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Research Article

The Truncated q-Bernstein Polynomials in the Case $q > 1$

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The truncated q-Bernstein polynomials $B_{n,m,q}(f;x)$, $n \in \mathbb{N}$, and $m \in \mathbb{N}_0$ emerge naturally when the q-Bernstein polynomials of functions vanishing in some neighbourhood of 0 are considered. In this paper, the convergence of the truncated q-polynomials on $[0,1]$ is studied. To support the theoretical results, some numerical examples are provided.

1. Introduction

Let $q > 0$. For any nonnegative integer n, the *q-integer* $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1} \quad (n = 1, 2, \ldots), \ [0]_q := 0, \tag{1}$$

and the *q-factorial* $[n]_q!$ is defined by

$$[n]_q! := [1]_q[2]_q \cdots [n]_q \quad (n = 1, 2, \ldots), \ [0]_q! := 1. \tag{2}$$

For integers $0 \le k \le n$, the *q-binomial* coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{3}$$

Clearly, for $q = 1$,

$$[n]_1 = n, \qquad [n]_1! = n!, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}. \tag{4}$$

We also use the following standard notations:

$$(a;q)_0 := 1, \qquad (a;q)_k := \prod_{s=0}^{k-1} (1 - aq^s),$$
$$(a;q)_\infty := \prod_{s=0}^{\infty} (1 - aq^s). \tag{5}$$

These notations and definitions can be seen in [1, Chapter 10]. The following generalization of the *Bernstein polynomials* based on the q-integers has been introduced by Phillips in [2].

Definition 1. For any $f \in C[0,1]$, the *q-Bernstein polynomials* of f are defined by

$$B_{n,q}(f;x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{nk}(x), \quad n \in \mathbb{N}, \tag{6}$$

where

$$p_{nk}(x) := \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (x;q)_{n-k}, \quad k = 0, 1, \ldots, n. \tag{7}$$

Whereas one can obtain the classical Bernstein polynomials for $q = 1$, the name "q-Bernstein polynomials" will be used to refer to the case $q \ne 1$. For more information and open problems related to the q-Bernstein polynomials the reader is referred to [3].

The q-Bernstein polynomials as well as the classical Bernstein polynomials have some common properties (cf. [4] and references therein), such as the end-point interpolation property,

$$B_{n,q}(f;0) = f(0), \quad B_{n,q}(f;1) = f(1) \quad \forall q > 0, \ n \in \mathbb{N}, \tag{8}$$

and the shape-preserving properties in the case $0 < q < 1$. Similar to the Bernstein polynomials, the q-Bernstein polynomials are degree reducing on the set of polynomials.

As for the convergence properties of the q-Bernstein polynomials, they demonstrate a striking difference from those of the Bernstein polynomials. In general terms, this is true for various Bernstein-type operators based on the q-integers, and it is exactly the occurrence of new phenomena and insights that makes the study of the convergence of these operators attractive and challenging. During the last decade, the approximation by operators based on the q-integers has been investigated by many researchers (see, e.g., [5–10]).

In this paper, the convergence of the truncated q-Bernstein polynomials is addressed. Such polynomials come into the picture once the q-Bernstein polynomials are considered for the functions vanishing in some neighbourhood of 0. The truncation of functions and operators is a well-known and widely used tool in functional analysis and approximation theory. See, for example, a recent paper [11] where it has been used to prove a Daugavet-type inequality. As for the truncation of the classical Bernstein polynomials, it has been used by Cooper and Waldron in [12].

Definition 2 (see [13]). For any $f : [0, 1] \to \mathbb{R}$, $m \in \mathbb{N}_0$, and $n \geq m$, the *m-truncated q-Bernstein polynomials* of f are

$$B_{n,m,q}(f; x) := \sum_{k=n-m}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{n,k}(x). \tag{9}$$

In this paper, the properties of polynomials (9) are studied only in the case $q > 1$. Throughout the paper, and unless stated otherwise, it is assumed that $q > 1$ is fixed. For the sake of simplicity, the notations

$$\mathbb{J}_q := \{0\} \cup \left\{q^{-j}\right\}_{j=0}^{\infty} \tag{10}$$

and $x_k = [k]_q/[n]_q$ for $k = 0, \ldots, n$ will be used.

