Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 105719, 22 pages
http://dx.doi.org/10.1155/2014/105719

Research Article

Positive Solutions and Mann Iterations of a Fourth Order Nonlinear Neutral Delay Differential Equation

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Received 16 December 2013; Accepted 20 February 2014; Published 29 May 2014

Academic Editor: Allan Peterson

This paper deals with a fourth order nonlinear neutral delay differential equation. By using the Banach fixed point theorem, we establish the existence of uncountably many bounded positive solutions for the equation, construct several Mann iterative sequences with mixed errors for approximating these positive solutions, and discuss some error estimates between the approximate solutions and these positive solutions. Seven nontrivial examples are given.

1. Introduction and Preliminaries

The oscillation, nonoscillation, and existence of solutions for various kinds of second order and third order neutral delay differential equations have been extensively studied over the last decades; for example, see [1–12]. Elbert [2] and Huang [3] established a few oscillation and nonoscillation criteria for the second order linear differential equation

$$x''(t) + q(t)x(t) = 0, \quad t \ge 0, \tag{1}$$

where $q \in C([0,+\infty),\mathbb{R}^+)$. Tang and Liu [9] studied the existence of bounded oscillation for the second order linear delay differential equation of unstable type

$$x''(t) = p(t)x(t-\tau), \quad t \ge t_0, \tag{2}$$

where $\tau > 0$, $p \in C([t_0,+\infty),\mathbb{R}^+)$, and $p(t) \not\equiv 0$ on any interval of length τ. Using the Banach fixed point theorem, Kulenović and Hadžiomerspahić [4] deduced the existence of a nonoscillatory solution for the second order linear neutral delay differential equation with positive and negative coefficients

$$\begin{aligned}(x(t) + cx(t-\tau))'' &+ Q_1(t)x(t-\sigma_1)\\ &- Q_2(t)x(t-\sigma_2) = 0, \quad t \ge t_0,\end{aligned} \tag{3}$$

where $c \in \mathbb{R} \setminus \{-1,1\}$, $\tau > 0$, $\sigma_1, \sigma_2 \in [0,+\infty)$, $Q_1, Q_2 \in C([t_0,+\infty),\mathbb{R}^+)$. Lin [5] suggested a few sufficient conditions for oscillation and nonoscillation of the second order nonlinear neutral differential equation

$$(x(t) + p(t)x(t-\tau))'' + q(t)f(x(t-\sigma)) = 0, \quad t \ge t_0, \tag{4}$$

where $\tau > 0$, $\sigma \ge 0$, $p \in C([t_0,+\infty),\mathbb{R}^+)$, $q \in C(\mathbb{R},\mathbb{R}^+)$, $f \in C(\mathbb{R},\mathbb{R})$, f is nondecreasing, and $xf(x) > 0$, $x \ne 0$. Qin et al. [8] and Yang et al. [11] developed several oscillation criteria for the second order differential equation

$$\begin{aligned}\left(r(t)\left(x(t) + p(t)x(t-\tau)\right)'\right)'&\\ + q(t)f(x(t-\delta)) = 0, \quad t \ge t_0,&\end{aligned} \tag{5}$$

where τ and δ are nonnegative constants, $r, p, q \in C([t_0, +\infty), \mathbb{R})$, and $f \in C(\mathbb{R}, \mathbb{R})$. By utilizing Krasnoselskii's fixed point theorem, Zhou [12] discussed the existence of nonoscillatory solutions of the second order nonlinear neutral differential equation

$$\begin{aligned}&\left(r(t)(x(t)+p(t)x(t-\tau))'\right)' \\ &\quad+\sum_{i=1}^{m} Q_i(t) f_i(x(t-\sigma_i))=0, \quad t \ge t_0,\end{aligned} \tag{6}$$

