$\begin{aligned} \beta_1 &= 0, \\ \beta_2 &= 1 - \frac{c}{k}. \end{aligned} \tag{81}$$

Hence in our case,

$$\begin{aligned} \beta_1 &= 0, \\ \beta_2 &= \frac{c-a-b}{k}. \end{aligned} \tag{82}$$

3.4. Analysis of the Solutions in terms of the Difference "(c–a–b)/k" of the Two Roots. Let us now find out the solutions. In the following we replace each y by $(1/k) - z$.

Case 1 ("$(c-a-b)/k$" not an integer). Let $(c-a-b)/k$ be not an integer. Then

$$\begin{aligned} w = {} & A\,{}_2F_{1,k}\left((a,k),(b,k);(a+b-c+k,k);\frac{1}{k}-z\right) \\ & + B\left(\frac{1}{k}-z\right)^{(c-a-b)/k} \\ & \times {}_2F_{1,k}\left((c-a,k),(c-b,k);(c-a-b+k,k);\frac{1}{k}-z\right). \end{aligned} \tag{83}$$

Case 2 (“$(c-a-b)/k=0$”). Let $(c-a-b)/k=0$. Then

$$w = C\,{}_2F_{1,k}\left((a,k),(b,k);(k,k);\frac{1}{k}-z\right) + D\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(k)_{n,k}} \times\left[\ln\left(\frac{1}{k}-z\right) +\sum_{j=0}^{n-1}\left(\frac{k}{a+jk}+\frac{k}{b+jk}-\frac{2k^n}{1+j}\right)\right] \times\frac{((1/k)-z)^n}{n!}. \tag{84}$$

Case 3 (“$(c-a-b)/k$” an integer and “$(c-a-b)/k\neq 0$”). Here, we discuss further two cases:

(i) “$(c-a-b)/k>0$.” Let $(c-a-b)/k>0$. Then

$$w = \frac{E}{(a+b-c+k)_{(c-a-b)/k,k}} \times\sum_{n=(c-a-b)/k}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(k)_{n+((a+b-c)/k),k}}\frac{((1/k)-z)^n}{(1)_n} + F(c-a-b)\times\left(\frac{1}{k}-z\right)^{(c-a-b)/k} \times\sum_{n=0}^{\infty}\frac{(c-a)_{n,k}(c-b)_{n,k}}{(c-a-b+k)_{n,k}} \times\left[\ln\left(\frac{1}{k}-z\right)+\frac{k}{c-a-b} +k\sum_{j=0}^{n-1}\left(\frac{1}{(c-a)+jk}+\frac{1}{(c-b)+jk} -\frac{1}{(c-a-b+k)+jk} -\frac{1}{(1+j)k}\right)\right]\frac{((1/k)-z)^n}{n!}. \tag{85}$$

(ii) “$(c-a-b)/k<0$.” Let $(c-a-b)/k<0$. Then

$$w = \frac{G}{(c-a-b+k)_{((a+b-c)/k)-1,k}}\left(\frac{1}{k}-z\right)^{(c-a-b)/k} \times\sum_{n=((a+b-c)/k)}^{\infty}\frac{(c-a)_{n,k}(c-b)_{n,k}}{(k)_{n+((c-a-b)/k),k}} \times\frac{((1/k)-z)^n}{(1)_n} + H(a+b-c) \times\sum_{n=0}^{\infty}\frac{(a)_{n,k}(b)_{n,k}}{(a+b-c+k)_{n,k}} \times\left[\ln\left(\frac{1}{k}-z\right)+\frac{k}{a+b-c} +\sum_{j=0}^{n-1}\left(\frac{k}{a+jk}+\frac{k}{b+jk} -\frac{1}{1+j}-\frac{k}{(a+b-c+k)+jk}\right)\right] \times\frac{((1/k)-z)^n}{n!}. \tag{86}$$

3.5. Solution Around “∞”. Finally we study the singularity as $z\to\infty$. Since we cannot study this directly, therefore we let $z=1/ks$; then the solution of the equation as $z\to\infty$ is identical to the solution of the modified equation when $s=0$.