For $f : [0, 1] \to \mathbb{R}$, $m \in \mathbb{N}_0$, the function

$$f_m(x) := \begin{cases} 0 & \text{for } x \in \left[0, q^{-(m+1)}\right), \\ f(x) & \text{for } x \in \left[q^{-(m+1)}, 1\right], \end{cases} \tag{11}$$

will be called the *m-truncation* of f. Clearly, $f(x) = f_m(x)$ if and only if $f(x) = 0$ for $x \in [0, q^{-(m+1)})$.

The paper is organized as follows. The next section is devoted to the main results, which, finally, are illustrated in Section 3 using numerical examples.

2. Results

In this section, the results are presented pertaining the convergence of the truncated q-Bernstein polynomials. The following simple assertion reveals that the truncated q-Bernstein polynomials appear naturally when we consider those functions vanishing in some neighbourhood of 0.

Theorem 3. *If $n \geq \ln([m + 1]_q)/\ln q$, then $B_{n,m,q}(f; x) = B_{n,q}(f_m; x)$, where f_m is given by* (11).

Proof. Since $f_m(x_k) = 0$, for $x_k < q^{-(m+1)}$, one has

$$B_{n,q}(f_m; x) = \sum_{\{k : x_k \geq q^{-(m+1)}\}} f(x_k) p_{n,k}(x) \tag{12}$$

while

$$B_{n,m,q}(f; x) = \sum_{k \geq n-m} f(x_k) p_{n,k}(x). \tag{13}$$

The right sides of (12) and (13) are identical if and only if

$$q^{-(m+1)} \leq x_{n-m} < q^{-m}. \tag{14}$$

The inequality on the right holds for all $q > 1$, while the one on the left is true if and only if $n \geq \ln([m + 1]_q)/\ln q$. □

Remark 4. If $q \geq 2$, then $B_{n,q}(f_m, x) = B_{n,m,q}(f; x)$ for all $n \geq m$.

Corollary 5. *If $f(x) = 0$ for $x \in [0, q^{-(m+1)})$, then $B_{n,q}(f; x) = B_{n,m,q}(f; x)$.*

Theorem 2.1 of [7] states that, for any $f \in C[0, 1]$, $j \in \mathbb{N}_0$, and $q > 1$, $\lim_{n\to\infty} B_{n,q}(f; q^{-j}) = f(q^{-j})$. That is, when $q > 1$, the sequence of polynomials $B_{n,q}(f)$ converges to f on $\mathbb{J}_q$. The following lemma, which can be inferred from Lemma 3.1 of [13], is needed to investigate the convergence of the truncated q-Bernstein polynomials.

Lemma 6. *Let $x_0 \in [0, 1] \setminus \mathbb{J}_q$. Then*

$$\lim_{n\to\infty} \frac{p_{n,n-k}(q; x_0)}{(q^k x_0)^n} = C_k(x_0), \tag{15}$$

where $C_k(x_0) = (-1)^k q^{-k(k-1)/2}(x_0; q)_k/[(q; q)_k x_0^k] \neq 0$.

Note that $C_k(x) = 0$ if and only if $x \in \mathbb{J}_q \setminus \{0\}$. Thus, the following holds.