where $m \ge 1$ is an integer, $\tau > 0$, $\sigma_i \ge 0$, $r, p, Q_i \in C([t_0, +\infty), \mathbb{R})$, and $f_i \in C(\mathbb{R}, \mathbb{R})$ for $i \in \{1, 2, \ldots, m\}$. Yu and Wang [10] studied the existence of a nonoscillatory solution for the second order nonlinear neutral delay differential equations with positive and negative coefficients

$$\begin{aligned}&\left(r(t)(x(t)+P(t)x(t-\tau))'\right)'+Q_1(t) f(x(t-\sigma_1)) \\ &\quad-Q_2(t) g(x(t-\sigma_2))=0, \quad t \ge t_0,\end{aligned} \tag{7}$$

where $\tau > 0$, $\sigma_1, \sigma_2 \in [0, +\infty)$, $P, Q_1, Q_2, r \in C([t_0, +\infty), \mathbb{R})$, $f, g \in C(\mathbb{R}, \mathbb{R})$. Liu and Kang [7] investigated the existence of nonoscillatory solutions of the second order nonlinear neutral delay differential equation

$$\begin{aligned}&\left(a(t)(x(t)+b(t)x(t-\tau))'\right)' \\ &\quad+\left(h(t, x(h_1(t)), x(h_2(t)), \ldots, x(h_k(t)))\right)' \\ &\quad+f(t, x(f_1(t)), x(f_2(t)), \ldots, x(f_k(t)))=g(t), \\ &\quad t \ge t_0,\end{aligned} \tag{8}$$

where $\tau > 0$, $a, b, g \in C([t_0, +\infty), \mathbb{R})$ with $a(t) > 0$ for $t \ge t_0$, $h \in C^1([t_0, +\infty) \times \mathbb{R}^k, \mathbb{R})$, $f \in C([t_0, +\infty) \times \mathbb{R}^k, \mathbb{R})$, $h_l \in C^1([t_0, +\infty), \mathbb{R})$, and $f_l \in C([t_0, +\infty), \mathbb{R})$ with

$$\lim_{t \to +\infty} h_l(t) = \lim_{t \to +\infty} f_l(t) = +\infty, \quad l = 1, \ldots, k. \tag{9}$$

Kang et al. [13] discussed the existence of nonoscillatory solutions of the third order nonlinear neutral delay differential equation

$$\begin{aligned}&\left(\alpha(t)\left(\beta(t)(x(t)+p(t)x(t-\tau))'\right)'\right)' \\ &\quad+f(t, x(\sigma_1(t)), x(\sigma_2(t)), \ldots, x(\sigma_n(t)))=0, \\ &\quad t \ge t_0,\end{aligned} \tag{10}$$

where $n \ge 1$ is an integer, $\tau > 0$, $\alpha, \beta \in C([t_0, +\infty), \mathbb{R}^+ \setminus \{0\})$, $p \in C([t_0, +\infty), \mathbb{R})$, and $f \in C([t_0, +\infty) \times \mathbb{R}^n, \mathbb{R})$.

Motivated by the papers mentioned above, in this paper, we investigate the following fourth order nonlinear neutral delay differential equation:

$$\begin{aligned}&\left(\alpha(t)\left(\beta(t)\left(\gamma(t)(x(t)+p(t)x(t-\tau))'\right)'\right)'\right)' \\ &\quad+\left(h(t, x(h_1(t)), \ldots, x(h_k(t)))\right)' \\ &\quad+f(t, x(f_1(t)), x(f_2(t)), \ldots, x(f_k(t)))=g(t), \\ &\quad t \ge t_0,\end{aligned} \tag{11}$$

where $k \in \mathbb{N}$, $\tau > 0$, $\alpha, \beta, \gamma \in C([t_0, +\infty), \mathbb{R} \setminus \{0\})$, $p, g \in C([t_0, +\infty), \mathbb{R})$, $h \in C^1([t_0, +\infty) \times \mathbb{R}^k, \mathbb{R})$, $f \in C([t_0, +\infty) \times \mathbb{R}^k, \mathbb{R})$, $h_l \in C^1([t_0, +\infty), \mathbb{R})$, and $f_l \in C([t_0, +\infty), \mathbb{R})$ with

$$\lim_{t \to +\infty} h_l(t) = \lim_{t \to +\infty} f_l(t) = +\infty, \quad l = 1, 2, \ldots, k. \tag{12}$$

Utilizing the contraction mapping principle, we show the existence of uncountably many bounded positive solutions for (11), construct a few Mann type iterative schemes with mixed errors for these positive solutions, and discuss error estimates between the approximate solutions and the bounded positive solutions. Seven nontrivial examples are considered to illustrate our results.

