

Research Article

Bifurcation Analysis and Control of a Differential-Algebraic Predator-Prey Model with Allee Effect and Time Delay

Xue Zhang and Qing-ling Zhang

Department of Science, Northeastern University, No. 3-11, Wenhua Road, Heping District, Shenyang, Liaoning 110004, China

Correspondence should be addressed to Xue Zhang; zhangxueer@gmail.com

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This paper studies systematically a differential-algebraic prey-predator model with time delay and Allee effect. It shows that transcritical bifurcation appears when a variation of predator handling time is taken into account. This model also exhibits singular induced bifurcation as the economic revenue increases through zero, which causes impulsive phenomenon. It can be noted that the impulsive phenomenon can be much weaker by strengthening Allee effect in numerical simulation. On the other hand, at a critical value of time delay, the model undergoes a Hopf bifurcation; that is, the increase of time delay destabilizes the model and bifurcates into small amplitude periodic solution. Moreover, a state delayed feedback control method, which can be implemented by adjusting the harvesting effort for biological populations, is proposed to drive the differential-algebraic system to a steady state. Finally, by using Matlab software, numerical simulations illustrate the effectiveness of the results.

1. Introduction

In recent years, the growing human needs for more food and more energy have led to increased exploitation of these resources. The problems related to many fields like fishery, forestry, and wildlife. Therefore, mankind is facing the dual problems of resource shortages and environmental degradation. Concerning the conservation for the long-term benefits of humanity, there is a wide-range of interest in analysis and modelling of bioeconomic systems. In many earlier studies, it has been shown that harvesting has a strong impact on population dynamics, ranging from rapid depletion to complete preservation of biological populations. Two main kinds of harvesting were focused on nonzero constant harvesting [1–3] and constant harvesting effort [4–7]. With constant harvesting, the generalized Gause prey-predator model is found to exhibit saddle-node bifurcations, Hopf bifurcation, heteroclinic bifurcation, and nilpotent saddle bifurcation [1]. Xiao et al. [2] have investigated the dynamical properties of a ratio-dependent predator-prey model with nonzero constant rate predator harvesting. These results reveal far richer dynamics compared to the model with no harvesting.

The literature [3] shows that harvesting effort as control parameter is not only possible to control the cyclic behavior of populations leading to the persistence of all species, but other desired stable equilibrium including disease-free can also be obtained. Das et al. [4] discussed the bioeconomic harvesting of a prey-predator fishery in which both species are infected by some toxicant. Ji and Wu [5] studied a predator-prey model with a constant-rate prey harvesting incorporating a constant prey refuge, where the influence of harvesting effort on the density of two species was discussed. Chakraborty et al. [6] describes a prey-predator model with stage structure for predator and selective harvesting effort on predator population. Geometric approach is used to drive the sufficient conditions for global stability of the system, and fishing effort is used to investigate the optimal utilization of the resource. Xiang et al. [7] consider a Lotka-Volterra model with impulsive harvest for the prey and investigate globally attractive periodic solution. However, most of these discussions are only based on differential equations or difference equations.

In power systems, neural networks, and genetic networks [8–12], differential-algebraic equations have been studied

widely and a lot of results have been obtained, such as local stability, optimal control, singularity induced bifurcation, and feasibility regions. From 2009, several differential-algebraic biological models were reported [13–16]. Surprisingly enough, all the existing differential-algebraic biological modelling literature considers only the simplest case of a logistic prey growth function. However, numerous examples demonstrate that the growth of natural populations can exhibit Allee effect, which is a phenomenon in biology named after Allee [17]. Allee effect describes a positive relation between population density and the per capita growth rate. In other words, for smaller populations, the reproduction and survival of individuals decrease. This effect usually saturates or disappears as populations get larger. The effect may be due to any number of causes, for example, mate finding, social dysfunction, inbreeding depression, food exploitation, and predator avoidance or defense.

On the other hand, since reproduction of predator after consuming prey is not instantaneous in most cases, some time lag for gestation is required. Therefore, in this paper, we consider a differential-algebraic prey-predator model with time delay and the Allee effect on the growth of the prey population. We analyze the stability properties and bifurcation behavior of this model. A state delayed feedback control method is also proposed, which can eliminate Hopf bifurcation and drive the differential-algebraic prey-predator model to stay at a steady state.

2. Model Equations

The general predator-prey model in its classical form is represented by

$$\frac{d\tilde{x}}{d\tilde{t}} = \tilde{x}p(\tilde{x}) - \tilde{y}q(\tilde{x}), \quad \frac{d\tilde{y}}{d\tilde{t}} = \beta\tilde{y}q(\tilde{x}) - d_2\tilde{y}, \quad (1)$$

where \tilde{x} and \tilde{y} represent the prey density and predator density at time \tilde{t} , respectively; $p(\tilde{x})$ is the per capita growth rate of prey in absence of predator; d_2 is the intrinsic mortality rate of predator in the absence of food; $q(\tilde{x})$ is predator's functional response, defined as the amount of prey catch per predator per unit of time; β ($0 < \beta < 1$) is the rate of conversion of nutrients into the reproduction rate of the predator.

In 1954, Gordon [18] studied the effect of harvest effort on ecosystem from an economic perspective and proposed the following economic theory:

$$\begin{aligned} & \text{Net Economic Revenue (NER)} \\ & = \text{Total Revenue (TR)} - \text{Total Cost (TC)}. \end{aligned} \quad (2)$$

Based on this theory, this paper studies a class of delayed differential-algebraic predator-prey model with Allee effect

on prey species and Holling-II functional response, which is written in the following form:

$$\begin{aligned} \frac{d\tilde{x}}{d\tilde{t}} &= \tilde{x}(s(\tilde{x}) - m_1 - a\tilde{x}(\tilde{t} - \tilde{\tau})) - \frac{\alpha\tilde{x}\tilde{y}}{1 + \alpha\tilde{h}\tilde{x}}, \\ \frac{d\tilde{y}}{d\tilde{t}} &= \frac{\beta\alpha\tilde{x}(\tilde{t} - \tilde{\tau})\tilde{y}}{1 + \alpha\tilde{h}\tilde{x}(\tilde{t} - \tilde{\tau})} - m_2\tilde{y} - \tilde{E}\tilde{y}, \\ 0 &= \tilde{E}(p\tilde{y} - c) - m, \end{aligned} \quad (3)$$

where $s(\tilde{x})$ is the fertility rate of prey species. m_1 and m_2 are the intrinsic mortality rate of the prey and predator species, respectively. a is the strength of intracompetition of prey population. α is the attack coefficient and \tilde{h} is the handling time. β denotes food utilization efficiency. The predator takes time $\tilde{\tau}$ to convert the food into its growth; \tilde{E} is harvesting effort for predator, $p > 0$, $c > 0$, and $m > 0$ are harvesting reward per unit harvesting effort for unit weight of predator, harvesting cost per unit harvesting effort for predator, and the net economic revenue per unit harvesting effort, respectively. All the parameters are positive constants.