Lemma 7. *For any $f \in C[0, 1]$ and $x \in [0, q^{-m})$, $\lim_{n\to\infty} B_{n,m,q}(f; x) = 0$. Moreover, the convergence is uniform on any compact subset of $[0, q^{-m})$.*

Proof. Since f is continuous on $[0, 1]$, there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [0, 1]$. In addition, from the definition of $p_{n,k}$, one can obtain

$$\begin{aligned} \left|p_{n,n-k}(x)\right| &\leq \frac{q^{k(n-k)}}{(1 - 1/q)(1 - 1/q^2)\cdots(1 - 1/q^k)} \\ &\quad \cdot x^{n-k} \cdot (1 + q)\cdots\left(1 + q^{k-1}\right) \\ &\leq \frac{(-1/q; 1/q)_\infty}{(1/q; 1/q)_\infty} \left(q^k x\right)^{n-k} \\ &\leq \frac{(-1/q; 1/q)_\infty}{(1/q; 1/q)_\infty} (q^m x)^{n-m} \end{aligned} \tag{16}$$

for $n \geq 2m$. Hence, for any $x \in [0, q^{-m})$,

$$\begin{aligned} \left|B_{n,m,q}(f; x)\right| &= \left|\sum_{k=0}^{m} f(x_k) p_{n,n-k}(x)\right| \\ &\leq M_q (m + 1) (q^m x)^{n-m}, \end{aligned} \tag{17}$$

where $M_q = M(-1/q; 1/q)_\infty/(1/q; 1/q)_\infty$. This yields the desired result. □

Clearly, for $m = 0$, the truncated q-Bernstein polynomials have the form

$$B_{n,0,q}(f;x) = f(1)\,x^n, \tag{18}$$

whence

$$\lim_{n\to\infty} B_{n,0,q}(f;x) = \begin{cases} 0 & \text{for } x \in [0,1), \\ f(1) & \text{for } x = 1. \end{cases} \tag{19}$$

As the next theorem reveals, the case $m \neq 0$ is not straightforward.

Theorem 8. *Let $f \in C[0,1]$, $m \geq 1$. If $f(q^{-m}) \neq 0$, then*

$$\lim_{n\to\infty} B_{n,m,q}(f;x) = \begin{cases} 0 & \text{for } x \in [0,q^{-m}), \\ f(x) & \text{for } x = 1, q^{-1}, \ldots, q^{-m}, \end{cases} \tag{20}$$

$$\lim_{n\to\infty} \left|B_{n,m,q}(f;x)\right| = 0 \quad \text{for } x \in [q^{-m},1] \setminus \mathbb{J}_q. \tag{21}$$

Proof. The assertion (20) is obvious by Lemma 7 and the convergence on $\mathbb{J}_q$. To see (21), let $x \in [q^{-m},1] \setminus \mathbb{J}_q$. Then, Lemma 6 implies that

$$p_{n,n-k}(q;x) \sim C_k(x)\left(q^k x\right)^n \quad \text{as } n \longrightarrow \infty. \tag{22}$$

Moreover, it is easy to see that $x_{n-k} \to q^{-k}$ as $n \to \infty$. Therefore,

$$\begin{aligned} B_{n,m,q}(f;x) &= \sum_{k=0}^{m} f(x_{n-k})\, p_{n,n-k}(q;x) \\ &\sim \sum_{k=0}^{m} f\left(q^{-k}\right) C_k(x)\left(q^k x\right)^n \\ &\sim f\left(q^{-m}\right) C_m(x)\left(q^m x\right)^n \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{23}$$

Since $f(q^{-m}) \neq 0$ and $C_m(x) \neq 0$, the statement follows. □

In the case when $f(q^{-m}) = 0$, the truncated q-Bernstein polynomials behave differently as the following theorem indicates.

