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*Research Article*

# New Derivative Based Open Newton-Cotes Quadrature Rules

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Some new families of open Newton-Cotes rules which involve the combinations of function values and the evaluation of derivative at uniformly spaced points of the interval are presented. The order of accuracy of these numerical formulas is higher than that of the classical open Newton-Cotes formulas. An extensive comparison of the computational cost, order of accuracy, error terms, coefficients of the error terms, observed order of accuracy, CPU usage time, and results obtained from these formulas is given. The comparisons show that we have been able to define some new open Newton-Cotes rules which are superior to classical open rules for less number of nodes and less computational cost with increased order of accuracy.

## 1. Introduction

In the recent past, the numerical quadrature rules were widely addressed in terms of their higher order of accuracy. The general form of any numerical integration formula is

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i). \tag{1}$$

If the weights $w_i$ are constant and the $n+1$ nodes $x_0, x_1, \ldots, x_n$ are equispaced within the interval $[a, b]$, the formula (1) gives rise to the Newton-Cotes quadrature rules. In open Newton-Cotes formulae, function evaluation at the endpoints of the interval is excluded from the quadrature rule. That is,

$$\int_a^b f(x)\,dx = \int_{x_{-1}}^{x_{n+1}} f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i), \tag{2}$$

where $x_0, x_1, \ldots, x_n$ are distinct $n+1$ integration points and $w_i$ are $n+1$ weights within the interval $(a, b)$ with $x_i = a + (i+1)h$, $i = 0, 1, 2, \ldots, n$, and $h = (b-a)/(n+2)$.

The classical open rules are given as follows.

For $n = 0$,

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2hf(x_0) + \frac{h^3}{3} f^{(2)}(\xi), \tag{3}$$

where $\xi \in (x_{-1}, x_1)$.

For $n = 1$,

$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3h}{2}\left(f(x_0) + f(x_1)\right) + \frac{3h^3}{4} f^{(2)}(\xi), \tag{4}$$

where $\xi \in (x_{-1}, x_2)$.

For $n = 2$,

$$\int_{x_{-1}}^{x_3} f(x)\,dx = \frac{4h}{3}\left(2f(x_0) - f(x_1) + 2f(x_2)\right) + \frac{14h^5}{45} f^{(4)}(\xi), \tag{5}$$

where $\xi \in (x_{-1}, x_3)$.

For $n = 3$,

$$\int_{x_{-1}}^{x_4} f(x)\,dx = \frac{5h}{24}\left(11f(x_0) + f(x_1) + f(x_2) + 11f(x_3)\right) + \frac{95h^5}{144} f^{(4)}(\xi), \tag{6}$$

where $\xi \in (x_{-1}, x_4)$.

For $n = 4$,

$$\int_{x_{-1}}^{x_5} f(x)\,dx = \frac{3h}{10}\left(11f(x_0) - 14f(x_1) + 26f(x_2) - 14f(x_3) + 11f(x_4)\right) + \frac{41h^7}{140}f^{(6)}(\xi), \quad (7)$$

where $\xi \in (x_{-1}, x_5)$.

The Newton-Cotes rules can be constructed by replacing the integrand with an interpolating polynomial of appropriate degree. Lagrangian interpolating polynomials are extensively used to construct the Newton-Cotes rules; however, it was noted later that if, instead of Lagrangian polynomials, Hermite's interpolating polynomial is used which interpolates not only the function but also the first derivatives of the function at the selected nodes, a new class of integration rules are introduced called as the corrected Newton-Cotes formulas which include the first order derivative of the function as well.

The corrected Newton-Cotes formulas have greater degree of precision and are more accurate than the classical ones. For example, the Trapezoidal rule is improved by the corrected Trapezoidal rule with greater accuracy. Markoff also noticed that the Gaussian formulas can also be constructed by integrating a Hermitian interpolating polynomial. Numerous new quadratures were found which satisfy optimality properties for various classes of functions [1]. These numerical quadratures have gained a lot of importance in recent years. A lot of work has been done in the numerical improvement of Newton-Cotes formulas. In 2003, Moawwad gave a unified approach for Newton-Cotes formulas of open as well as closed type by solving directly linear systems with coefficients matrices of Vandermonde type [2]. In 2005, Sermutlu presented a detailed comparison of Newton-Cotes and Gaussian methods of quadrature [3]. In 2005 and 2006, Dehghan and his companions [4–6] presented some new numerical integration formulas as an improvement of open, semiopen, and closed Newton-Cotes quadrature formulas. The new formulas by Dehghan et al. have an increased order of accuracy as compared to classical rules. The locations of boundaries along with weights were included as two additional parameters. The concept of degree of precision was used to set a system of nonlinear equations to obtain the approximate values of the parameters involved by solving the system approximately. In 2006, these authors introduced improvements of first and second kind Chebyshev-Newton-Cotes quadrature rules [7, 8]. In 2012, Burg introduced a new family of closed Newton-Cotes numerical integration formulas using first derivative values as well as functional values using the concept of degree of precision [9] to include additional parameters and thus to obtain new closed Newton-Cotes rules with increased order of accuracy. Recently, Burg and Degny [10] have also presented a new corrected family of midpoint rule involving odd order derivatives at the endpoints of the interval of integration. Also, in [11] Zhao and Li presented some new classes of midpoint derivatives based closed Newton-Cotes rules.

We, in this paper, construct a family of open Newton-Cotes formulas involving the first order derivatives at some or all nodes. The weights of the first derivative evaluation serve as additional parameters to increase the order of accuracy of these rules. The weights are determined by using the concept of degree of precision from a system of linear equations. The new formulas have increased order of accuracy for less number of nodes and less computational cost in comparison with the classical open Newton-Cotes rules.

## 2. Methodology and Derivation of Derivative Based Open Quadrature Formulas

Let $f \in C^{2n+2}[a,b]$ be a real valued function. Let the interval $[a,b]$ be subdivided into $n+2$ subintervals with $x_{-1}, x_0, \ldots, x_{n+1}$ as distinct $n+3$ nodes.

*2.1. Derivative Based Schemes.* We introduce the following derivative based schemes for the open Newton-Cotes formulas.

(1) Function value and first derivative at each point $x_i \in (a,b)$.

The general form of the scheme is

$$\int_{x_{-1}}^{x_{n+1}} f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i) + \sum_{i=0}^{n} u_i f'(x_i)\,h, \quad (8)$$

where $w_i$ and $u_i$ are the weights for the function and the derivative, respectively, and $n$ is the number of subintervals used in the basic formula.

(2) Function value at each point $x_i \in (a,b)$ and first derivative at each point $x_i \in [a,b]$.

The general form of the scheme is

$$\int_{x_{-1}}^{x_{n+1}} f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i) + \sum_{i=-1}^{n+1} u_i f'(x_i)\,h, \quad (9)$$

where $w_i$ and $u_i$ are as defined above.

(3) Function value at each point $x_i \in (a,b)$ and first derivative at interior endpoints $x_0, x_n$.

The general form of the scheme is

$$\int_{x_{-1}}^{x_{n+1}} f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i) + c_0 f'(x_0)\,h - c_1 f'(x_n)\,h, \quad (10)$$

where $w_i$ and $c_i$ are as defined above.

(4) Function value at each point $x_i \in (a,b)$ and first derivative at $x_{-1} = a, x_{n+1} = b$.

The general form of the scheme is

$$\int_{x_{-1}}^{x_{n+1}} f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i) + u_0 f'(x_{-1})\,h - u_1 f'(x_{n+1})\,h, \quad (11)$$

where $w_i$ and $u_i$ are as defined above.

We, now, derive the formulas for these cases.

*2.1.1. Open Newton-Cotes Rules with Derivative at All Interior Nodes (ONC1).* We can generate a scheme with a higher precision than that of the open Newton-Cotes scheme by using the first derivative of the integrand at all nodes, except at the endpoints, within the numerical integration scheme. We assume that the interval $[a, b]$ is subdivided into $n + 2$ subintervals out of which $n + 1$ points are included in this problem. The procedure is given as

$$\int_{x_{-1}}^{x_{n+1}} f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i) + \sum_{i=0}^{n} u_i f'(x_i)\,h, \tag{12}$$

where $w_i$ and $u_i$ are the weights for the function and the derivative, respectively, and $n$ is the number of subintervals used in the basic formula. We assume that degree of precision of (8) can be at most $2n + 1$. This imparts the condition on (8) that it will be exact for all monomials $x^k$ from $k = 0$ to $k = 2n + 1$. From the concept, we form a system of $2n + 2$ equations.