Let the fertility rate $s(\tilde{x})$ increase with population density and be described by

$$s(\tilde{x}) = \frac{r\tilde{x}}{\tilde{A} + \tilde{x}}, \quad (4)$$

where $r > 0$ and \tilde{A} are the per capita maximum fertility rate and the Allee effect constant of the prey species, respectively. If $\tilde{A} > 0$, the fertility of the species is zero when \tilde{x} is zero and approaches to r when \tilde{x} becomes very large. The increasing of $s(\tilde{x})$ depends on the parameter \tilde{A} . The larger \tilde{A} is, the stronger Allee effect will be. In particular, the fertility rate is density independent when $\tilde{A} = 0$; that is, $s(\tilde{x}) = r$.

When the prey population is subject to Allee effect above, the predator-prey model (3) becomes

$$\begin{aligned} \frac{d\tilde{x}}{d\tilde{t}} &= \tilde{x} \left(\frac{r\tilde{x}}{\tilde{A} + \tilde{x}} - m_1 - a\tilde{x}(\tilde{t} - \tilde{\tau}) \right) - \frac{\alpha\tilde{x}\tilde{y}}{1 + \alpha\tilde{h}\tilde{x}}, \\ \frac{d\tilde{y}}{d\tilde{t}} &= \frac{\beta\alpha\tilde{x}(\tilde{t} - \tilde{\tau})\tilde{y}}{1 + \alpha\tilde{h}\tilde{x}(\tilde{t} - \tilde{\tau})} - m_2\tilde{y} - \tilde{E}\tilde{y}, \\ 0 &= \tilde{E}(p\tilde{y} - c) - m. \end{aligned} \quad (5)$$

Assume that the per capita maximum fertility rate of prey must exceed its death rate, that is, $r > m_1$; otherwise, both prey and predator will become extinct. In addition, the maximum growth rate of predator population must exceed its death rate, that is, $\beta/\tilde{h} > m_2$. If not, prey population will never be able to sustain predator population.

3. Qualitative Analysis

We nondimensionalize the model (5) with the following scaling:

$$x = \frac{a\tilde{x}}{r}, \quad y = \frac{\alpha\tilde{y}}{r}, \quad t = r\tilde{t}, \quad (6)$$

and then obtain the following form:

$$\begin{aligned} \frac{dx}{dt} &= x \left(\frac{x}{A+x} - d_1 - x(t-\tau) \right) - \frac{xy}{1+hx}, \\ \frac{dy}{dt} &= y \left(\frac{bx(t-\tau)}{1+hx(t-\tau)} - d_2 \right) - Ey, \\ 0 &= E(vy - w) - m, \end{aligned} \tag{7}$$

where the nondimensional parameters are defined as

$$\begin{aligned} A &= \frac{a\tilde{A}}{r}, & d_1 &= \frac{m_1}{r}, & h &= \frac{\alpha r \tilde{h}}{a}, & b &= \frac{\alpha \beta}{a}, \\ d_2 &= \frac{m_2}{r}, & E &= \frac{\tilde{E}}{r}, & v &= \frac{pr^2}{\alpha}, & w &= rc, \\ & & & & \tau &= r\tilde{\tau}. \end{aligned} \tag{8}$$

From the assumption mentioned above, the following reasonable condition must be satisfied:

$$0 < d_1 < 1. \tag{9}$$

For simplicity of computation, we consider the above model (7) instead of the model (5). Hence, we will perform a qualitative analysis of the model (7).

Let

$$\begin{aligned} f(X, E, \mu) &= \begin{pmatrix} f_1(X, E, \mu) \\ f_2(X, E, \mu) \end{pmatrix} \\ &= \begin{pmatrix} x \left(\frac{x}{A+x} - d_1 - x(t-\tau) \right) - \frac{xy}{1+hx} \\ y \left(\frac{bx(t-\tau)}{1+hx(t-\tau)} - d_2 \right) - Ey \end{pmatrix}, \\ g(X, E, \mu) &= E(vy - w) - m = 0, \end{aligned} \tag{10}$$

where $X = (x, y)^T$ is the state and μ denotes bifurcation parameter.

$$J = D_X f - D_E f (D_E g)^{-1} D_X g = \begin{pmatrix} \frac{x(x+2A)}{(A+x)^2} - d_1 - x - xe^{-\lambda\tau} - \frac{y}{(1+hx)^2} & -\frac{x}{1+hx} \\ \frac{bye^{-\lambda\tau}}{(1+hx)^2} & \frac{bx}{1+hx} - d_2 - E + \frac{vEy}{vy-w} \end{pmatrix}. \tag{15}$$

Hence, the Jacobian matrix of the model (11) at Q_0 is

$$J_{Q_0} = \begin{pmatrix} -d_1 & 0 \\ 0 & -d_2 \end{pmatrix}. \tag{16}$$

3.1. *The Model (7) with Zero Economic Profit.* Considering zero economic revenue, the model (7) can be reduced as follows:

$$\begin{aligned} \frac{dx}{dt} &= x \left(\frac{x}{A+x} - d_1 - x(t-\tau) \right) - \frac{xy}{1+hx}, \\ \frac{dy}{dt} &= y \left(\frac{bx(t-\tau)}{1+hx(t-\tau)} - d_2 \right) - Ey, \\ 0 &= E(vy - w). \end{aligned} \tag{11}$$

By the analysis of roots for the model (11), we obtain the following result.