We have the k-hypergeometric differential equation

$$kz(1-kz)w'' + [c-(a+b+k)kz]w' - abw = 0,$$
$$\frac{dw}{dz}=\frac{dw}{ds}\frac{ds}{dz}=-ks^2\frac{dw}{ds}=-ks^2\dot{w}, \tag{87}$$
$$\frac{d^2w}{dz^2}=k^2\left(2s^3\dot{w}+s^4\ddot{w}\right).$$

Hence the k-hypergeometric differential equation (7) takes the new form

$$k^2\left(s^3-s^2\right)\ddot{w}+k\left[(2k-c)s^2+(a+b-k)s\right]\dot{w}-abw=0. \tag{88}$$

Let

$$P_0(s) = -ab$$
$$P_1(s) = k\left[(2k-c)s^2+(a+b-k)s\right], \tag{89}$$
$$P_2(s) = k^2\left(s^3-s^2\right).$$

Here, we only study the solutions when $s=0$. As we can see that this is a singular point because $P_2(0)=0$.

Let us now see that $s = 0$ is a regular singular point, for this

$$\lim_{s\to s_0}\frac{(s-s_0)P_1(s)}{P_2(s)} = \lim_{s\to 0}\frac{(s-0)k\left[(2k-c)s^2+(a+b-k)s\right]}{k^2(s^3-s^2)} = \frac{k-a-b}{k},$$
$$\lim_{s\to s_0}\frac{(s-s_0)^2P_0(s)}{P_2(s)} = \lim_{s\to 0}\frac{(s-0)^2(-ab)}{k^2(s^3-s^2)} = ab. \tag{90}$$

As both limits exist, therefore $s = 0$ is a regular singular point.

Thus, we assume the solution of the form

$$w = \sum_{n=0}^{\infty} d_n s^{n+\beta} \tag{91}$$

with $d_0 \neq 0$.

Hence, we set the following:

$$w^{\cdot} = \sum_{n=0}^{\infty} d_n (n+\beta) s^{n+\beta-1},$$
$$w^{\cdot\cdot} = \sum_{n=0}^{\infty} d_n (n+\beta)(n+\beta-1) s^{n+\beta-2}. \tag{92}$$

By substituting these into the modified k-hypergeometric differential equation (88), we get

$$k^2\sum_{n=0}^{\infty} d_n (n+\beta)(n+\beta-1)s^{n+\beta+1} - k^2\sum_{n=0}^{\infty} d_n (n+\beta)(n+\beta-1)s^{n+\beta} + k(2k-c)\sum_{n=0}^{\infty} d_n (n+\beta)s^{n+\beta+1} + k(a+b-k)\sum_{n=0}^{\infty} d_n (n+\beta)s^{n+\beta} - ab\sum_{n=0}^{\infty} d_n s^{n+\beta} = 0. \tag{93}$$

In order to simplify this equation, we need all powers of s to be the same, equal to $n + \beta$, the smallest power. Hence, we switch the indices as follows:

$$k^2\sum_{n=1}^{\infty} d_{n-1}(n+\beta-1)(n+\beta-2)s^{n+\beta} - k^2\sum_{n=0}^{\infty} d_n (n+\beta)(n+\beta-1)s^{n+\beta} + k(2k-c)\sum_{n=1}^{\infty} d_{n-1}(n+\beta-1)s^{n+\beta} + k(a+b-k)\sum_{n=0}^{\infty} d_n (n+\beta)s^{n+\beta} - ab\sum_{n=0}^{\infty} d_n s^{n+\beta} = 0. \tag{94}$$