For example, for $n = 0$, the two equations which are generated by this approach are

$$\begin{aligned} &\text{For } f(x) = 1 : \int_{x_{-1}}^{x_1} dx = x_1 - x_{-1} = w_0, \\ &\text{For } f(x) = x : \int_{x_{-1}}^{x_1} x\,dx = \frac{\left(x_1^2 - x_{-1}^2\right)}{2} = w_0 x_0 + u_0 h. \end{aligned} \tag{13}$$

Solving these equations for the coefficients $w_0$, and $u_0$, the following numerical quadrature rule is obtained:

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2hf(x_0) + \frac{1}{3}h^3 f^{(2)}(\xi), \tag{14}$$

for $\xi \in (x_{-1}, x_1)$ which is the classical midpoint rule. The precision of this method is 1 and it is 3rd order accurate.

For $n = 1$, the four equations generated are

$$\begin{aligned} \int_{x_{-1}}^{x_2} dx &= x_2 - x_{-1} = w_0 + w_1, \\ \int_{x_{-1}}^{x_2} x\,dx &= \frac{\left(x_2^2 - x_{-1}^2\right)}{2} = w_0 x_0 + w_1 x_1 + u_0 h + u_1 h, \\ \int_{x_{-1}}^{x_2} x^2\,dx &= \frac{\left(x_2^3 - x_{-1}^3\right)}{3} = w_0 x_0^2 + w_1 x_1^2 + 2u_0 h x_0 \\ &\quad + 2u_1 h x_1, \\ \int_{x_{-1}}^{x_2} x^3\,dx &= \frac{\left(x_2^4 - x_{-1}^4\right)}{4} = w_0 x_0^3 + w_1 x_1^3 + 3u_0 h x_0^2 \\ &\quad + 3u_1 h x_1^2. \end{aligned} \tag{15}$$

Solving these equations simultaneously for the unknown coefficients $w_0, w_1, u_0$, and $u_1$, the numerical quadrature rule becomes

$$\begin{aligned} \int_{x_{-1}}^{x_2} f(x)\,dx &\approx \frac{3}{2}h\left[f(x_0) + f(x_1)\right] \\ &\quad - \frac{3}{4}h^2\left[f'(x_0) - f'(x_1)\right] + \frac{7}{80}h^5 f^{(4)}(\xi), \end{aligned} \tag{16}$$

for $\xi \in (x_{-1}, x_2)$. The precision of this method is 3 and it is 5th order accurate and the associated composite method is of the 4th order. For this method in its basic form two function evaluations and two first derivative evaluations are required. In composite form, where $N$ is an integer divisible by 3, the method requires $2N/3$ function evaluations and $2N/3$ derivative evaluations.

For $n = 2$, the numerical quadrature rule is

$$\begin{aligned} \int_{x_{-1}}^{x_3} f(x)\,dx &= h\left[\frac{-16}{15}f(x_0) + \frac{92}{15}f(x_1) - \frac{16}{15}f(x_2)\right] \\ &\quad - \frac{28}{15}h^2\left[f'(x_0) - f'(x_2)\right] \\ &\quad + \frac{107}{4725}h^7 f^{(6)}(\xi), \end{aligned} \tag{17}$$

for $\xi \in (x_{-1}, x_3)$. The precision of this method is 5 with order local truncation error of the 7th order. The associated composite rule is 6th order accurate. For this method in its basic form three function evaluations and two first derivative evaluations are required. In composite form, where $N$ is an integer divisible by 4, the method requires $3N/4$ function evaluations and $N/2$ derivative evaluations.

For $n = 3$, the numerical quadrature rule is

$$\begin{aligned} \int_{x_{-1}}^{x_4} f(x)\,dx &= h\left[-\frac{1245}{224}f(x_0) + \frac{1805}{224}f(x_1)\right. \\ &\quad \left. + \frac{1805}{224}f(x_2) - \frac{1245}{224}f(x_3)\right] \\ &\quad + h^2\left[-\frac{6605}{2016}f'(x_0) - \frac{1315}{224}f'(x_1)\right. \\ &\quad \left. + \frac{1315}{224}f'(x_2) + \frac{6605}{2016}f'(x_3)\right] \\ &\quad + \frac{5951}{1016064}h^9 f^{(8)}(\xi), \end{aligned} \tag{18}$$

for $\xi \in (x_{-1}, x_4)$. The precision of this method is 7 with local truncation error of the 9th order. The associated composite rule is 8th order accurate. For this method in its basic form four function evaluations and two first derivative evaluations are required. In composite form, where $N$ is an integer divisible by 5, the method requires $4N/5$ function evaluations and $4N/5$ derivative evaluations.

For $n = 4$, the numerical quadrature rule is

$$\begin{aligned}\int_{x_{-1}}^{x_5} f(x)\,dx = h\Big[&\frac{-2509}{210} f(x_0) - \frac{757}{105} f(x_1) + \frac{3102}{70} f(x_2) \\ &- \frac{757}{105} f(x_3) - \frac{2509}{210} f(x_4)\Big] \\ &+ h^2\Big[-\frac{347}{70} f'(x_0) - \frac{773}{35} f'(x_1) \\ &+ \frac{773}{35} f'(x_3) + \frac{347}{70} f'(x_4)\Big] \\ &+ \frac{587}{388080} h^{11} f^{(10)}(\xi),\end{aligned} \tag{19}$$

for $\xi \in (x_{-1}, x_5)$. The precision of this method is 9 with local truncation error of the 11th order and the associated composite rule is 10th order accurate. For this method in its basic form five function evaluations and two first derivative evaluations are required. In composite form, where $N$ is an integer divisible by 6, the method requires $5N/6$ function evaluations and $2N/3$ derivative evaluations.

*2.1.2. Open Newton-Cotes Rules with Derivative at All Nodes including Endpoints (ONC2).* The procedure is given as

$$\int_{x_{-1}}^{x_{n+1}} f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i) + \sum_{i=-1}^{n+1} u_i f'(x_i)\,h, \tag{20}$$

where $w_i$ and $u_i$ are the weights for the function and the derivative, respectively, and $n$ is the number of subintervals used in the basic formula. We assume that degree of precision of (9) can be at most $2n + 3$. This imparts the condition on (9) that it will be exact for all monomials $x^k$ from $k = 0$ to $k = 2n + 3$. From the concept, we form a system of $2n + 4$ equations.

For example, for $n = 0$, the following numerical quadrature rule is obtained:

$$\begin{aligned}\int_{x_{-1}}^{x_1} f(x)\,dx = 2hf(x_0) &- \frac{h^2}{6}\left[f'(x_{-1}) - f'(x_1)\right] \\ &- \frac{7}{180} h^5 f^{(4)}(\xi),\end{aligned} \tag{21}$$

for $\xi \in (x_{-1}, x_1)$. The precision of this method is 3 and it is 5th order accurate. The associated composite rule is 4th order accurate. For this method in its basic form one function evaluation and two first derivative evaluations are required. In composite form, the method requires $N/2$ function evaluations and only two additional derivative evaluations due to cancellation of the derivatives at the endpoints in contrast to one point classical open method.

For $n = 1$, the quadrature rule becomes

$$\begin{aligned}\int_{x_{-1}}^{x_2} f(x)\,dx \approx \frac{3}{2}h\left[f(x_0) + f(x_1)\right] &- \frac{1}{80}h^2\Big[7f'(x_{-1}) + 39f'(x_0) \\ &-39f'(x_1) - 7f'(x_2)\Big] \\ &- \frac{31}{6720}h^7 f^{(6)}(\xi),\end{aligned} \tag{22}$$

for $\xi \in (x_{-1}, x_2)$. The precision of this method is 5 and it is 7th order accurate. The associated composite rule is 6th order accurate. For this method in its basic form two function evaluations and four first derivative evaluations are required. In composite form, where $N$ is an integer divisible by 3, the method requires $2N/3$ function evaluations and $2N/3 + 2$ derivative evaluations due to cancellation of the derivatives at the endpoints.