Theorem 1. (1) *The model (11) has a trivial equilibrium point $Q_0(0, 0, 0)$ for any positive parameters.*

(2) *There exist two boundary equilibrium points $Q_1(x_1, 0, 0)$ and $Q_2(x_2, 0, 0)$ if $1 - A - d_1 > 2\sqrt{Ad_1}$, where*

$$\begin{aligned} x_1 &= \frac{1}{2} \left(1 - A - d_1 - \sqrt{(1 - A - d_1)^2 - 4Ad_1} \right), \\ x_2 &= \frac{1}{2} \left(1 - A - d_1 + \sqrt{(1 - A - d_1)^2 - 4Ad_1} \right). \end{aligned} \tag{12}$$

(3) *There exists another boundary equilibrium point $Q_3(x_3, y_3, 0)$, where $x_3 = d_2/\delta$, $y_3 = b[\delta d_2 - (d_1\delta + d_2)(A\delta + d_2)]/\delta^2(A\delta + d_2)$, $\delta = b - d_2h$, if the following conditions are satisfied:*

$$\begin{aligned} 0 < d_1 < 1, & \quad b > \frac{d_2(1 + (1 - d_1)h)}{1 - d_1}, \\ 0 < A < & \frac{d_2(\delta(1 - d_1) - d_2)}{\delta(d_2 + d_1\delta)}. \end{aligned} \tag{13}$$

(4) *The model (11) has a positive equilibrium point $Q_4(x_4, y_4, E_4)$, where $y_4 = w/v$, $E_4 = (bx_4)/(1 + hx_4) - d_2$, and x_4 is the root of the following equation:*

$$\begin{aligned} vhx^3 &+ v(h(d_1 + A - 1) + 1)x^2 \\ &+ (v(d_1Ah + d_1 + A - 1) + w)x \\ &+ d_1Av + wA = 0. \end{aligned} \tag{14}$$

The Jacobian matrix of the model (11) takes the following form:

Theorem 2. Q_0 is always a locally stable node.

From Theorem 2, it can be seen that both prey and predator populations will become extinct when their population densities lie in the attraction region of Q_0 .

Note that $x_1/(A + x_1) - d_1 - x_1 = 0$, then the Jacobian matrix of the model (11) at Q_1 is

$$J_{Q_1} = \begin{pmatrix} \frac{x_1}{A + x_1} (1 - d_1 - x_1 - (A + x_1)e^{-\lambda\tau}) & -\frac{x_1}{1 + hx_1} \\ 0 & \frac{bx_1}{1 + hx_1} - d_2 \end{pmatrix}. \tag{17}$$

Since

$$1 - d_1 - x_1 - (A + x_1)e^{-\lambda\tau} > 1 - d_1 - 2x_1 - A = \sqrt{(1 - A - d_1)^2 - 4Ad_1} > 0 \tag{18}$$

and the other eigenvalue of J_{Q_1} is $bx_1/(1 + hx_1) - d_2$, we obtain the following results on the stability of Q_1 .

Theorem 3. Assume that $0 < d_1 < 1$ and $0 < A < (1 - \sqrt{d_1})^2$. Then, for any time delay $\tau \geq 0$, one has the following.

(1) Q_1 is a saddle point if

$$h > \frac{b}{d_2} - \frac{2}{1 - A - d_1 - \sqrt{(1 - A - d_1)^2 - 4Ad_1}}. \tag{19}$$

(2) Q_1 is an unstable node if

$$0 < h < \frac{b}{d_2} - \frac{2}{1 - A - d_1 - \sqrt{(1 - A - d_1)^2 - 4Ad_1}}. \tag{20}$$

Next, we analyze the stability of the equilibrium point Q_2 . The Jacobian matrix of the model (11) at Q_2 is

$$J_{Q_2} = \begin{pmatrix} \frac{x_2}{A + x_2} (1 - d_1 - x_2 - (A + x_2)e^{-\lambda\tau}) & -\frac{x_2}{1 + hx_2} \\ 0 & \frac{bx_2}{1 + hx_2} - d_2 \end{pmatrix}. \tag{21}$$

For Q_2 , one eigenvalue is $bx_2/(1 + hx_2) - d_2$, and the other is given by the equation

$$\lambda - \frac{x_2}{A + x_2} (1 - d_1 - x_2 - (A + x_2)e^{-\lambda\tau}) = 0. \tag{22}$$

By simple analysis, the root of (22) is negative if the time delay satisfies $\tau < \tau_0$, where

$$\tau_0 = \frac{A + x_2}{x_2 \sqrt{(1 - d_1 + A)(A - 1 + d_1 + 2x_2)}} \times \arccos \frac{1 - d_1 - x_2}{A + x_2}. \tag{23}$$

Then, we have the following results on the stability of Q_2 .

Theorem 4. Assume that $0 < d_1 < 1$, $0 < A < (1 - \sqrt{d_1})^2$, and $\tau < \tau_0$. Then, one has the following.

(1) Q_2 is a saddle point if

$$0 < h < \frac{b}{d_2} - \frac{2}{1 - A - d_1 + \sqrt{(1 - A - d_1)^2 - 4Ad_1}}. \tag{24}$$

(2) Q_2 is locally asymptotically stable if

$$h > \frac{b}{d_2} - \frac{2}{1 - A - d_1 + \sqrt{(1 - A - d_1)^2 - 4Ad_1}}. \tag{25}$$

Based on Theorem 4, the following bifurcation result can be obtained.

Theorem 5. Assume that $0 < d_1 < 1$, $0 < A < (1 - \sqrt{d_1})^2$, and $\tau < \tau_0$. Then, the model (11) undergoes transcritical bifurcation at the equilibrium point Q_2 when bifurcation parameter h increases through $h_0 = b/d_2 - 2/(1 - A - d_1 + \sqrt{(1 - A - d_1)^2 - 4Ad_1})$.