Theorem 9. *Let $f \in C[0,1]$ with $f(q^{-m}) = 0$, and suppose that, for each $k = 1,\ldots,m$, there exists $\alpha_k > 0$ such that*

$$\lim_{n\to\infty} \frac{f(x_{n-k})}{\left(q^{-k} - x_{n-k}\right)^{\alpha_k}} = \beta_k \neq 0. \tag{24}$$

Setting $s = \max_{k=1,\ldots,m}\{k - \alpha_k\}$, one has the following:

(i) *if $s > 0$, then*

$$\begin{aligned} &\lim_{n\to\infty} B_{n,m,q}(f;x) = 0 \quad \text{when } x \in [0,q^{-s}], \\ &\lim_{n\to\infty} \left|B_{n,m,q}(f;x)\right| = \infty \quad \text{when } x \in (q^{-s},1) \setminus \mathbb{J}_q, \end{aligned} \tag{25}$$

except possibly at most at m points, outside of $\mathbb{J}_q$;

(ii) *if $s \leq 0$, then $\lim_{n\to\infty} B_{n,m,q}(f;x) = 0$ for all $x \in [0,1]$.*

Proof. From (24), for each $k = 1,\ldots,n$, one has $f(x_{n-k}) \sim \beta_k(q^{-k} - x_{n-k})^{\alpha_k}$ as $n \to \infty$, which together with Lemma 6 and the fact that $q^{-k} - x_{n-k} = (q^k - 1)/[q^k(q^n - 1)]$ yields that

$$\begin{aligned} B_{n,m,q}(f;x) - f(1)\,x^n &= \sum_{k=1}^{m} f(x_{n-k})\, p_{n,n-k}(q;x) \\ &\sim \sum_{k=1}^{m} \beta_k\left(q^{-k} - x_{n-k}\right)^{\alpha_k} C_k(x)\left(q^k x\right)^n \\ &\sim \sum_{k=1}^{m} \widetilde{\beta}_k C_k(x)\left(q^{k-\alpha_k} x\right)^n \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{26}$$

where $\widetilde{\beta}_k = \beta_k(1 - q^{-k})^{\alpha_k}$. Now, suppose that $k - \alpha_k$ takes on its maximum value s at $k_1, k_2, \ldots, k_r \in \{1,\ldots,m\}$ with $k_1 < k_2 < \cdots < k_r$. Then,

$$\begin{aligned} &B_{n,m,q}(f;x) - f(1)\,x^n \\ &\quad \sim \sum_{k=1}^{m} \widetilde{\beta}_k C_k(x)\left(q^{k-\alpha_k} x\right)^n \\ &\quad \sim \left(q^s x\right)^n \sum_{j=1}^{r} \widetilde{\beta}_{k_j} C_{k_j}(x) := \left(q^s x\right)^n \varphi(x) \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{27}$$

Since $x^{k_r}\varphi(x) = x^{k_r}\sum_{j=1}^{r} \widetilde{\beta}_{k_j} C_{k_j}(x)$ is a polynomial of degree at most m, $\varphi(x)$ can vanish at most at m points. This completes the proof. □

3. Numerical Examples

In this part, some numerical examples are given to demonstrate the theoretical results.

Example 10. Let $q = 2$, $m = 2$. Consider the function $f : [0,1] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0, & x < \dfrac{1}{8}, \\ \left(x - \dfrac{1}{8}\right)^2, & \dfrac{1}{8} \leq x < \dfrac{1}{4}, \\ \dfrac{1}{64}, & x \geq \dfrac{1}{4}. \end{cases} \tag{28}$$

Since $f(x) = 0$ for $x \in [0, 2^{-3})$ and $f(2^{-2}) \neq 0$, by Theorem 8,

$$\lim_{n\to\infty} B_{n,2,2}(f;x) = \begin{cases} 0, & \text{if } x \in \left[0, \dfrac{1}{4}\right), \\ f(x), & \text{if } x = 1, \dfrac{1}{2}, \dfrac{1}{4}, \\ \pm\infty, & \text{otherwise}. \end{cases} \tag{29}$$