Thus, by isolating the first terms of the sums starting from 0, we get

$$d_0\left[-k^2\beta(\beta-1)+k(a+b-k)\beta-ab\right]s^{\beta} + k^2\sum_{n=1}^{\infty} d_{n-1}(n+\beta-1)(n+\beta-2)s^{n+\beta} - k^2\sum_{n=1}^{\infty} d_n (n+\beta)(n+\beta-1)s^{n+\beta} + k(2k-c)\sum_{n=1}^{\infty} d_{n-1}(n+\beta-1)s^{n+\beta} + k(a+b-k)\sum_{n=1}^{\infty} d_n (n+\beta)s^{n+\beta} - ab\sum_{n=1}^{\infty} d_n s^{n+\beta} = 0. \tag{95}$$

Now from the linear independence of all powers of s, that is, of the functions 1, s, s^2, and so forth, the coefficients of s^r vanish for all r. Hence, from the first term, we have

$$d_0\left[-k^2\beta(\beta-1)+k(a+b-k)\beta-ab\right] = 0 \tag{96}$$

which is the indicial equation.

Since $d_0 \neq 0$, we have

$$-k^2\beta(\beta-1)+k(a+b-k)\beta-ab = 0. \tag{97}$$

Hence, we get two solutions of this indicial equation:

$$\beta_1 = \frac{a}{k},$$
$$\beta_2 = \frac{b}{k}. \tag{98}$$

Also, from the rest of the terms we have

$$\begin{aligned}&\left[k^2 (n+\beta-1)(n+\beta-2)\right.\\&\quad\left.+k(2k-c)(n+\beta-1)\right]d_{n-1}\\&\quad+\left[-k^2(n+\beta)(n+\beta-1)\right.\\&\qquad\left.+k(a+b-k)(n+\beta)-ab\right]d_n=0.\end{aligned} \tag{99}$$

Hence, we get the recurrence relation of the following form:

$$d_n=\frac{(k(n+\beta)-c)\,k(n+\beta-1)}{(k(n+\beta)-a)(k(n+\beta)-b)}d_{n-1} \tag{100}$$

for $n \geq 1$.

Let us now simplify this relation by giving d_n in terms of d_0 instead of d_{n-1}.

From the recurrence relation, we have

$$d_n=\frac{(k\beta)_{n,k}(k(\beta+1)-c)_{n,k}}{(k(\beta+1)-a)_{n,k}(k(\beta+1)-b)_{n,k}}d_0 \tag{101}$$

for $n \geq 1$.

Hence, our assumed solution in equation (91) takes the form

$$w=d_0\sum_{n=0}^{\infty}\frac{(k\beta)_{n,k}(k(\beta+1)-c)_{n,k}}{(k(\beta+1)-a)_{n,k}(k(\beta+1)-b)_{n,k}}s^{n+\beta}. \tag{102}$$

3.6. Analysis of the Solutions in terms of the Difference "(a/k)–(b/k)" of the Two Roots. Now, we study the solutions corresponding to the different cases for the expression $\beta_1-\beta_2=(a/k)-(b/k)$ (this reduces to studying the nature of the parameter $(a/k)-(b/k)$ whether it is an integer or not).

Case 1 ("$(a/k)-(b/k)$" not an integer). Let $(a/k)-(b/k)$ be not an integer. Then

$$\begin{aligned}w_1&=w|_{\beta=a/k},\\w_2&=w|_{\beta=b/k}.\end{aligned} \tag{103}$$

Since

$$w=d_0\sum_{n=0}^{\infty}\frac{(k\beta)_{n,k}(k(\beta+1)-c)_{n,k}}{(k(\beta+1)-a)_{n,k}(k(\beta+1)-b)_{n,k}}s^{n+\beta}, \tag{104}$$

therefore, we have

$$\begin{aligned}w_1&=d_0(kz)^{-(a/k)}\sum_{n=0}^{\infty}\frac{(a)_{n,k}(a+k-c)_{n,k}}{(a+k-b)_{n,k}}\frac{1}{(k^2z)^n(1)_n}\\&=d_0(kz)^{-(a/k)}\\&\quad\times{}_2F_{1,k}\left((a,k),(a+k-c,k);(a+k-b,k);\frac{1}{k^2z}\right),\\w_2&=d_0(kz)^{-(b/k)}\\&\quad\times{}_2F_{1,k}\left((b,k),(b+k-c,k);(b+k-a,k);\frac{1}{k^2z}\right).\end{aligned} \tag{105}$$