Proof. When $\tau < \tau_0$ and the bifurcation parameter $h = h_0$, the characteristic polynomial at the equilibrium point Q_2 has a simple zero eigenvalue with left null vector

$$u_{\text{trans}} = (0 \ 1) \tag{26}$$

and right null vector

$$v_{\text{trans}} = \left(\frac{x_2}{1 + hx_2} \ \frac{x_2}{A + x_2} (1 - d_1 - 2x_2 - A) \right)^T,$$

$$u_{\text{trans}}(D_X D_h f_R)v_{\text{trans}}|_{Q_2}$$

$$= (0 \ 1) \begin{pmatrix} \frac{2x_2 y_2}{(1 + hx_2)^3} & \frac{x_2^2}{(1 + hx_2)^2} \\ -\frac{2bx_2 y_2}{(1 + hx_2)^3} & -\frac{bx_2^2}{(1 + hx_2)^2} \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{x_2}{1 + hx_2} \\ \frac{x_2(1 - d_1 - 2x_2 - A)}{A + x_2} \end{pmatrix}$$

$$= -\frac{bx_2^3(1 - d_1 - 2x_2 - A)}{(1 + hx_2)^2(A + x_2)},$$

$$u_{\text{trans}} D_X^2 f_R(v_{\text{trans}}, v_{\text{trans}})|_{P_0} = (0 \ 1) \begin{pmatrix} -\frac{x_2}{(1 + hx_2)^3} \\ \frac{bx_2}{(1 + hx_2)^3} \end{pmatrix} = \frac{bx_2}{(1 + hx_2)^3}. \tag{27}$$

According to the literature [19], the model (11) undergoes transcritical bifurcation at the equilibrium point Q_2 . This completes the proof. \square

Remark 6. Transcritical bifurcation implies that the equilibrium point Q_2 remains stable if the handling time is longer than the critical point h_0 , otherwise the stability of Q_2 is lost. From the view of biological explanation, stable equilibrium

point Q_2 means that predator population is to be extinct, which results from longer handling time reducing the amount of prey catch per predator per unit of time.

The Jacobian matrix of the model (11) at Q_3 is

$$J_{Q_3} = \begin{pmatrix} \frac{x_3(x_3 + 2A)}{(A + x_3)^2} - d_1 - x_3 - \frac{y_3}{(1 + hx_3)^2} - x_3 e^{-\lambda\tau} & -\frac{x_3}{1 + hx_3} \\ \frac{by_3 e^{-\lambda\tau}}{(1 + hx_3)^2} & 0 \end{pmatrix}. \tag{28}$$

The corresponding characteristic equation is

$$\lambda^2 - \left(\frac{Ax_3}{(A + x_3)^2} + \frac{hx_3 y_3}{(1 + hx_3)^2} \right) \lambda + \left(x_3 \lambda + \frac{bx_3 y_3}{(1 + hx_3)^3} \right) e^{-\lambda\tau} = 0. \tag{29}$$

In the absence of delay, since the constant term of the above equation is always positive, the sign of the following expression:

$$x_3 - \frac{Ax_3}{(A + x_3)^2} - \frac{hx_3 y_3}{(1 + hx_3)^2} \tag{30}$$

determines the stability of equilibrium Q_3 . Moreover, if the following condition:

$$\delta(4bd_1 d_2 h - (b + d_2 h)^2) + 4bd_2(b + d_2 h) > 0 \tag{31}$$

is satisfied, it is clear that Q_3 is always locally asymptotically stable for the model (11) in the absence of delay.

In the presence of delay, assume that a purely imaginary solution of the form $\lambda = i\omega_0$ exists for the above characteristic equation, where ω_0 is the root of the following equation:

$$\omega^4 + \left[\left(\frac{Ax_3}{(A + x_3)^2} + \frac{hx_3 y_3}{(1 + hx_3)^2} \right)^2 - x_3^2 \right] \omega^2 - \frac{b^2 x_3^2 y_3^2}{(1 + hx_3)^6} = 0. \tag{32}$$

Obviously, this equation has a positive solution ω_0 . Therefore, the system undergoes Hopf bifurcation at the equilibrium point Q_3 when

$$\tau_k^0 = \frac{1}{\omega_0} \arccos \left(\left(\omega_0^2 \left(\frac{by_3}{(1 + hx_3)^3} + \frac{Ax_3}{(A + x_3)^2} + \frac{hx_3 y_3}{(1 + hx_3)^2} \right) \right) \times \left(x_3 \omega_0^2 + \frac{b^2 x_3 y_3^2}{(1 + hx_3)^6} \right)^{-1} \right) + \frac{2k\pi}{\omega_0}. \tag{33}$$

Then the following result is obtained.

Theorem 7. Assume that the condition (31) holds. The model (11) is stable for the time delay $\tau < \tau_0^0$ and undergoes Hopf bifurcation at the equilibrium point Q_3 when $\tau = \tau_0^0$.

In succession, we discuss the bifurcation behavior regarding m as the bifurcation parameter; that is, $\mu = m$.

Theorem 8. If $A/(A + x_4)^2 + hy_4/(1 + hx_4)^2 \neq 1$, the model (11) undergoes singular induced bifurcation at the equilibrium point Q_4 when the bifurcation parameter μ increases through 0. Moreover, the stability of the equilibrium point Q_4 changes, that is, from stable to unstable.

Proof. We define a new variable as $\Delta = D_E g = \nu y - w$. Then, Δ has a simple zero eigenvalue at the equilibrium point $Q_4(x_4, y_4, E_4)$ as follows:

$$\begin{vmatrix} D_X f & D_E f \\ D_X g & D_E g \end{vmatrix}_{Q_4} = \begin{vmatrix} \frac{Ax_4}{(A + x_4)^2} - x_4 + \frac{hx_4 y_4}{(1 + hx_4)^2} & -\frac{x_4}{1 + hx_4} & 0 \\ \frac{by_4}{(1 + hx_4)^2} & 0 & -y_4 \\ 0 & \nu E_4 & \nu y_4 - w \end{vmatrix}$$

$$\begin{aligned}
 &= vx_4y_4E_4 \left(\frac{A}{(A+x_4)^2} - 1 + \frac{hy_4}{(1+hx_4)^2} \right), \\
 \text{trace}(D_E f \text{adj}(D_E g) D_X g)|_{Q_4} &= \text{trace} \left(\begin{pmatrix} 0 \\ -y_4 \end{pmatrix} (0 \ vE_4) \right) \\
 &= -wE_4, \\
 \begin{vmatrix} D_X f & D_E f & D_\mu f \\ D_X g & D_E g & D_\mu g \\ D_X \Delta & D_E \Delta & D_\mu \Delta \end{vmatrix}_{Q_4} &= \begin{vmatrix} \frac{Ax_4}{(A+x_4)^2} - x_4 + \frac{hx_4y_4}{(1+hx_4)^2} & -\frac{x_4}{1+hx_4} & 0 & 0 \\ \frac{by_4}{(1+hx_4)^2} & 0 & -y_4 & 0 \\ 0 & vE_4 & vp_4 - w & -1 \\ 0 & v & 0 & 0 \end{vmatrix} \\
 &= vx_4y_4 \left(\frac{A}{(A+x_4)^2} - 1 + \frac{hy_4}{(1+hx_4)^2} \right). \tag{34}
 \end{aligned}$$