For $n = 2$, the numerical quadrature rule is

$$\begin{aligned}\int_{x_{-1}}^{x_3} f(x)\,dx = \frac{h}{77}&\left[32f(x_0) + 244f(x_1) + 32f(x_2)\right] \\ &- h^2\Big[\frac{428}{6930} f'(x_{-1}) + \frac{496}{495} f'(x_0) \\ &-\frac{496}{495} f'(x_2) - \frac{428}{6930} f'(x_3)\Big] \\ &- \frac{1601}{2182950} h^9 f^{(8)}(\xi),\end{aligned} \tag{23}$$

for $\xi \in (x_{-1}, x_3)$. The precision of this method is 7 with local truncation error of the 9th order. The associated composite rule is 8th order accurate. For this method in its basic form three function evaluations and four first derivative evaluations are required. In composite form, where $N$ is an integer divisible by 4, the method requires $3N/4$ function evaluations and $N/2 + 2$ derivative evaluations due to cancellation of the derivatives at the endpoints.

For $n = 3$, the numerical quadrature rule is

$$\begin{aligned}&\int_{x_{-1}}^{x_4} f(x)\,dx \\ &= \frac{h}{192}\left[-217f(x_0) + 697f(x_1) + 697f(x_2) - 217f(x_3)\right] \\ &\quad - h^2\Big[\frac{5951}{120960} f'(x_{-1}) + \frac{37603}{24192} f'(x_0) \\ &\quad + \frac{11701}{6048} f'(x_1) - \frac{11701}{6048} f'(x_2) \\ &\quad -\frac{37603}{24192} f'(x_3) - \frac{5951}{120960} f'(x_4)\Big] \\ &\quad - \frac{63817}{479001600} h^{11} f^{(10)}(\xi),\end{aligned} \tag{24}$$

for $\xi \in (x_{-1}, x_4)$. The precision of this method is 9 with local truncation error of the 11th order. The associated composite rule is 10th order accurate. For this method in its basic form four function evaluations and six first derivative evaluations are required. In composite form, where $N$ is an integer divisible by 5, the method requires $4N/5$ function evaluations and $4N/5+2$ derivative evaluations due to cancellation of the derivatives at the endpoints.

For $n = 4$, the numerical quadrature rule is

$$\begin{aligned}\int_{x_{-1}}^{x_5} f(x)\,dx = {} & \frac{h}{15070}\left[-46578 f(x_0) + 3162 f(x_1)\right. \\ & + 177252 f(x_2) + 3162 f(x_3) \\ & \left. -46578 f(x_4)\right] \\ & - h^2\left[\frac{3522}{84392} f'(x_{-1}) + \frac{226494}{105490} f'(x_0)\right. \\ & + \frac{2598138}{421960} f'(x_1) - \frac{2598138}{421960} f'(x_3) \\ & \left. - \frac{226494}{105490} f'(x_4) - \frac{3522}{84392} f'(x_5)\right] \\ & - \frac{300929}{1151950800} h^{13} f^{(12)}(\xi),\end{aligned} \tag{25}$$

for $\xi \in (x_{-1}, x_5)$. The precision of this method is 11 with local truncation error of the 13th order and the associated composite rule is 12th order accurate. For this method in its basic form five function evaluations and six first derivative evaluations are required. In composite form, where $N$ is an integer divisible by 6, the method requires $5N/6$ function evaluations and $2N/3 + 2$ derivative evaluations due to cancellation of the derivatives at the endpoints of the basic form.

*2.1.3. Open Newton-Cotes Rules with Derivative at Interior Endpoints (ONC3).* In this approach, the evaluation of the derivative is restricted for the two points only, while maintaining the improvement of order of accuracy. These numerical integration formulas follow the same behaviour as the Newton-Cotes formulas with respect to $n$ being even or odd.

These numerical integration formulas are two order of magnitude greater than the corresponding open Newton-Cotes formulas.

(i) If $n$ is even, the precision is $n + 3$.

(ii) If $n$ is odd, the precision is $n + 2$.

The general form of the scheme is

$$\int_{x_{-1}}^{x_{n+1}} f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i) + c_0 f'(x_0)\,h - c_1 f'(x_n)\,h. \tag{26}$$

In these composite forms, all of the interior derivative calculations cancel so that the only 2 derivative calculations are required for the composite form.

For $n = 0$, the quadrature formula is given as

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2hf(x_0) + \frac{1}{24} h^3 f^{(2)}(\xi), \tag{27}$$

which is the midpoint rule.

The four equations are shown below for $n = 1$:

$$\begin{aligned}\int_{x_{-1}}^{x_2} dx &= x_2 - x_{-1} = w_0 + w_1, \\ \int_{x_{-1}}^{x_2} x\,dx &= \frac{\left(x_2^2 - x_{-1}^2\right)}{2} = w_0 x_0 + w_1 x_1 + c_0 h + c_1 h, \\ \int_{x_{-1}}^{x_2} x^2\,dx &= \frac{\left(x_2^3 - x_{-1}^3\right)}{3} = w_0 x_0^2 + w_1 x_1^2 + 2c_0 x_0 h + 2c_1 x_1 h, \\ \int_{x_{-1}}^{x_2} x^3\,dx &= \frac{\left(x_2^4 - x_{-1}^4\right)}{4} = w_0 x_0^3 + w_1 x_1^3 + 3c_0 x_0^2 h + 3c_1 x_1^2 h.\end{aligned} \tag{28}$$

Solving these equations for unknown parameters we get

$$\begin{aligned}\int_{x_{-1}}^{x_2} f(x)\,dx = {} & \frac{3}{2} h\left[f(x_0) + f(x_1)\right] \\ & - \frac{3}{4} h^2\left[f'(x_0) - f'(x_1)\right] \\ & + \frac{7h^5}{80} f^{(4)}(\xi).\end{aligned} \tag{29}$$

This scheme has precision 3 and 5th order local truncation error and the associated composite rule is 4th order accurate which is the same as (16).

For $n = 2$, the optimal quadrature formula is

$$\begin{aligned}\int_{x_{-1}}^{x_3} f(x)\,dx = {} & h\left[\frac{-16}{15} f(x_0) + \frac{92}{15} f(x_1) - \frac{16}{15} f(x_2)\right] \\ & - \frac{28}{15} h^2\left[f'(x_0) - f'(x_2)\right] \\ & + \frac{107}{4725} h^7 f^{(6)}(\xi).\end{aligned} \tag{30}$$

This scheme has precision 5 and 7th order local truncation error and the associated composite rule is 6th order accurate. For this method in its basic form three function evaluations and two first derivative evaluations are required. In composite form, where $N$ is an integer divisible by 4, the method requires $3N/4$ function evaluations and $N/2$ derivative evaluations.

For $n = 3$, the optimal quadrature formula is

$$\int_{x_{-1}}^{x_4} f(x)\,dx = \frac{5}{16}h\left[f(x_0) + 7f(x_1) + 7f(x_2) + f(x_3)\right] - \frac{95}{72}h^2\left[f'(x_0) - f'(x_3)\right] + \frac{263}{4032}h^7 f^{(6)}(\xi). \tag{31}$$

This scheme has precision 5 and 7th order local truncation error and the associated composite rule is 6th order accurate. For this method in its basic form four function evaluations and two first derivative evaluations are required. In composite form, where $N$ is an integer divisible by 5, the method requires $4N/5$ function evaluations and $2N/5$ derivative evaluations.

For $n = 4$,

$$\int_{x_{-1}}^{x_5} f(x)\,dx = h\left[\frac{-73}{40}f(x_0) + \frac{263}{35}f(x_1) - \frac{753}{140}f(x_2) + \frac{263}{35}f(x_3) - \frac{73}{40}f(x_4)\right] - \frac{123}{56}h^2\left[f'(x_0) - f'(x_4)\right] + \frac{773}{39200}h^9 f^{(8)}(\xi). \tag{32}$$

This scheme has precision 7 and 9th order local truncation error and the associated composite rule is 8th order accurate. For this method in its basic form five function evaluations and two first derivative evaluations are required. In composite form, where $N$ is an integer divisible by 6, the method requires $5N/6$ function evaluations and $N/3$ derivative evaluations.

*2.1.4. Open Newton-Cotes Rules with Derivative at Endpoints (ONC4).* The general form of the scheme is

$$\int_{x_{-1}}^{x_{n+1}} f(x)\,dx \approx \sum_{i=0}^{n} w_i f(x_i) + u_0 f'(x_{-1})h - u_1 f'(x_{n+1})h. \tag{33}$$

For $n = 0$, the quadrature rule is

$$\int_{x_{-1}}^{x_1} f(x)\,dx = 2hf(x_0) - \frac{1}{6}h^2\left[f'(x_{-1}) - f'(x_1)\right] - \frac{7}{180}h^5 f^{(4)}(\xi), \tag{34}$$

for $\xi \in (x_{-1}, x_1)$ which is the same as (21).