Hence

$$w=A'w_1+B'w_2. \tag{106}$$

Let

$$\begin{aligned}A'd_0&=A,\\B'd_0&=B.\end{aligned} \tag{107}$$

Then

$$\begin{aligned}w&=A(kz)^{-(a/k)}\\&\quad\times{}_2F_{1,k}\left((a,k),(a+k-c,k);(a+k-b,k);\frac{1}{k^2z}\right)\\&\quad+B(kz)^{-(b/k)}\\&\quad\times{}_2F_{1,k}\left((b,k),(b+k-c,k);(b+k-a,k);\frac{1}{k^2z}\right).\end{aligned} \tag{108}$$

Case 2 ("$(a/k)-(b/k)=0$"). Let $(a/k)-(b/k)=0$. Then

$$w_1=w|_{\beta=a/k}. \tag{109}$$

Also, we have

$$w=d_0\sum_{n=0}^{\infty}\frac{(k\beta)_{n,k}(k(\beta+1)-c)_{n,k}}{(k(\beta+1)-a)_{n,k}(k(\beta+1)-b)_{n,k}}s^{n+\beta} \tag{110}$$

which can be written as

$$w=d_0\sum_{n=0}^{\infty}\frac{(\beta)_n(\beta+1-(c/k))_n}{(\beta+1-(a/k))_n(\beta+1-(b/k))_n}s^{n+\beta}. \tag{111}$$

Since we have $(a/k)=(b/k)$, therefore

$$w_1=d_0\sum_{n=0}^{\infty}\frac{(\beta)_n(\beta+1-(c/k))_n}{(\beta+1-(a/k))_n^2}s^{n+\beta}. \tag{112}$$

At $\beta=a/k$,

$$\begin{aligned}w_1&=d_0(kz)^{-(a/k)}\\&\quad\times{}_2F_{1,k}\left((a,k),(a+k-c,k);(k,k);\frac{1}{k^2z}\right),\\w_2&=\left.\frac{\partial w}{\partial\beta}\right|_{\beta=a/k}.\end{aligned} \tag{113}$$

To calculate this derivative, let

$$M_n=\frac{(\beta)_n(\beta+1-(c/k))_n}{(\beta+1-(a/k))_n^2}, \tag{114}$$

and then we get

$$\frac{\partial M_n}{\partial \beta} = \frac{(\beta)_n(\beta+1-(c/k))_n}{(\beta+1-(a/k))_n^2} \times \sum_{j=0}^{n-1}\left[\frac{1}{\beta+j}+\frac{1}{\beta+1-(c/k)+j} - \frac{2}{\beta+1-(a/k)+j}\right],$$

$$\frac{\partial w}{\partial \beta} = d_0 s^{\beta}\sum_{n=0}^{\infty}\frac{(\beta)_n(\beta+1-(c/k))_n}{(\beta+1-(a/k))_n^2} \times \left[\ln s + \sum_{j=0}^{n-1}\left(\frac{1}{\beta+j}+\frac{1}{\beta+1-(c/k)+j} - \frac{2}{\beta+1-(a/k)+j}\right)\right]s^n. \tag{115}$$

For $\beta = a/k$, we get

$$w_2 = d_0(kz)^{-(a/k)} \times \sum_{n=0}^{\infty}\frac{(a)_{n,k}(a+k-c)_{n,k}}{k^n(k)_{n,k}} \times \left[\ln (kz)^{-1} + \sum_{j=0}^{n-1}\left(\frac{k}{a+jk}+\frac{k}{a+k-c+jk}-\frac{2}{1+j}\right)\right] \times \frac{(kz)^{-n}}{n!}. \tag{116}$$