On the other hand, we get

$$\begin{aligned}
 \xi_1 &= -\text{trace}(D_E f \text{adj}(D_E g) D_X g)|_{Q_4} = wE_4 > 0, \\
 \xi_2 &= \left(D_\mu \Delta - (D_X \Delta \ D_E \Delta) \begin{pmatrix} D_X f & D_E f \\ D_X g & D_E g \end{pmatrix}^{-1} \begin{pmatrix} D_\mu f \\ D_\mu g \end{pmatrix} \right)_{Q_4} \\
 &= \frac{1}{E_4} > 0. \tag{35}
 \end{aligned}$$

According to [20], all conditions of singular induced bifurcation are satisfied. Hence, the model (11) undergoes singular induced bifurcation at the equilibrium point Q_4 if the economic revenue is zero. When economic revenue m increases through 0, one eigenvalue of the model (11) moves from C^- (the open complex left half plane) to C^+ (the open complex right half plane) along the real axis by diverging through ∞ , which causes impulsive phenomenon of differential-algebraic system, that is, rapid expansion of the population from the view of biological explanation. Therefore, the stability of the equilibrium point Q_4 changes, that is, from stable to unstable. This completes the proof. \square

For the model (11) without time delay, eigenvalues of Jacobian matrix J at the equilibrium point Q_4 are the roots of the following equation:

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0, \tag{36}$$

where

$$\begin{aligned}
 a_{11} &= \frac{Ax_4}{(A+x_4)^2} - x_4 + \frac{hx_4y_4}{(1+hx_4)^2}, \\
 a_{12} &= -\frac{x_4}{1+hx_4}, \\
 a_{21} &= \frac{by_4}{(1+hx_4)^2}, \quad a_{22} = \frac{vy_4E_4}{vy_4 - w}. \tag{37}
 \end{aligned}$$

Remark 9. After simple computation, it can be seen that the Jacobian matrix J at the equilibrium point Q_4 has two eigenvalues. One is

$$\begin{aligned}
 \lambda_1 &= \frac{1}{2} \left[(a_{11} + a_{22}) + \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}} \right] \\
 &\rightarrow +\infty (a_{11}), \tag{38}
 \end{aligned}$$

and the other is

$$\begin{aligned}
 \lambda_2 &= \frac{1}{2} \left[(a_{11} + a_{22}) - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}} \right] \\
 &\rightarrow a_{11} (-\infty) \tag{39}
 \end{aligned}$$

since $a_{22} \rightarrow +\infty(-\infty)$ as $m \rightarrow 0^+(0^-)$.

3.2. The Model (7) with Positive Economic Profit. When the economic profit is positive, the model (7) has positive equilibrium point $Q_5(x_5, y_5, E_5)$, where coordinates $E_5 = m/(vy_5 - w)$, $y_5 = (1 + hx_5)(x_5/(A + x_5) - d_1 - x_5)$, and x_5 are the root of the following equation:

$$B(x) = B_1x^4 + B_2x^3 + B_3x^2 + B_4x + B_5 = 0, \tag{40}$$

where

$$\begin{aligned}
 B_1 &= vh(b - d_2h), \\
 B_2 &= v(1 + h(d_1 + A - 1))(b - d_2h) - d_2vh, \\
 B_3 &= (b - d_2h)(vhd_1A + (d_1 + A - 1)v + w) \\
 &\quad - d_2v(1 + h(d_1 + A - 1)) + mh, \\
 B_4 &= (b - d_2h)(vd_1 + w)A \\
 &\quad - d_2(vhd_1A + (d_1 + A - 1)v + w) \\
 &\quad + m(1 + hA), \\
 B_5 &= mA - d_2A(vd_1 + w). \tag{41}
 \end{aligned}$$

The Jacobian matrix of the model (7) at the equilibrium point Q_5 is

$$J_{Q_5} = \begin{pmatrix} \frac{Ax_5}{(A+x_5)^2} - x_5e^{-\lambda\tau} + \frac{hx_5y_5}{(1+hx_5)^2} & -\frac{x_5}{1+hx_5} \\ \frac{by_5e^{-\lambda\tau}}{(1+hx_5)^2} & \frac{vy_5E_5}{vy_5 - w} \end{pmatrix}. \tag{42}$$

For the sake of simplicity, let

$$\begin{aligned} u_1 &= u_{11} + u_{12}, & u_{11} &= -x_5, \\ u_{12} &= \frac{Ax_5}{(A + x_5)^2} + \frac{hx_5y_5}{(1 + hx_5)^2}, \\ u_2 &= -\frac{x_5}{1 + hx_5}, & u_3 &= \frac{by_5}{(1 + hx_5)^2}, \\ u_4 &= \frac{vy_5E_5}{vy_5 - w}. \end{aligned} \tag{43}$$

The characteristic polynomial for the model (7) without time delay at the equilibrium point Q_5 takes the following form:

$$R_1(\lambda) = \lambda^2 - (u_1 + u_4)\lambda + u_1u_4 - u_2u_3 = 0. \tag{44}$$

By using the Routh-Hurwitz criteria, Q_5 is locally asymptotically stable for the model (7) without time delay if

$$u_1 + u_4 < 0, \quad u_1u_4 - u_2u_3 > 0. \tag{45}$$

According to the Jacobian matrix J_{Q_5} , we can obtain the characteristic equation of the differential-algebraic model (7) at Q_5 , which can be expressed as follows:

$$R_2(\lambda) = \lambda^2 + p_1\lambda + p_2 + (p_3\lambda + p_4)e^{-\lambda\tau} = 0, \tag{46}$$

where

$$\begin{aligned} p_1 &= -u_{12} - u_4, & p_2 &= u_{12}u_4, \\ p_3 &= -u_{11}, & p_4 &= u_{11}u_4 - u_2u_3. \end{aligned} \tag{47}$$