For $n = 1$, the four equations are given below:

$$\begin{aligned}
\int_{x_{-1}}^{x_2} dx &= x_2 - x_{-1} = w_0 + w_1, \\
\int_{x_{-1}}^{x_2} x\,dx &= \frac{\left(x_2^2 - x_{-1}^2\right)}{2} = w_0x_0 + w_1x_1 + u_0h - u_1h, \\
\int_{x_{-1}}^{x_2} x^2\,dx &= \frac{\left(x_2^3 - x_{-1}^3\right)}{3} = w_0x_0^2 + w_1x_1^2 + 2u_0x_{-1}h - 2u_1x_2h, \\
\int_{x_{-1}}^{x_2} x^3\,dx &= \frac{\left(x_2^4 - x_{-1}^4\right)}{4} = w_0x_0^3 + w_1x_1^3 + 3u_0x_{-1}^2h + -3u_1x_2^2h.
\end{aligned} \tag{35}$$

By solving these equations, we get the following rule:

$$\int_{x_{-1}}^{x_2} f(x)\,dx = \frac{3}{2}h\left[f(x_0) + f(x_1)\right] - \frac{h^2}{4}\left[f'(x_{-1}) - f'(x_2)\right] - \frac{13}{80}h^5 f^{(4)}(\xi), \tag{36}$$

for $\xi \in (x_{-1}, x_2)$. This scheme has precision 3 and 5th order local truncation error and the associated composite rule is 4th order accurate.

For $n = 2$, the quadrature rule is

$$\int_{x_{-1}}^{x_3} f(x)\,dx = \frac{1}{15}h\left[32f(x_0) - 4f(x_1) + 32f(x_2)\right] - \frac{2h^2}{15}\left[f'(x_{-1}) - f'(x_3)\right] - \frac{124}{4725}h^7 f^{(6)}(\xi), \tag{37}$$

for $\xi \in (x_{-1}, x_3)$. This scheme has precision 5 and 7th order local truncation error and the associated composite rule is 6th order accurate.

For $n = 3$, the quadrature rule is

$$\int_{x_{-1}}^{x_4} f(x)\,dx = \frac{1}{48}h\left[91f(x_0) + 29f(x_1) + 29f(x_2) + 91f(x_3)\right] - \frac{19h^2}{120}\left[f'(x_{-1}) - f'(x_4)\right] - \frac{4567}{60480}h^7 f^{(6)}(\xi), \tag{38}$$

for $\xi \in (x_{-1}, x_4)$. This scheme has precision 5 and 7th order local truncation error and the associated composite rule is 6th order accurate.

For $n = 4$, the quadrature rule is

$$\begin{aligned}\int_{x_{-1}}^{x_5} f(x)\,dx = h\Big[&\frac{12906}{5495}f(x_0) - \frac{837}{785}f(x_1) + \frac{18876}{5495}f(x_2) \\ &- \frac{837}{785}f(x_3) + \frac{12906}{5495}f(x_4)\Big] \\ &- \frac{246h^2}{2198}\left[f'(x_{-1}) - f'(x_5)\right] \\ &- \frac{28341}{1538600}h^9 f^{(8)}(\xi),\end{aligned} \tag{39}$$

for $\xi \in (x_{-1}, x_5)$. This scheme has precision 7 and 9th order local truncation error and the associated composite rule is 8th order accurate.

It may be observed that all these methods require only two additional first derivative evaluations in their basic as well as in the composite form due to cancellation of the derivatives at the endpoints in contrast to the classical open rules of the respective domain.

### 2.2. Error Analysis

*Error Analysis Approach 1.* To find error term, several methods can be used. Using the precision, the error term is related to the difference between the quadrature formula for the polynomial $x^{p+1}/(p+1)!$ and the exact result

$$\frac{1}{(p+1)!}\int_{x_{-1}}^{x_{n+1}} x^{p+1}dx, \tag{40}$$

where $p$ is the precision of the quadrature formula. This difference yields the coefficient for the $(p+1)$th derivative of $f(x)$.

*Error Analysis Approach 2.* The third approach is based on the interpolating polynomials. Since it is difficult to justify why to ignore the higher order monomials in the above mentioned approaches. The error analysis is carried out by using the classical approach of examining the integral of the remainder term of the Lagrange interpolating polynomial. Thus, we conjecture the following theorem for the error analysis of open derivative based quadrature by following an approach similar to [9].

**Theorem 1** (see [9]). *Suppose a function $f(x) \in C^{(M+n)}[a,b]$. Let $P(x)$ be the $(M+n)$th degree polynomial such that $f(x) = P(x)$ at $n+1$ locations $x_i \in [a,b]$ with $x_0 = a$ and $x_n = b$. At each location $x_i$, let $\alpha_i$ be the number of derivatives that are equal to zero at $x_i$ starting with the first derivative through the $\alpha_i$th derivative or*

$$\begin{aligned}&f'(x_i) = P'(x_i),\ f''(x_i) = P''(x_i), \ldots, \\ &f^{(\alpha_i)}(x_i) = P^{(\alpha_i)}(x_i),\end{aligned} \tag{41}$$

*with $M = \sum_{i=0}^{n}\alpha_i$. Then, there exists $\xi \in (a,b)$ such that*

$$f(x) = P(x) + \frac{f^{(M+n+1)}(\xi(x))}{(M+n+1)!}\prod_{i=0}^{n}(x-x_i)^{\alpha_i+1}. \tag{42}$$

**Theorem 2** (conjectured error theorem for derivative based numerical quadrature formulas). *Suppose $f \in C^{(M+n)}(a,b)$. Let $P(x)$ be a $(M+n)$th degree polynomial such that $f(x) = P(x)$ at $n+1$ locations $x_i \in (a,b)$ where $x_{-1} = a$ and $x_{n+1} = b$ at each location $x_i$; let $\alpha_i$ be the number of the first derivative of $f(x)$ and $P(x)$ that agree at $x_i$ starting with the first derivative through the $\alpha_i$th derivative or*

$$\begin{aligned}&f'(x_i) = P'(x_i);\ f''(x_i) = P''(x_i); \ldots; \\ &f^{(\alpha_i)}(x_i) = P^{(\alpha_i)}(x_i),\end{aligned} \tag{43}$$

*with $M = \sum_{i=0}^{n}\alpha_i$. Let $w_{M+n}(x) = \prod_{i=0}^{n}(x-x_i)^{\alpha_i+1}$. If $\int_a^b w_{(M+n)}(x)dx \neq 0$, then there exists a location $\xi \in (a,b)$ such that*

$$\int_a^b f(x)\,dx = \int_a^b P(x)\,dx + \frac{f^{(M+n+1)}(\xi)}{(M+n+1)!}\int_a^b w_{M+n}(x)\,dx. \tag{44}$$

*Let $w_{M+n}(x) = \prod_{i=0}^{n}(x-x_i)^{\alpha_i+1}$. If $\int_a^b w_{M+n}(x)dx = 0$, then there exists a location $\xi \in (a,b)$ such that*

$$\int_a^b f(x)\,dx = \int_a^b P(x)\,dx + \frac{f^{(M+n+2)}(\xi)}{(M+n+2)!}\int_a^b x w_{M+n}(x)\,dx. \tag{45}$$

This theorem can be used to calculate the error terms of our scheme as follows.