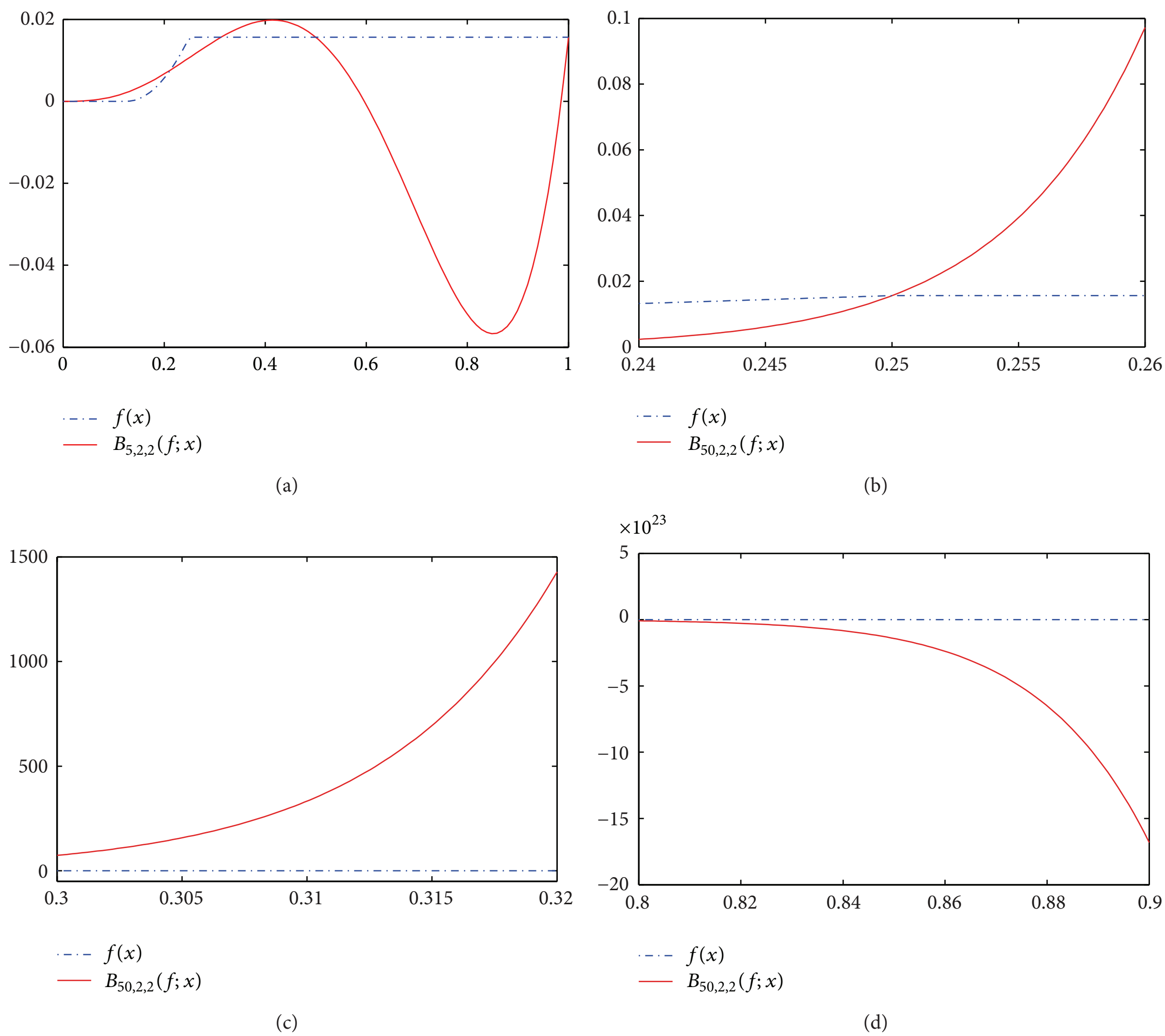

FIGURE 1: Graphs of f and $B_{n,2,2}(f)$ on different intervals for some values of n.