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}$ denotes the set of positive integers, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, and

$$a = \min\left\{t_0 - \tau, \inf\left\{h_l(t), f_l(t) : t \in [t_0, +\infty), 1 \le l \le k\right\}\right\}. \tag{13}$$

By a solution of (11), we mean a function $x \in C([a, +\infty), \mathbb{R})$ for some $T \ge t_0 + |a| + \tau$, such that $\alpha(t)(\beta(t)(\gamma(t)(x(t) + p(t)x(t-\tau))')')'$ is continuously differentiable in $[T, +\infty)$ and (11) is satisfied for $t \ge T$.

Let $CB([a, +\infty), \mathbb{R})$ denote the Banach space of all continuous and bounded functions on $[a, +\infty)$ with the norm $\|x\| = \sup_{t \ge a} |x(t)|$ for each $x \in CB([a, +\infty), \mathbb{R})$ and

$$\begin{aligned}A(N, M) = \{x \in CB([a, +\infty), \mathbb{R}) : N \le x(t) \le M, t \ge a\} \\ \text{for } M > N > 0.\end{aligned} \tag{14}$$

It is easy to see that $A(N, M)$ is a bounded closed and convex subset of $CB([a, +\infty), \mathbb{R})$.

The following lemma plays an important role in this paper.

Lemma 1 (see [6]). *Let* $\{\alpha_n\}_{n\in\mathbb{N}_0}$, $\{\beta_n\}_{n\in\mathbb{N}_0}$, $\{\gamma_n\}_{n\in\mathbb{N}_0}$, *and* $\{t_n\}_{n\in\mathbb{N}_0}$ *be four nonnegative real sequences satisfying the inequality*

$$\alpha_{n+1} \le (1-t_n)\alpha_n + t_n\beta_n + \gamma_n, \quad n \in \mathbb{N}_0, \tag{15}$$

where $\{t_n\}_{n\in\mathbb{N}_0} \subset [0,1]$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty}\beta_n = 0$, *and* $\sum_{n=0}^{\infty}\gamma_n < +\infty$. *Then* $\lim_{n\to\infty}\alpha_n = 0$.

2. Uncountably Many Bounded Positive Solutions and Iterative Approximations

Now we study the solvability of (11).

Theorem 2. *Assume that there exist constants* M, N, c_1, c_2, *and* $T_0 \ge t_0$ *and functions* Q_1, Q_2, R_1, *and* $R_2 \in C([t_0,+\infty), R^+)$ *satisfying*

$$\begin{gathered} \left|f(t,u_1,u_2,\ldots,u_k) - f(t,\bar{u}_1,\bar{u}_2,\ldots,\bar{u}_k)\right| \\ \le Q_1(t)\max\{|u_l-\bar{u}_l| : 1\le l\le k\}, \\ \left|h(t,u_1,u_2,\ldots,u_k) - h(t,\bar{u}_1,\bar{u}_2,\ldots,\bar{u}_k)\right| \\ \le Q_2(t)\max\{|u_l-\bar{u}_l| : 1\le l\le k\} \\ \text{for } t\in[t_0,+\infty), \quad u_l,\bar{u}_l\in[N,M], \quad 1\le l\le k; \end{gathered} \tag{16}$$

$$\begin{gathered} \left|f(t,u_1,u_2,\ldots,u_k)\right| \le R_1(t), \\ \left|h(t,u_1,u_2,\ldots,u_k)\right| \le R_2(t), \\ \text{for } t\in[t_0,+\infty),\ u_l\in[N,M], \\ 1\le l\le k; \end{gathered} \tag{17}$$