**Theorem 3** (function value and first derivative at each point $x_i \in (a,b)$). *Suppose $f \in C^{2n+2}(a,b)$. Let $P(x)$ be a $2n+1$ degree polynomial that agrees with $f(x)$ and the first derivative of $f(x)$ at $n+1$ locations $x_i \in (a,b)$ where $x_{-1} = a$ and $x_{n+1} = b$. Then, there exists a location $\xi \in (a,b)$ such that*

$$\int_a^b f(x)\,dx = \int_a^b P(x)\,dx + \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\int_a^b \prod_{i=0}^{n}(x-x_i)^2 dx. \tag{46}$$

*Proof.* Putting $M = n+1$ in (44) and using Theorem 1, we get the required result. □

**Theorem 4** (function value at each point $x_i \in (a,b)$ and first derivative at each point $x_i \in [a,b]$). *Suppose $f \in C^{2n+4}(a,b)$. Let $P(x)$ be a $2n+3$ degree polynomial that agrees with $f(x)$ and the first derivative of $f(x)$ at $n+3$ locations $x_i \in [a,b]$ where $x_{-1} = a$ and $x_{n+1} = b$. Then, there exists a location $\xi \in (a,b)$ such that*

$$\int_a^b f(x)\,dx = \int_a^b P(x)\,dx + \frac{f^{(2n+4)}(\xi)}{(2n+4)!}\int_a^b \prod_{i=-1}^{n+1}(x-x_i)^2 dx. \tag{47}$$

*Proof.* Putting $M = n+3$ in (44) and using Theorem 1, we get the required result. □

**Theorem 5** (function value at each point $x_i \in (a,b)$ and first derivative at $x_{-1} = a, x_{n+1} = b$). *Let $n$ be odd. Suppose $f \in$

Table 1: Coefficients of error term for each order of accuracy.

| Order | $N_0$ | ONC | $N_0$ | ONC1 | $N_0$ | ONC2 | $N_0$ | ONC3 | $N_0$ | ONC4 |
|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 3 | $\frac{14}{45} \approx 0.31$ | 2 | $\frac{7}{80} \approx 0.08$ | 1 | $\frac{7}{180} \approx 0.04$ | 2 | $\frac{7}{80} \approx 0.08$ | 1 | $\frac{7}{180} \approx 0.04$ |
| 6 | 5 | $\frac{41}{140} \approx 0.29$ | 3 | $\frac{107}{4725} \approx 0.02$ | 2 | $\frac{31}{6720} \approx 0.005$ | 3 | $\frac{107}{4725} \approx 0.02$ | 3 | $\frac{124}{4725} \approx 0.03$ |
| 8 | 7 | $\frac{3956}{14175} \approx 0.28$ | 4 | $\frac{5951}{1016064} \approx 0.01$ | 3 | $\frac{1601}{2182950} \approx 0.001$ | 5 | $\frac{773}{39200} \approx 0.02$ | 5 | $\frac{28341}{1538600} \approx 0.02$ |

*$C^{n+3}(a,b)$. Let $P(x)$ be a $n+2$ degree polynomial that agrees with $f(x)$ and the first derivative of $f(x)$ at 2 locations $x_{-1} = a$ and $x_{n+1} = b$. Then, there exists a location $\xi \in (a,b)$ such that*

$$\int_a^b f(x)\,dx = \int_a^b P(x)\,dx + \frac{f^{(n+3)}(\xi)}{(n+3)!}\int_a^b (x-a)(x-b) \times \prod_{i=-1}^{n+1}(x-x_i)\,dx. \tag{48}$$

*Let $n$ be even and suppose $f \in C^{n+4}(a,b)$. Let $P(x)$ be a $n+3$ degree polynomial; then*

$$\int_a^b f(x)\,dx = \int_a^b P(x)\,dx + \frac{f^{(n+4)}(\xi)}{(n+4)!} \times \int_a^b x(x-a)(x-b)\prod_{i=-1}^{n+1}(x-x_i)\,dx. \tag{49}$$

*Proof.* Putting $M = 2$ in (45) Theorem 1, we get the required result. □

**Theorem 6** (function value and higher order derivative at a single interval; that is, $x_{-1} = a, x_1 = b$.). *Suppose $f \in C^{2D+1}(a,b)$. Let $P(x)$ be a $2D$ degree polynomial that agrees with $f(x)$ and the first $D$ derivative of $f(x)$ at 2 locations $x_{-1} = a$ and $x_1 = b$. Then, there exists a location $\xi \in (a,b)$ such that*

$$\int_a^b f(x)\,dx = \int_a^b P(x)\,dx + \frac{f^{(2D+1)}(\xi)}{(2D+1)!} \times \int_a^b (x-a)^{D+1}(x-b)^{D+1}dx. \tag{50}$$

*Proof.* Using Theorem 1, the above theorem can be proved easily. □

We, now, show that these higher order derivatives based quadrature formulas are computationally efficient. The numerical results are obtained in Maple 13 using a precision of twenty decimal digits.

## 3. Computational Efficiency of Derivative Based Open Rules

Now, we will use the following notations in the comparison tables.

ONC for classical open numerical quadrature rule.

ONC1 for open numerical quadrature with derivative at all interior points.

ONC2 for open numerical quadrature with derivative at all points including endpoints.

ONC3 for open numerical quadrature with derivative used only at interior endpoints.

ONC4 for open numerical quadrature with derivative used only at exterior endpoints.

Table 1 compares the coefficient of the classical open Newton-Cotes formulas with newly developed open Newton-Cotes formulas with derivatives. The order described in the first column is the order of the composite formula; $N_0$ represents the number of nodes. The columns below each heading of the open type quadrature rules represent the coefficients of the error terms. It may be observed from the table that the new method ONC2 has prominently smaller error term coefficients.

Tables 2, 3, 4, 5 and 6 show the number of function and derivative evaluations for the classical open Newton-Cotes rules and the new derivative based open Newton-Cotes formulas, as well as their orders of accuracy. These results will be used in Tables 7 and 8 to determine the number of function and derivative evaluations sufficient to reduce the error in the calculations below some threshold. The results in Table 4 show that ONC2 is a highly attractive option because it has the highest order of accuracy in contrast to the other methods and is twice the value of N. Additionally, as shown in Table 1, the coefficients for the leading order error term for ONC2 decrease more rapidly than the other formulas. The results in Table 6 show that ONC4 is also a better option due to the increase in the order of accuracy by a factor of two over the classical open Newton-Cotes for mulas but includes only two additional derivative evaluations, regardless of the number of intervals for a certain scheme. Because ONC2 and ONC4 have clear advantages to ONC1 and ONC3, we will focus our analysis on these two methods.

We, now, give the following examples to compare the computational costs of the new rules and the classical rules.

*Example 1.*

$$\int_0^1 \frac{x\ln(1+x)}{1+x^2}dx, \tag{51}$$

Table 2: Computational cost for ONC (FE—Number of Function Evaluations, DE—Number of Derivative Evaluations).

| $N$ | Order | FE in basic form | DE in basic form | FE in composite form | DE in composite form |
|---|---|---|---|---|---|
| 2 | 2 | 1 | 0 | $\frac{N}{2}$ | 0 |
| 3 | 2 | 2 | 0 | $\frac{2N}{3}$ | 0 |
| 4 | 4 | 3 | 0 | $\frac{3N}{4}$ | 0 |
| 5 | 4 | 4 | 0 | $\frac{4N}{5}$ | 0 |
| 6 | 6 | 5 | 0 | $\frac{5N}{6}$ | 0 |

Table 3: Computational cost for ONC1.

| $N$ | Order | FE in basic form | DE in basic form | FE in composite form | DE in composite form |
|---|---|---|---|---|---|
| 2 | 2 | 1 | 0 | $\frac{N}{2}$ | 0 |
| 3 | 4 | 2 | 2 | $\frac{2N}{3}$ | $\frac{2N}{3}$ |
| 4 | 6 | 3 | 2 | $\frac{3N}{4}$ | $\frac{N}{2}$ |
| 5 | 8 | 4 | 4 | $\frac{4N}{5}$ | $\frac{4N}{5}$ |
| 6 | 10 | 5 | 4 | $\frac{5N}{6}$ | $\frac{2N}{3}$ |

Table 4: Computational cost for ONC2.

| $N$ | Order | FE in basic form | DE in basic form | FE in composite form | DE in composite form |
|---|---|---|---|---|---|
| 2 | 4 | 1 | 2 | $\frac{N}{2}$ | 2 |
| 3 | 6 | 2 | 4 | $\frac{2N}{3}$ | $\frac{2N}{3}+2$ |
| 4 | 8 | 3 | 4 | $\frac{3N}{4}$ | $\frac{N}{2}+2$ |
| 5 | 10 | 4 | 6 | $\frac{4N}{5}$ | $\frac{4N}{5}+2$ |
| 6 | 12 | 5 | 6 | $\frac{5N}{6}$ | $\frac{2N}{3}+2$ |

Table 5: Computational cost for ONC3.