Hence

$$w = C'w_1 + D'w_2. \tag{117}$$

Let

$$C'd_0 = C, \quad D'd_0 = D. \tag{118}$$

Then

$$w = C(kz)^{-(a/k)}\sum_{n=0}^{\infty}\frac{(a)_{n,k}(a+k-c)_{n,k}}{k^n(k)_{n,k}}\frac{(kz)^{-n}}{n!} + D(kz)^{-(a/k)} \times \sum_{n=0}^{\infty}\frac{(a)_{n,k}(a+k-c)_{n,k}}{k^n(k)_{n,k}} \times \left[\ln (kz)^{-1} + \sum_{j=0}^{n-1}\left(\frac{k}{a+jk}+\frac{k}{a+k-c+jk}-\frac{2}{1+j}\right)\right] \times \frac{(kz)^{-n}}{n!}. \tag{119}$$

Case 3 ("$(a/k)-(b/k)$" an integer and "$(a/k)-(b/k)\neq 0$"). Here, we have further two cases:

(i) "$(a/k)-(b/k)>0$." Let $(a/k)-(b/k)>0$.

Then, from the recurrence relation

$$d_n = \frac{(k(n+\beta)-c)(k(n+\beta-1))}{(k(n+\beta)-a)(k(n+\beta)-b)}d_{n-1}, \tag{120}$$

we see that, when $\beta = b/k$ (the smaller root), $d_{(a/k)-(b/k)} \to \infty$. Thus, we must make the substitution

$$d_0 = g_0(\beta-\beta_i)k, \tag{121}$$

where β_i is the root for which our solution is infinite.

Hence we take

$$d_0 = g_0\left(\beta-\frac{b}{k}\right)k \tag{122}$$

and our assumed solution in equation (110) takes the new form

$$w_g = g_0 s^{\beta}\sum_{n=0}^{\infty}\frac{((\beta-(b/k))k)(k(\beta+1)-c)_{n,k}(\beta k)_{n,k}}{(k(\beta+1)-a)_{n,k}(k(\beta+1)-b)_{n,k}}s^n. \tag{123}$$

Then

$$w_1 = w_g\big|_{\beta=b/k}. \tag{124}$$

As we can see, all terms before

$$\frac{((\beta-(b/k))k)(\beta k)_{(a/k)-(b/k),k}(k(\beta+1)-c)_{(a/k)-(b/k),k}}{(k(\beta+1)-a)_{(a/k)-(b/k),k}(k(\beta+1)-b)_{(a/k)-(b/k),k}} \times s^{(a/k)-(b/k)} \tag{125}$$

vanish because of the $(\beta-(b/k))$ in the numerator.

Starting from this term, however, the $(\beta-(b/k))$ in the numerator vanishes. To see this, note that

$$\left(\beta+1-\frac{a}{k}\right)_{(a/k)-(b/k)} = \left(\beta+1-\frac{a}{k}\right)\left(\beta+2-\frac{a}{k}\right)\cdots\left(\beta-\frac{b}{k}\right). \tag{126}$$

Hence, our solution takes the form

$$\begin{aligned} w_1 &= \frac{g_0}{(b+k-a)_{(a/k)-(b/k)-1,k}} \times \sum_{n=(a/k)-(b/k)}^{\infty} \frac{(b)_{n,k}(b+k-c)_{n,k}}{k^{n+(b/k)-(a/k)}(1)_{n+(b/k)-(a/k)}} \frac{s^n}{(1)_n} \\ &= \frac{g_0}{(b+k-a)_{(a/k)-(b/k)-1,k}} \times \sum_{n=(a/k)-(b/k)}^{\infty} \frac{(b)_{n,k}(b+k-c)_{n,k}}{(k)_{n+(b/k)-(a/k),k}} \frac{(kz)^{-n}}{(1)_n}. \end{aligned} \tag{127}$$