Assume that a purely imaginary solution of the form $\lambda = i\omega$ exists in (46). Substituting it into (46) and separating the real and imaginary parts, we have

$$\begin{aligned} p_2 - \omega^2 + p_4 \cos \omega\tau + p_3\omega \sin \omega\tau &= 0, \\ p_1\omega + p_3\omega \cos \omega\tau - p_4 \sin \omega\tau &= 0. \end{aligned} \tag{48}$$

Taking square on both sides of (48) and summing them up, we obtain

$$\omega^4 + (p_1^2 - 2p_2 - p_3^2)\omega^2 + p_2^2 - p_4^2 = 0. \tag{49}$$

From the condition (45), we have $p_1^2 - 2p_2 - p_3^2 < 0$ and $p_2 + p_4 > 0$. Therefore, (49) has at least one real root ω_m if

$$p_2 - p_4 < 0. \tag{50}$$

The critical value of the delay corresponding to ω_m is given by

$$\begin{aligned} \tau_k^m &= \frac{1}{\omega_m} \arccos \frac{p_4\omega_m^2 - p_2p_4 - p_1p_3\omega_m^2}{p_3^2\omega_m^2 + p_4^2} + \frac{2k\pi}{\omega_m}, \\ &k = 0, 1, 2, \dots \end{aligned} \tag{51}$$

Now, differentiating (46) with respect to τ , we obtain

$$\frac{d\lambda}{d\tau} = \frac{(p_3\lambda + p_4)\lambda e^{-\lambda\tau}}{2\lambda + p_1 + [p_3 - \tau(p_3\lambda + p_4)]e^{-\lambda\tau}}. \tag{52}$$

Substituting the eigenvalue $i\omega_m$ and noticing that the derivative of (49) at ω_m^2 is positive, it follows that

$$\text{sign} \left(\frac{d}{d\tau} \text{Re}(\lambda) \right)_{\tau_k^m} = \text{sign} \left(\omega_m^2 (2\omega_m^2 + p_1^2 - 2p_2 - p_3^2) \right) > 0. \tag{53}$$

Theorem 10. Assume that conditions (45) and (50) hold. For the model (7), there exists a $\tau_0^m > 0$ such that Q_5 is locally asymptotically stable when $\tau \in [0, \tau_0^m)$ and is unstable when $\tau > \tau_0^m$. Furthermore, the model (7) undergoes Hopf bifurcation at the equilibrium point Q_5 when $\tau = \tau_k^m$, $k = 0, 1, 2, \dots$

From Theorem 10, it can be seen that Hopf bifurcation results in oscillation of population density; that is, the differential-algebraic model (7) becomes unstable when the time delay exceeds the critical value. From the view of ecological managers, it may be desirable to have stable population densities in order to keep sustainable development of ecosystem. However, food conversion delay is generally regarded as an inherence of biological populations. It is difficult to adjust the food conversion delay. Therefore, we propose the following state delayed feedback control method:

$$\begin{aligned} \frac{dx}{dt} &= x \left(\frac{x}{A+x} - d_1 - x(t-\tau) \right) - \frac{xy}{1+hx} \\ &\quad + L_1(x - x(t-\tau)), \\ \frac{dy}{dt} &= y \left(\frac{bx(t-\tau)}{1+hx(t-\tau)} - d_2 \right) - Ey + L_2(y - y(t-\tau)), \\ 0 &= E(vy - w) - m, \end{aligned} \tag{54}$$

where $L = (L_1 \ L_2)^T$ is the feedback gain.

Under the state delayed feedback control, the Jacobian matrix at the positive equilibrium point Q_5 is

$$J_{Q_5}^c = \begin{pmatrix} u_{11}e^{-\lambda\tau} + u_{12} + L_1(1 - e^{-\lambda\tau}) & u_2 \\ u_3e^{-\lambda\tau} & u_4 + L_2(1 - e^{-\lambda\tau}) \end{pmatrix}. \tag{55}$$

Then the character equation corresponding to the matrix $J_{Q_5}^c$ is

$$\begin{aligned} &[\lambda^2 - (u_{12} + u_4 + L_1 + L_2)\lambda + (u_{12} + L_1)(L_2 + u_4)] \\ &\quad + [(L_2 - u_{11} + L_1)\lambda + (u_{11} - L_1)(u_4 + L_2) \\ &\quad \quad - L_2(u_{12} + L_1) - u_2u_3]e^{-\lambda\tau} - L_2(u_{11} - L_1)e^{-2\lambda\tau} \\ &= 0. \end{aligned} \tag{56}$$

For the simplicity of computation, the feedback gain term L_1 is denoted as u_{11} . Then, the character equation (56) is written as

$$\begin{aligned} & [\lambda^2 - (u_1 + u_4 + L_2)\lambda + u_1(L_2 + u_4)] \\ & + e^{-\lambda\tau} [L_2\lambda - L_2u_1 - u_2u_3] = 0. \end{aligned} \tag{57}$$

Assume that (57) has a purely imaginary root $\lambda = i\omega$, and substitute it into the equation, then we obtain the following equation:

$$\begin{aligned} & \omega^4 + (u_1^2 + (u_4 + L_2)^2)\omega^2 \\ & + (u_1u_4 - u_2u_3)(u_1u_4 + u_2u_3 + 2L_2u_1) = 0. \end{aligned} \tag{58}$$

By simple analysis, we can obtain the following result.

Theorem 11. *Under the conditions of Theorem 10, for any time delay $\tau > 0$, the equilibrium point Q_5 of the model (54) is locally asymptotically stable if the feedback gain satisfies the following conditions:*

$$\begin{aligned} & L_1 = u_{11}, \\ & u_1u_4 + u_2u_3 + 2L_2u_1 > 0. \end{aligned} \tag{59}$$

In fact, the two differential equations of the differential-algebraic system (54) can be rewritten as

$$\begin{aligned} \frac{dx}{dt} &= x \left(\frac{x}{A+x} - d_1 - x(t-\tau) \right) - \frac{xy}{1+hx} + E_x x, \\ \frac{dy}{dt} &= y \left(\frac{bx(t-\tau)}{1+hx(t-\tau)} - d_2 \right) + E_y y, \end{aligned} \tag{60}$$

where $E_x = L_1(1 - x(t-\tau)/x)$ and $E_y = L_2(1 - y(t-\tau)/y) - E$.