| $N$ | Order | FE in basic form | DE in basic form | FE in composite form | DE in composite form |
|---|---|---|---|---|---|
| 2 | 2 | 1 | 0 | $\frac{N}{2}$ | 0 |
| 3 | 4 | 2 | 2 | $\frac{2N}{3}$ | $\frac{2N}{3}$ |
| 4 | 6 | 3 | 2 | $\frac{3N}{4}$ | $\frac{N}{2}$ |
| 5 | 6 | 4 | 2 | $\frac{4N}{5}$ | $\frac{2N}{5}$ |
| 6 | 8 | 5 | 2 | $\frac{5N}{6}$ | $\frac{N}{3}$ |

Table 6: Computational cost for ONC4.

| $N$ | Order | FE in basic form | DE in basic form | FE in composite form | DE in composite form |
|---|---|---|---|---|---|
| 2 | 4 | 1 | 2 | $\frac{N}{2}$ | 2 |
| 3 | 4 | 2 | 2 | $\frac{2N}{3}$ | 2 |
| 4 | 6 | 3 | 2 | $\frac{3N}{4}$ | 2 |
| 5 | 6 | 4 | 2 | $\frac{4N}{5}$ | 2 |
| 6 | 8 | 5 | 2 | $\frac{5N}{6}$ | 2 |

Table 7: Computational cost for evaluating $\int_0^1 (x\ln(1+x)/(1+x^2))dx$.

| Formula | Order | $N$ | FE in composite form | DE in composite form | Total |
|---|---|---|---|---|---|
| ONC | | | | | |
| $n=0$ | 2 | 18258 | 9129 | 0 | 9129 |
| $n=1$ | 2 | 22362 | 14908 | 0 | 14908 |
| $n=2$ | 4 | 188 | 141 | 0 | 141 |
| $n=3$ | 4 | 215 | 172 | 0 | 172 |
| $n=4$ | 6 | 60 | 50 | 0 | 50 |
| ONC2 | | | | | |
| $n=0$ | 4 | 134 | 67 | 2 | 69 |
| $n=1$ | 6 | 33 | 22 | 24 | 46 |
| $n=2$ | 8 | 20 | 15 | 12 | 27 |
| $n=3$ | 10 | 15 | 12 | 14 | 26 |
| $n=4$ | 12 | 18 | 15 | 14 | 29 |
| ONC4 | | | | | |
| $n=0$ | 4 | 134 | 67 | 2 | 69 |
| $n=1$ | 4 | 174 | 116 | 2 | 118 |
| $n=2$ | 6 | 40 | 30 | 2 | 32 |
| $n=3$ | 6 | 50 | 40 | 2 | 42 |
| $n=4$ | 8 | 24 | 20 | 2 | 22 |

*Example 2.*

$$\int_0^2 e^{-x^2}dx. \tag{52}$$

Tables 7 and 8 show the total number of evaluations required to guarantee that the numerical results for these integrals have an error of less than $10^{-9}$. These results are obtained from the error terms calculated previously. Here, $N$ represents the number of intervals involved in obtaining the required accuracy.

Assuming that the computational cost of evaluating the derivative of the function is the same as the cost of evaluating the function, the fourth order method in terms of computational cost is ONC2 ($n = 0$) which is the same formula as ONC4 ($n = 0$), which has roughly half the cost of the original fourth order open Newton-Cotes method. The best sixth order accurate method is ONC4 ($n = 2$), which is roughly 60% of the cost of the sixth order ONC method. ONC4 ($n = 4$) is slightly better than ONC2 ($n = 2$) as the best 8th order accurate method.

Tables 9 and 10 show the average CPU usage time required to process 100 simulations for the two above mentioned integrals using the classical open and new derivative based open methods ONC2 and ONC4. The time is measured in seconds.

## 4. Numerical Results

We, in this section, approximate the values of the following examples using the newly derived open derivative based formulas. These integrals contain rational, exponential, trigonometric, and logarithmic functions as integrands.

*Example 1.*

$$\int_0^1 \frac{x\ln(1+x)}{1+x^2}dx. \tag{53}$$

Table 8: Computational cost for evaluating $\int_0^2 e^{-x^2}dx$.

| Formula | Order | $N$ | FE in composite form | DE in composite form | Total |
|---|---|---|---|---|---|
| ONC | | | | | |
| $n = 0$ | 2 | 51640 | 25820 | 0 | 25820 |
| $n = 1$ | 2 | 63246 | 42164 | 0 | 42164 |
| $n = 2$ | 4 | 416 | 312 | 0 | 312 |
| $n = 3$ | 4 | 475 | 380 | 0 | 380 |
| $n = 4$ | 6 | 96 | 80 | 0 | 80 |
| ONC2 | | | | | |
| $n = 0$ | 4 | 294 | 147 | 2 | 149 |
| $n = 1$ | 6 | 54 | 36 | 38 | 74 |
| $n = 2$ | 8 | 28 | 21 | 16 | 37 |
| $n = 3$ | 10 | 20 | 16 | 18 | 34 |
| $n = 4$ | 12 | 18 | 15 | 14 | 29 |
| ONC4 | | | | | |
| $n = 0$ | 4 | 294 | 147 | 2 | 149 |
| $n = 1$ | 4 | 381 | 254 | 2 | 256 |
| $n = 2$ | 6 | 72 | 54 | 2 | 56 |
| $n = 3$ | 6 | 80 | 64 | 2 | 66 |
| $n = 4$ | 8 | 36 | 30 | 2 | 32 |

Table 9: Average CPU usage time for evaluating $\int_0^1 (x\ln(1+x)/(1+x^2))dx$.

| Formulas | Level of accuracy | |
|---|---|---|
| | $10^{-6}$ | $10^{-9}$ |
| ONC | | |
| $n = 0$ | 0.468 | 1769.468 |
| $n = 1$ | 0.525 | 2768.875 |
| $n = 2$ | 0.003 | 0.087 |
| $n = 3$ | 0.004 | 0.103 |
| $n = 4$ | 0.002 | 0.006 |
| ONC2 | | |
| $n = 0$ | 0.005 | 0.178 |
| $n = 1$ | 0.002 | 0.007 |
| $n = 2$ | 0.001 | 0.003 |
| $n = 3$ | 0.000 | 0.004 |
| $n = 4$ | 0.001 | 0.003 |
| ONC4 | | |
| $n = 0$ | 0.007 | 0.178 |
| $n = 1$ | 0.006 | 0.168 |
| $n = 2$ | 0.001 | 0.006 |
| $n = 3$ | 0.001 | 0.006 |
| $n = 4$ | 0.001 | 0.002 |

*Example 2.*

$$\int_0^2 e^{-x^2}dx. \tag{54}$$

Table 10: Average CPU usage time for evaluating $\int_0^2 e^{-x^2}dx$.

| Formulas | Level of accuracy | |
|---|---|---|
| | $10^{-6}$ | $10^{-9}$ |
| ONC | | |
| $n = 0$ | 0.315 | 3139.375 |
| $n = 1$ | 0.418 | 2485.484 |
| $n = 2$ | 0.002 | 0.107 |
| $n = 3$ | 0.006 | 0.120 |
| $n = 4$ | 0.002 | 0.003 |
| ONC2 | | |
| $n = 0$ | 0.005 | 0.172 |
| $n = 1$ | 0.002 | 0.002 |
| $n = 2$ | 0.001 | 0.003 |
| $n = 3$ | 0.001 | 0.003 |
| $n = 4$ | 0.000 | 0.002 |
| ONC4 | | |
| $n = 0$ | 0.006 | 0.182 |
| $n = 1$ | 0.007 | 0.193 |
| $n = 2$ | 0.002 | 0.002 |
| $n = 3$ | 0.001 | 0.004 |
| $n = 4$ | 0.001 | 0.003 |