Now

$$w_2 = \left.\frac{\partial w_g}{\partial \beta}\right|_{\beta=a/k}. \tag{128}$$

To calculate this derivative, let

$$M_n = \frac{((\beta-(b/k))k)(\beta)_n(\beta+1-(c/k))_n}{(\beta+1-(a/k))_n(\beta+1-(b/k))_n}, \tag{129}$$

and then we get

$$\begin{aligned} \frac{\partial M_n}{\partial \beta} &= M_n\left[\frac{1}{\beta-(b/k)} + \sum_{j=0}^{n-1}\left(\frac{1}{\beta+1-(c/k)+j}+\frac{1}{\beta+j} - \frac{1}{\beta+1-(a/k)+j} - \frac{1}{\beta+1-(b/k)+j}\right)\right], \\ \frac{\partial w_g}{\partial \beta} &= g_0 s^{\beta}\sum_{n=0}^{\infty}\frac{(\beta-(b/k))k(\beta)_n(\beta+1-(c/k))_n}{(\beta+1-(a/k))_n(\beta+1-(b/k))_n} \times\left[\ln s+\frac{1}{\beta-(b/k)} + \sum_{j=0}^{n-1}\left(\frac{1}{\beta+j}+\frac{1}{\beta+1-(c/k)+j} - \frac{1}{\beta+1-(a/k)+j} - \frac{1}{\beta+1-(b/k)+j}\right)\right]s^n. \end{aligned} \tag{130}$$

At $\beta = a/k$, we get

$$w_2 = g_0 s^{a/k}\sum_{n=0}^{\infty}\frac{(a-b)(a)_{n,k}(a+k-c)_{n,k}}{(a+k-b)_{n,k}(k)_{n,k}} \times\left[\ln s+\frac{k}{a-b} + \sum_{j=0}^{n-1}\left(\frac{k}{a+jk}+\frac{k}{a+k-c+jk} - \frac{k}{a+k-b+jk} - \frac{1}{1+j}\right)\right]s^n. \tag{131}$$

Replacing s by $(kz)^{-1}$, we get w_2 in terms of z:

$$w_2 = g_0(kz)^{-(a/k)}\sum_{n=0}^{\infty}\frac{(a-b)(a)_{n,k}(a+k-c)_{n,k}}{(a+k-b)_{n,k}} \times\left[\ln(kz)^{-1}+\frac{k}{a-b} + \sum_{j=0}^{n-1}\left(\frac{k}{a+jk}+\frac{k}{a+k-c+jk} - \frac{1}{1+j} - \frac{k}{a+k-b+jk}\right)\right] \times\frac{(kz)^{-n}}{(k)_{n,k}}. \tag{132}$$

Let

$$w = E' w_1 + F' w_2. \tag{133}$$

Let

$$\begin{aligned} E' g_0 &= E, \\ F' g_0 &= F. \end{aligned} \tag{134}$$

Then

$$\begin{aligned} w &= \frac{E}{(b+k-a)_{(a/k)-(b/k)-1,k}} \times \sum_{n=(a/k)-(b/k)}^{\infty}\frac{(b)_{n,k}(b+k-c)_{n,k}}{(k)_{n+(b/k)-(a/k),k}}\frac{(kz)^{-n}}{(1)_n} \\ &\quad + F(kz)^{-(a/k)}\sum_{n=0}^{\infty}\frac{(a-b)(a)_{n,k}(a+k-c)_{n,k}}{(a+k-b)_{n,k}} \times\left[\ln(kz)^{-1}+\frac{k}{a-b} + \sum_{j=0}^{n-1}\left(\frac{k}{a+jk}+\frac{k}{a+k-c+jk} - \frac{1}{1+j} - \frac{k}{a+k-b+jk}\right)\right] \times\frac{(kz)^{-n}}{(k)_{n,k}}. \end{aligned} \tag{135}$$

(ii) "$(a/k) - (b/k) < 0$." Let $(a/k) - (b/k) < 0$.