Remark 12. From the practical point of view, E_x and E_y can be regarded as harvesting efforts for the prey and predator species. From the expression above, it is clear that E_x and E_y are related to the equilibrium, the past, and present population density. Hence, population densities can be kept stable by changing the strength of harvesting efforts; that is, the state delayed feedback control can be completed by adjusting harvesting efforts for two species. Hence, the state feedback controller can be used to eliminate Hopf bifurcation and drive the model system to stabilize at the interior equilibrium point.

4. Numerical Simulation

In this section, we firstly assign some parameter values of the model (7) and provide some numerical simulations to illustrate the effectiveness of the results which have been established in the previous sections. For the purpose of simulation experiments, we mainly use the software MATLAB 7.0.

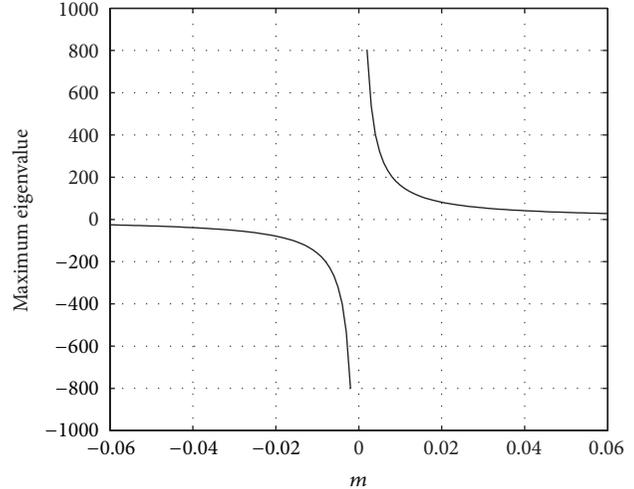


FIGURE 1: The maximum eigenvalue of the model (61) w.r.t. economic revenue m .

The differential-algebraic model (7) with given values takes the following form:

$$\begin{aligned} \frac{dx}{dt} &= x \left(\frac{x}{A+x} - 0.01 - x(t-\tau) \right) - \frac{xy}{1+1.25x}, \\ \frac{dy}{dt} &= y \left(\frac{7.5x(t-\tau)}{1+1.25x(t-\tau)} - 2 \right) - Ey, \\ 0 &= E(35y - 1) - m. \end{aligned} \tag{61}$$

4.1. Existence of Singular Induced Bifurcation. The existence of singularity induced bifurcation of the model (61) is clearly shown in Table 1. From Table 1, it can be seen that as economic revenue m increases through zero, one eigenvalue remains almost constant and the other moves from C^- to C^+ along the real axis by diverging through ∞ for the fixed Allee effect constant. Moreover, for the fixed Allee effect $A = 0.02$, we draw a figure to illustrate the movement of maximum eigenvalue with respect to the economic revenue m (see Figure 1). From Figure 1, it can be seen that singularity induced bifurcation for the model (61) occurs at $m = 0$, which is corresponding to the theoretical analysis. Hence, the stability of the model (61) at positive equilibrium point changes from stable to unstable. Furthermore, the stronger Allee effect is, the weaker impulsive phenomenon will be; that is, Allee effect has an impact on the dynamical behavior of the proposed model.

4.2. Existence and Control of Hopf Bifurcation. For the fixed parameter $A = 0.02$, we draw a critical curve $\tau_0^m = f(m)$ w.r.t. parameter m (see Figure 2). From Figure 2, it is clear that the domain surrounded by the critical curve $f(m)$ and the two axes is stable for the model (7). Based on the analysis in Section 3.2, the equilibrium point $Q_5(0.4055, 0.8099, 0.0183)$ is stable for the model (7) without time delay. Furthermore, since $p_2 - p_4 = -0.7079 < 0$, there is a positive root $\omega_m = 0.8781$ for (49). The critical value of the delay corresponding

TABLE 1: Equilibrium points and eigenvalues of the model (7) for different Allee effect constants and economic revenue.

Allee effect	Economic profit	Equilibrium point	Eigenvalues
$A = 0.02$	$m = -0.01$	(0.957, 0.0284, 1.424)	-0.930, -201.571
	$m = 0.01$	(0.956, 0.0288, 1.264)	-0.929, 161.025
$A = 0.1$	$m = -0.01$	(0.874, 0.0285, 2.367)	-0.775, -559.217
	$m = 0.01$	(0.874, 0.0288, 1.142)	-0.774, 131.598
$A = 0.5$	$m = -0.01$	(0.442, 0.0265, 0.136)	-0.203, -1.684
	$m = 0.01$	(0.435, 0.0311, 0.115)	-0.162, 1.420

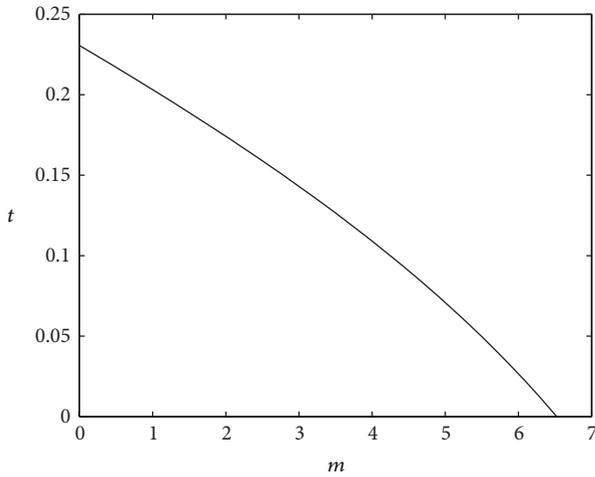


FIGURE 2: The critical curve of the delay τ w.r.t. economic revenue m .

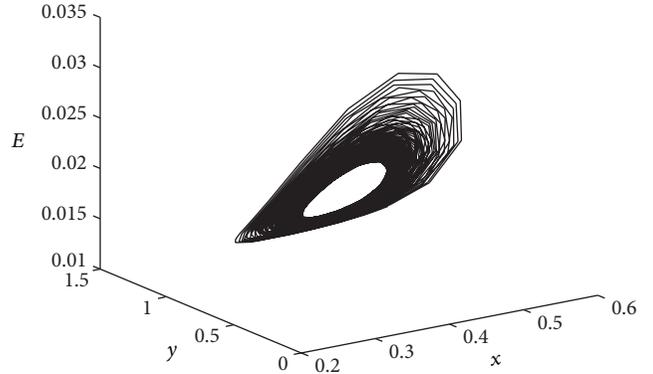


FIGURE 4: When $\tau = 0.23$, bifurcation period solutions from the equilibrium point Q_5 occur.