In Figure 1, the graphs of f and $B_{n,2,2}(f)$ on some different intervals are given.

Example 11. Let $q = 2$, $m = 3$, and consider a function $g \in C[0,1]$ satisfying

$$g(x) = \begin{cases} 0, & x \in \left[0, \frac{1}{16}\right), \\ \frac{72(1/8 - x)^2}{7}, & x \in \left[\frac{3}{32}, \frac{1}{8}\right), \\ \frac{1}{4} - x, & x \in \left[\frac{3}{16}, \frac{1}{4}\right), \\ 3\left(\frac{1}{2} - x\right), & x \in \left[\frac{3}{8}, \frac{1}{2}\right), \end{cases} \tag{30}$$

and on the rest of $[0,1]$, g is piecewise linear. It is easily seen that the conditions of Theorem 9 are fulfilled by $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 2$, $\beta_1 = 3$, $\beta_2 = 1$, and $\beta_3 = 72/7$. Here, $s = 1$ and $\varphi(x) = (1-x)(1-2x)(3-4x)/(64x^3)$ vanishes at $x = 1, 1/2$, and $3/4$. Therefore,

$$\lim_{n\to\infty} B_{n,3,2}(g;x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{2}\right) \cup \left\{\frac{3}{4}\right\}, \\ g(x), & \text{if } x = 1, \frac{1}{2}, \\ \pm\infty, & \text{otherwise}. \end{cases} \tag{31}$$

In Figure 2, the graphs of g and $B_{g,3,2}(g)$ on some different intervals are provided.