Then, from the symmetry of the situation here, we see that

$$w = \frac{G}{(a+k-b)_{(b/k)-(a/k)-1,k}} \times \sum_{n=(b/k)-(a/k)}^{\infty} \frac{(a)_{n,k}(a+k-c)_{n,k}}{(k)_{n+(a/k)-(b/k),k}} \frac{(kz)^{-n}}{(1)_n} + H(kz)^{-(b/k)} \sum_{n=0}^{\infty} \frac{(b-a)\,(b)_{n,k}(b+k-c)_{n,k}}{(b+k-a)_{n,k}} \times \left[\ln (kz)^{-1} + \frac{k}{b-a} + \sum_{j=0}^{n-1} \left(\frac{k}{b+jk} + \frac{k}{b+k-c+jk} - \frac{1}{1+j} - \frac{k}{b+k-a+jk} \right) \right] \times \frac{(kz)^{-n}}{(k)_{n,k}}. \tag{136}$$

4. Conclusion

In this research work, we derived the k-hypergeometric differential equation. Also, we obtained 24 solutions of k-hypergeometric differential equation around all its regular singular points by using Frobenius method. So, we conclude that, if $k \to 1$, we obtain Euler's hypergeometric differential equation. Similarly, by letting $k \to 1$, we find 24 solutions of hypergeometric differential equation around all its regular singular points.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] L. Euler, *Institutiones Calculi Integralis*, vol. 1 of *Opera Omnia Series*, 1769.

[2] C. F. Gauss, *Disquisitiones Generales Circa Seriem Infinitam*, vol. 3, Werke, 1813.

[3] E. E. Kummer, "Über die hypergeometrische Reihe," *Journal: Für die Reine und Angewandte Mathematik*, vol. 15, pp. 39–83, 1836.

[4] B. Dwork, "On Kummer's twenty-four solutions of the hypergeometric differential equation," *Transactions of the American Mathematical Society*, vol. 285, no. 2, pp. 497–521, 1984.

[5] R. Díaz and C. Teruel, "q,k-generalized gamma and beta functions," *Journal of Nonlinear Mathematical Physics*, vol. 12, no. 1, pp. 118–134, 2005.

[6] R. Díaz and E. Pariguan, "On hypergeometric functions and Pochhammer k-symbol," *Revista Matemática de la Universidad del Zulia: Divulgaciones Matemáticas*, vol. 15, no. 2, pp. 179–192, 2007.

[7] R. Díaz, C. Ortiz, and E. Pariguan, "On the k-gamma q-distribution," *Central European Journal of Mathematics*, vol. 8, no. 3, pp. 448–458, 2010.

[8] C. G. Kokologiannaki, "Properties and inequalities of generalized k-gamma, beta and zeta functions," *International Journal of Contemporary Mathematical Sciences*, vol. 5, no. 13–16, pp. 653–660, 2010.

[9] V. Krasniqi, "Inequalities and monotonicity for the ration of k-gamma functions," *Scientia Magna*, vol. 6, no. 1, pp. 40–45, 2010.

[10] V. Krasniqi, "A limit for the k-gamma and k-beta function," *International Mathematical Forum: Journal for Theory and Applications*, vol. 5, no. 33–36, pp. 1613–1617, 2010.

[11] S. Mubeen and G. M. Habibullah, "k-fractional integrals and application," *International Journal of Contemporary Mathematical Sciences*, vol. 7, no. 1–4, pp. 89–94, 2012.

[12] S. Mubeen and G. M. Habibullah, "An integral representation of some k-hypergeometric functions," *International Mathematical Forum: Journal for Theory and Applications*, vol. 7, no. 1–4, pp. 203–207, 2012.

[13] S. Mubeen, "Solution of some integral equations involving con uent K-hypergeometric functions," *Applied Mathematics*, vol. 4, no. 7A, pp. 9–11, 2013.

[14] I. N. Sneddon, *Special Functions of Mathematical Physics and Chemistry*, Oliver and Boyd, 1956.