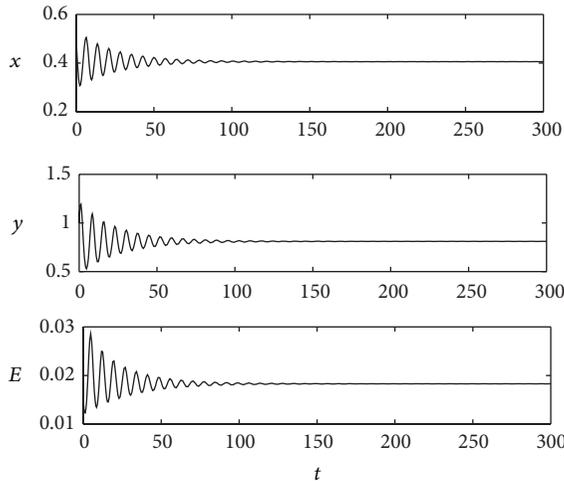


FIGURE 3: When $\tau = 0.12$, the equilibrium point Q_5 is stable.

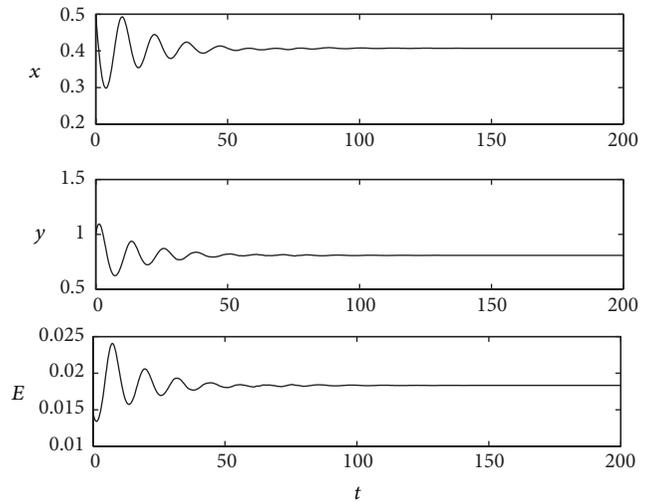


FIGURE 5: Dynamical responses of the controlled system (54) with the feedback gain $L = (-0.4055 \quad -7.5)^T$.

to ω_m is $\tau_0^m = 0.2172$. The interior equilibrium point Q_5 remains stable for $\tau < \tau_0^m$. In Figure 3, a random time delay $\tau = 0.12$ is selected in the interval $(0, 0.2172)$, which is enough to merit the above mathematical study. When the time delay τ passes through the critical value τ_0^m , the equilibrium point Q_5 loses its stability and Hopf bifurcation occurs. The bifurcating period solution from Q_5 at τ_0^m is

depicted in Figure 4. Next, a state feedback controller is applied to the model (7). We choose the feedback gain $L = (-0.4055 \quad -7.5)^T$, where the gain terms satisfy $L_1 = u_{11} = -0.4055$ and $u_1 u_4 + u_2 u_3 + 2L_2 u_1 = 1.9754 > 0$. Then, the model (7) is stable at Q_5 and the Hopf bifurcation is also eliminated. Figure 5 shows the dynamical responses of the differential-algebraic model (7).

5. Conclusions

Nowadays, much attention has been paid to preserving biological resources with the aim of stemming the damage and ensuring the balance of ecosystems, which inspires the introduction of harvesting in the biological system. In this paper, we analyze the dynamical behavior of a delayed predator-prey model with Allee effect and harvesting by using differential algebraic systems theory. From the analysis of the proposed model, we have obtained some interesting and useful results, which extend the work done in [21]. This paper is mainly divided into two parts.

In the first part, we consider a delayed differential-algebraic predator-prey model with zero economic revenue. It is observed that transcritical bifurcation and singular induced bifurcation phenomena take place when handling time of predator and economic revenue are regarded as bifurcation parameters, respectively.

As the handling time decreases through the critical point h_0 , the differential-algebraic model undergoes transcritical bifurcation and the stability of Q_2 is lost. Due to longer handling time h reducing the amount of prey catch per predator per unit of time, prey population density can stay at a positive value and predator population is to be extinct; that is, Q_2 is stable.

Singular induced bifurcation may cause impulsive phenomenon due to the variation of the economic revenues of harvesting. From a biological point of view, singular induced bifurcation implies rapid expansion of biological population, which may cause ecosystem unbalance and hamper the sustainable development. Hence, it is necessary to investigate the singular induced bifurcation in the presence of a reserve depending on the variation of economic revenue. The analysis of singular induced bifurcation can provide more information of forecasting so that ecological managers can lay down better management strategy. Furthermore, numerical simulation shows that the impulsive phenomenon can be much weaker by increasing Allee effect constant, which implies that Allee effect has an impact on the dynamical behavior of the proposed model. Therefore, ecological managers need to consider the inherent characters of the biological population and some external factors comprehensively.

In general, an individual prey killed does not contribute instantaneously to the growth of predator population. And differential equations with time delay always exhibit much more complicated dynamics than ordinary differential equations. Hence, in the second part of this paper, the effect of time delay on dynamical behavior of the differential-algebraic model is discussed. It shows that time delay plays an important role in the dynamical behavior of the differential-algebraic model. Hopf bifurcation occurs as time delay increases through a certain threshold, and time delay switches the stability of the proposed model. Furthermore, a state delayed feedback controller is designed to eliminate bifurcational phenomenon and keep population density at steady state. From numerical simulation, it is noted that the stronger Allee effect is not beneficial to the stability of biological species.

It should be noted that almost the existing bioeconomic models (see [13–16]) only investigate the simplest case of a logistic prey growth function. Compared with these works, the introduction of Allee effect makes the work studied in this paper novel.

Conflict of Interests

The authors do not have a conflict of interests with any commercial identities.

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