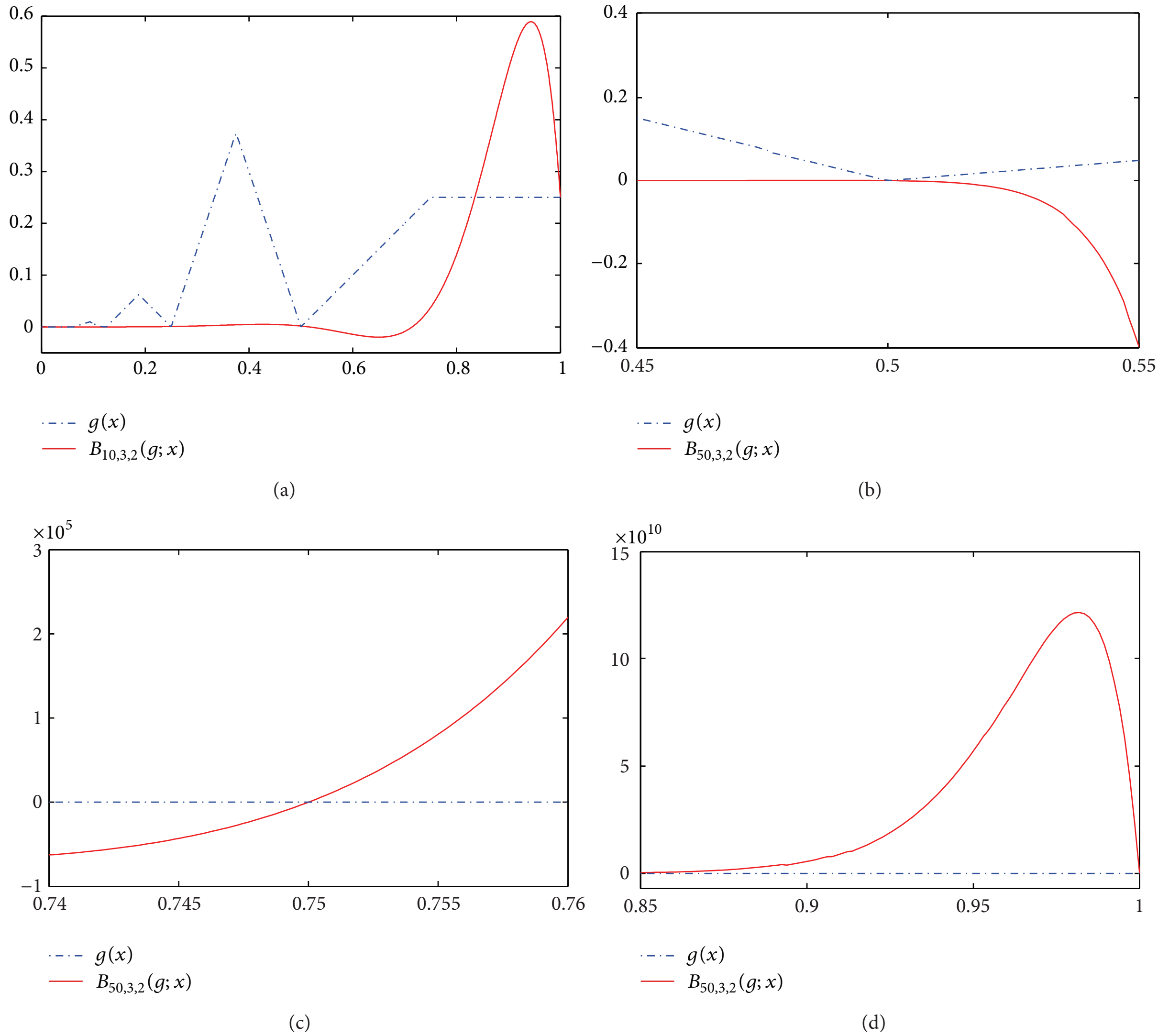

FIGURE 2: Graphs of g and $B_{n,3,2}(g)$ on different intervals for some values of n.

Example 12. Let $q = 2$, $m = 5$, and consider the function $h : [0,1] \to \mathbb{R}$ with

$$h(x) = \begin{cases} 0, & x \in \left[0, \dfrac{1}{64}\right), \\ \dfrac{73400320(1/32 - x)^3}{54777}, & x \in \left[\dfrac{3}{128}, \dfrac{1}{32}\right), \\ \dfrac{453376(1/16 - x)^2}{7695}, & x \in \left[\dfrac{3}{64}, \dfrac{1}{16}\right), \\ \dfrac{1}{8} - x, & x \in \left[\dfrac{3}{32}, \dfrac{1}{8}\right), \\ 2\left(\dfrac{1}{4} - x\right)^2, & x \in \left[\dfrac{3}{16}, \dfrac{1}{4}\right), \\ \dfrac{1}{2} - x, & x \in \left[\dfrac{3}{8}, \dfrac{1}{2}\right). \end{cases} \tag{32}$$

At any other point in $[0,1]$, h is defined in such a way that it is continuous. Obviously, the conditions of Theorem 9 are met by $\alpha_1 = 1$, $\alpha_2 = 1$, $\alpha_3 = 1$, $\alpha_4 = 2$, $\alpha_5 = 3$, $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 1$, $\beta_4 = 453376/7695$, and $\beta_5 = 73400320/54777$. Now, $s = 2$ and $\varphi(x) = (1-x)(1-2x)(1-4x)(25x^2-35x+12)/(98496x^5)$. Therefore,

$$\lim_{n\to\infty} B_{n,5,2}(h;x) = \begin{cases} 0, & \text{if } x \in \left[0, \dfrac{1}{4}\right) \cup \left\{\dfrac{3}{5}, \dfrac{4}{5}\right\}, \\ h(x), & \text{if } x = 1, \dfrac{1}{2}, \dfrac{1}{4}, \\ \pm\infty, & \text{otherwise.} \end{cases} \tag{33}$$

In Figure 3, the graphs of h and $B_{n,5,2}(h)$ on some different intervals are depicted.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