*Example 3.*

$$\int_0^3 e^{-2x}\sin(4x)\,dx. \tag{55}$$

Table 11: Results and observed orders for the rules with order 4, for $\int_0^1 (x\ln(1+x)/(1+x^2))dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Three point ONC | 0.160017948 | 0.162787385 | 0.162860841 | 0.162864751 | 0.162864990 |
| $p$ | NA | 5.19 | 4.22 | 4.03 | 3.99 |
| Four point ONC | 0.160959871 | 0.162811573 | 0.162862116 | 0.162864829 | 0.162864995 |
| $p$ | NA | 5.16 | 4.21 | 4.03 | 4.01 |
| Two point ONC1 | 0.159436033 | 0.162772635 | 0.162860067 | 0.162864704 | 0.162864987 |
| $p$ | NA | 5.27 | 4.16 | 4.03 | 4.01 |
| One point ONC2 | 0.172602710 | 0.163164138 | 0.162881573 | 0.162866024 | 0.162865069 |
| $p$ | NA | 5.02 | 4.17 | 4.02 | 4.01 |
| Two point ONC3 | 0.159436034 | 0.162772635 | 0.162860067 | 0.162864704 | 0.162864987 |
| $p$ | NA | 5.21 | 4.22 | 4.31 | 4.01 |
| One point ONC4 | 0.172602710 | 0.163164138 | 0.162881573 | 0.162866024 | 0.162865069 |
| $p$ | NA | 5.02 | 4.17 | 4.02 | 4.01 |
| Two point ONC4 | 0.167979592 | 0.163027931 | 0.162874109 | 0.162865566 | 0.162865041 |
| $p$ | NA | 4.97 | 4.16 | 4.02 | 4.01 |

Table 12: Results and observed orders for the rules with order 4, for $\int_0^2 e^{-x^2} dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Three point ONC | 0.933680383 | 0.882318596 | 0.882095283 | 0.882082261 | 0.882081446 |
| $p$ | NA | 7.76 | 8.10 | 4.21 | NA |
| Four point ONC | 0.915679501 | 0.882245499 | 0.882091045 | 0.882081995 | 0.882081429 |
| $p$ | NA | 7.67 | 8.10 | 4.31 | NA |
| Two point ONC1 | 0.944928707 | 0.882363060 | 0.882097854 | 0.882082422 | 0.882081455 |
| $p$ | NA | 7.80 | 4.09 | 4.01 | 4.19 |
| One point ONC2 | 0.723548456 | 0.881147401 | 0.882025799 | 0.882077911 | 0.882081173 |
| $p$ | NA | 7.41 | 4.07 | 3.99 | 3.95 |
| Two point ONC3 | 0.944928707 | 0.882363060 | 0.882097854 | 0.882082422 | 0.882081455 |
| $p$ | NA | 7.80 | 4.09 | 4.01 | 4.19 |
| One point ONC4 | 0.723548456 | 0.881147401 | 0.882025799 | 0.882077911 | 0.882081173 |
| $p$ | NA | 7.41 | 4.07 | 3.99 | 3.95 |
| Two point ONC4 | 0.802053420 | 0.881569701 | 0.882050796 | 0.882079476 | 0.882081271 |
| $p$ | NA | 7.29 | 4.06 | 3.99 | 3.91 |

Table 13: Results and observed orders for the rules with order 4, for $\int_0^3 e^{-2x} \sin(4x)dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Three point ONC | 0.086043984 | 0.367562601 | 0.201398942 | 0.199657135 | 0.199709254 |
| $p$ | NA | NA | 6.64 | NA | 3.41 |
| Four point ONC | 0.269179529 | 0.310254438 | 0.200767736 | 0.199673399 | 0.199710886 |
| $p$ | NA | 0.67 | 6.71 | NA | 3.44 |
| Two point ONC1 | −0.04987788 | 0.403160349 | 0.201798239 | 0.199647441 | 0.199708266 |
| $p$ | NA | NA | 6.61 | NA | 3.93 |
| One point ONC2 | −1.53759876 | −0.31986670 | 0.195705275 | 0.199975525 | 0.199736732 |
| $p$ | NA | 1.74 | 7.02 | NA | 3.56 |
| Two point ONC3 | −0.04987788 | 0.403160349 | 0.201798239 | 0.199647441 | 0.199708266 |
| $p$ | NA | NA | 6.61 | NA | 3.93 |
| One point ONC4 | −1.53759876 | −0.31986670 | 0.195705275 | 0.199975525 | 0.199736732 |
| $p$ | NA | 1.74 | 7.02 | NA | 3.56 |
| Two point ONC4 | −1.12369531 | −0.06440215 | 0.197908079 | 0.199862681 | 0.199726870 |
| $p$ | NA | 2.32 | 7.19 | NA | 3.59 |

Table 14: Results and observed orders for the rules with order 6, for $\int_0^1 (x\ln(1+x)/(1+x^2))dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Five point ONC | 0.163095991 | 0.162866594 | 0.162865018 | 0.0162865006 | 0.162865006 |
| $p$ | NA | 7.18 | 7.05 | 6.91 | Exact |
| Six point ONC | 0.163025802 | 0.162866109 | 0.162865014 | 0.162865006 | 0.162865006 |
| $p$ | NA | 7.19 | 7.02 | 5.41 | Exact |
| Three point ONC1 | 0.163172754 | 0.162867123 | 0.162865022 | 0.162865006 | 0.162865006 |
| $p$ | NA | 7.18 | 7.06 | 5.94 | 4.64 |
| Two point ONC2 | 0.162426279 | 0.162861989 | 0.162864982 | 0.162865006 | 0.162865006 |
| $p$ | NA | 7.18 | 6.99 | 7.90 | Exact |
| Three point ONC3 | 0.163172754 | 0.162867123 | 0.162865022 | 0.162865006 | 0.162865006 |
| $p$ | NA | 7.18 | 7.06 | 5.99 | Exact |
| Four point ONC3 | 0.163046710 | 0.162866251 | 0.162865016 | 0.162865006 | 0.162865006 |
| $p$ | NA | 7.08 | 7.02 | 5.82 | Exact |
| Three point ONC4 | 0.162534900 | 0.162862736 | 0.162864988 | 0.162865006 | 0.162865006 |
| $p$ | NA | 7.18 | 6.96 | 6.24 | Exact |
| Four point ONC4 | 0.162669270 | 0.162863659 | 0.162864995 | 0.162865006 | 0.162865006 |
| $p$ | NA | 7.18 | 6.94 | 6.19 | Exact |

Table 15: Results and observed orders for the rules with order 6, for $\int_0^2 e^{-x^2} dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Five point ONC | 0.874932787 | 0.882079118 | 0.882081395 | 0.882081390 | 0.882081392 |
| $p$ | NA | 11.61 | 8.02 | 0.35 | Exact |
| Six point ONC | 0.877191468 | 0.882079849 | 0.882081394 | 0.882081391 | 0.882081391 |
| $p$ | NA | 11.63 | 4.29 | Exact | Exact |
| Three point ONC1 | 0.872471734 | 0.882078323 | 0.882081396 | 0.882081393 | 0.882081393 |
| $p$ | NA | 11.61 | 9.96 | Exact | Exact |
| Two point ONC2 | 0.894922357 | 0.882085385 | 0.882081384 | 0.882081391 | 0.882081391 |
| $p$ | NA | 6.57 | 11.56 | Exact | Exact |
| Three point ONC3 | 0.872471734 | 0.882078323 | 0.882081396 | 0.882081393 | 0.882081393 |
| $p$ | NA | 11.61 | 9.96 | Exact | Exact |
| Four point ONC3 | 0.876567697 | 0.882079654 | 0.882081394 | 0.882081391 | 0.882081391 |
| $p$ | NA | 11.38 | 8.42 | Exact | Exact |
| Three point ONC4 | 0.891653998 | 0.882084357 | 0.882081386 | 0.882081391 | 0.882081391 |
| $p$ | NA | 11.66 | NA | Exact | Exact |
| Four point ONC4 | 0.887662361 | 0.882083108 | 0.882081388 | 0.882081391 | 0.882081391 |
| $p$ | NA | 11.67 | NA | Exact | Exact |