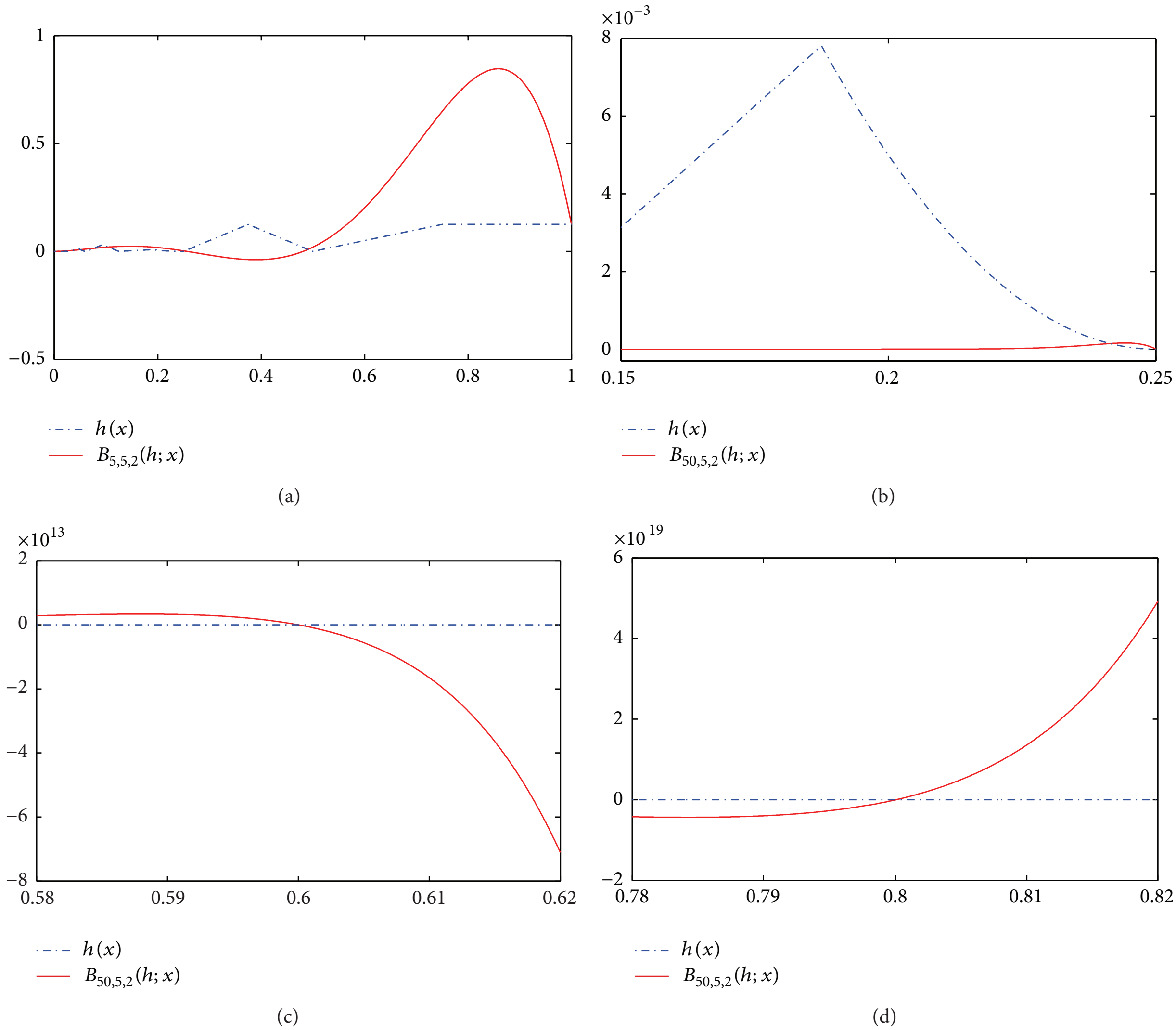

Figure 3: Graphs of h and $B_{n,5,2}(h)$ on different intervals for some values of n.

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