Table 16: Results and observed orders for the rules with order 6, for $\int_0^3 e^{-2x} \sin(4x)dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Five point ONC | 0.668675155 | 0.180576440 | 0.199344558 | 0.199710595 | 0.199714605 |
| $p$ | NA | NA | 5.69 | 6.50 | 6.15 |
| Six point ONC | 0.506621196 | 0.186032806 | 0.199455606 | 0.199711798 | 0.199714622 |
| $p$ | NA | NA | 5.72 | 6.50 | 6.17 |
| Three point ONC1 | 0.848922696 | 0.174588551 | 0.199223240 | 0.199709279 | 0.199714586 |
| $p$ | NA | NA | 5.67 | 6.51 | 6.14 |
| Two point ONC2 | −0.425713982 | 0.239513474 | 0.200436683 | 0.199722775 | 0.199714778 |
| $p$ | NA | NA | 5.78 | 6.47 | 6.13 |
| Three point ONC3 | 0.848922696 | 0.174588551 | 0.199223240 | 0.199709279 | 0.199714586 |
| $p$ | NA | NA | 5.67 | 6.51 | 6.14 |
| Four point ONC3 | 0.548955957 | 0.184222649 | 0.199422043 | 0.199714258 | 0.199714616 |
| $p$ | NA | NA | 5.72 | 9.49 | 6.13 |
| Three point ONC4 | −0.023868456 | 0.230076741 | 0.200260208 | 0.199720813 | 0.199714749 |
| $p$ | NA | NA | 5.79 | 6.47 | 6.15 |
| Four point ONC4 | −0.02248199 | 0.218175909 | 0.200040826 | 0.199718362 | 0.199714715 |
| $p$ | NA | NA | 5.82 | 6.46 | 6.13 |

Table 17: Results and observed orders for the rules with order 8, for $\int_0^1 (x\ln(1+x)/(1+x^2))dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Seven point ONC | 0.162856790 | 0.162864940 | 0.162865006 | 0.162865006 | 0.162865006 |
| $p$ | NA | 6.96 | Exact | Exact | Exact |
| Four point ONC1 | 0.162853419 | 0.1628649100 | 0.1628650058 | 0.1628650059 | 0.1628650059 |
| $p$ | NA | 6.92 | 9.41 | Exact | Exact |
| Three point ONC2 | 0.162877298 | 0.1628650905 | 0.1628650060 | 0.1628650059 | 0.1628650059 |
| $p$ | NA | 7.18 | 9.59 | Exact | Exact |
| Five point ONC3 | 0.162857063 | 0.162864944 | 0.162865006 | 0.162865006 | 0.162865006 |
| $p$ | NA | 7.01 | Exact | Exact | Exact |
| Five point ONC4 | 0.162873175 | 0.162865061 | 0.162865006 | 0.162865006 | 0.162865006 |
| $p$ | NA | 7.21 | Exact | Exact | Exact |

Table 18: Results and observed orders for the rules with order 8, for $\int_0^2 e^{-x^2}dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Seven point ONC | 0.882649378 | 0.882081616 | 0.882081390 | 0.882081396 | 0.882081393 |
| $p$ | NA | NA | NA | NA | 1.39 |
| Four point ONC1 | 0.882908200 | 0.882081724 | 0.882081389 | 0.882081391 | 0.882081391 |
| $p$ | NA | 11.31 | NA | Exact | Exact |
| Three point ONC2 | 0.881359024 | 0.882081118 | 0.882081391 | 0.882081391 | 0.882081391 |
| $p$ | NA | 11.33 | 11.37 | Exact | Exact |
| Five point ONC3 | 0.882611232 | 0.882081598 | 0.882081390 | 0.882081390 | 0.882081392 |
| $p$ | NA | 11.62 | 8.42 | Exact | Exact |
| Five point ONC4 | 0.881613287 | 0.882081217 | 0.882081391 | 0.882081391 | 0.882081391 |
| $p$ | NA | 11.32 | Exact | Exact | Exact |

Table 19: Results and observed orders for the rules with order 8, for $\int_0^3 e^{-2x}\sin(4x)dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Seven point ONC | 0.116217345 | 0.199107249 | 0.199722936 | 0.199714694 | 0.199714661 |
| $p$ | NA | 7.10 | NA | 8.02 | 7.98 |
| Four point ONC1 | 0.086930244 | 0.198791095 | 0.199726603 | 0.199714706 | 0.199714662 |
| $p$ | NA | 6.93 | NA | 8.07 | Exact |
| Three point ONC2 | 0.345139247 | 0.200290872 | 0.199703568 | 0.199714621 | 0.199714662 |
| $p$ | NA | 7.97 | 7.94 | 8.08 | Exact |
| Five point ONC3 | 0.114010828 | 0.199174615 | 0.199722444 | 0.199714691 | 0.199714662 |
| $p$ | NA | NA | 5.72 | 6.49 | Exact |
| Five point ONC4 | 0.297964032 | 0.200061133 | 0.199707419 | 0.199714636 | 0.199714662 |
| $p$ | NA | 8.14 | NA | 8.11 | Exact |

Table 20: Results and observed orders for the rules with order 10, for $\int_0^1 (x\ln(1+x)/(1+x^2))dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Nine point ONC | 1.295725718 | 1.143369947 | 1.097799574 | 1.075658649 | 1.064314363 |
| $p$ | NA | 0.20 | 0.07 | 0.03 | 0.02 |
| Five point ONC1 | 0.162863548 | 0.162865010 | 0.162865006 | 0.162865006 | 0.162865006 |
| $p$ | NA | NA | Exact | Exact | Exact |
| Four point ONC2 | 0.1628657293 | 0.1628650035 | 0.1628650059 | 0.1628650059 | 0.1628650059 |
| $p$ | NA | NA | Exact | Exact | Exact |

Table 21: Results and observed orders for the rules with order 10, for $\int_0^2 e^{-x^2} dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Nine point ONC | 3.470923242 | 5.116115937 | 5.430111149 | 5.567289203 | 5.634338380 |
| $p$ | NA | −0.71 | −0.10 | −0.04 | −0.02 |
| Five point ONC1 | 0.882047084 | 0.882081362 | 0.882081390 | 0.882081389 | 0.882081391 |
| $p$ | NA | 9.85 | 2.07 | 0.41 | Exact |
| Four point ONC2 | 0.882104655 | 0.882081409 | 0.882081391 | 0.882081391 | 0.882081391 |
| $p$ | NA | 11.12 | Exact | Exact | Exact |

Table 22: Results and observed orders for the rules with order 10, for $\int_0^3 e^{-2x} \sin(4x) dx$.

| $N$ | 1 | 2 | 4 | 8 | 16 |
|---|---|---|---|---|---|
| Nine point ONC | 0.526431102 | −0.041135432 | 1.51645849 | 1.38389260 | 1.31653544 |
| $p$ | NA | NA | NA | 0.15 | 0.08 |
| Five point ONC1 | 0.152040549 | 0.199941942 | 0.199714629 | 0.199714662 | 0.199714662 |
| $p$ | NA | NA | NA | Exact | Exact |
| Four point ONC2 | 0.219131222 | 0.199573932 | 0.199714685 | 0.199714662 | 0.199714662 |
| $p$ | NA | NA | 7.36 | Exact | Exact |

We evaluate Examples 1–3 in Tables 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, and 22 with their observed order of accuracy. The order of accuracy is calculated by the formula given as

$$p = \frac{\ln\left(\left|N(2h) - N(0)\right| / \left|N(h) - N(0)\right|\right)}{\ln 2}, \qquad (56)$$

where $N(h)$ means the numerical result with the step size $h$. $N(0)$ is the exact value of the definite integrals. Thus, $N(0)$ is 0.162865005 in the first example, 0.882081390 in the second, and 0.199714662 in the third example. The comparison is done with the classical open Newton-Cotes rules of the respective order of accuracy. The first row indicates the partitioning of the interval into subintervals. $p$ represents the observed order of accuracy and the results in line with each heading of Open Newton-Cotes rules represent the approximated value of the definite integral calculated through that formula.

## 5. Conclusion

We have used several approaches to construct the derivative based open Newton-Cotes formulas. These new schemes represent four different families of new derivative based open Newton-Cotes type quadrature formulas. Using the concept of precision, we determine the coefficients for the function and its derivative at different locations within and at the boundary of the integral and determine the error term for each method from $n = 0$ through $n = 4$ where $n$ is the number of interior points.

Because of the beneficial cancellation of the derivatives at the boundaries of the intervals, the most attractive methods are ONC2 (open Newton-Cotes with derivatives at all nodes including endpoints) and ONC4 (open Newton-Cotes with derivatives only at the end points). When comparing the computational cost of these two methods with that of the same order open Newton-Cotes methods (ONC), the ONC2 methods are consistently more efficient than the original ONC methods, and the ONC4 methods are the most computationally efficient in producing a numerical result that is guaranteed to be accurate to within $10^{-9}$. These conclusions are obtained from both the theoretical order of accuracy and error terms as well as from computational evidence.

Finally, the order of accuracy for all of the methods is confirmed computationally against several different integrals, showing that these methods are either comparable to or better than the classical open Newton-Cotes formula.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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