$$\begin{gathered} \int_{t_0}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\ \times\max\{Q_2(u),R_2(u)\}\,du\,dv\,dw < +\infty, \\ \int_{t_0}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\ \times\max\{Q_1(s),R_1(s),|g(s)|\}\,ds\,du\,dv\,dw < +\infty; \end{gathered} \tag{18}$$

$$\min\{c_1,c_2\}\ge 0, \quad c_1+c_2<1, \quad 0<N<(1-c_1-c_2)M; \tag{19}$$

$$-c_2 \le p(t) \le c_1, \quad t\ge T_0. \tag{20}$$

Then,

(a) for any $L\in(N+c_1M,(1-c_2)M)$, *there exist* $\theta\in(0,1)$ *and* $T > t_0 + |a| + \tau + |T_0|$ *such that for each* $x_0\in A(N,M)$, *the Mann iterative sequence* $\{x_m\}_{m\in\mathbb{N}_0}$ *with mixed errors generated by the scheme*

$$x_{m+1}(t) = \begin{cases} (1-\alpha_m-\beta_m)x_m(t) + \alpha_m \\ \quad\times\Big\{L - p(t)x_m(t-\tau) \\ \quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times h(u,x_m(h_1(u)),\ldots,x_m(h_k(u)))\,du\,dv\,dw \\ \quad - \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times[f(s,x_m(f_1(s)),\ldots,x_m(f_k(s))) \\ \quad - g(s)]\,ds\,du\,dv\,dw\Big\} \\ \quad +\beta_m\gamma_m(t), \quad t\ge T,\ m\in\mathbb{N}_0, \\ (1-\alpha_m-\beta_m)x_m(T) + \alpha_m \\ \quad\times\Big\{L - p(T)x_m(T-\tau) \\ \quad + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times h(u,x_m(h_1(u)),\ldots, \\ \qquad x_m(h_k(u)))\,du\,dv\,dw \\ \quad - \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times[f(s,x_m(f_1(s)),\ldots,x_m(f_k(s))) \\ \quad - g(s)]\,ds\,du\,dv\,dw\Big\} \\ \quad +\beta_m\gamma_m(T), \quad a\le t<T, m\in\mathbb{N}_0 \end{cases} \tag{21}$$

converges to a bounded positive solution $x\in A(N,M)$ *of* (11) *and has the following error estimate:*

$$\|x_{m+1}-x\| \le (1-(1-\theta)\alpha_m)\|x_m-x\| + 2M\beta_m, \quad m\in\mathbb{N}_0, \tag{22}$$

where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ *is an arbitrary sequence in* $A(N,M)$ *and* $\{\alpha_m\}_{m\in\mathbb{N}_0}$ *and* $\{\beta_m\}_{m\in\mathbb{N}_0}$ *are any sequences in* $[0,1]$ *such that*

$$\sum_{m=0}^{\infty}\alpha_m = +\infty, \tag{23}$$

$$\begin{gathered} \sum_{m=0}^{\infty}\beta_m < +\infty \text{ or there exists a sequence} \\ \{\xi_m\}_{m\in\mathbb{N}_0}\subseteq[0,+\infty) \text{ satisfying} \\ \beta_m = \xi_m\alpha_m,\ m\in\mathbb{N}_0,\ \lim_{m\to\infty}\xi_m = 0; \end{gathered} \tag{24}$$

(b) equation (11) *possesses uncountably many bounded positive solutions in* $A(N,M)$.

Proof. Firstly, we show that (a) holds. Let $L \in (N + c_1 M, (1 - c_2)M)$. It follows from (18) and (19) that there exist $\theta \in (0,1)$ and $T > t_0 + |a| + \tau + |T_0|$ satisfying

$$\theta = c_1 + c_2 + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw, \tag{25}$$

$$\begin{aligned}&\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw \\ &\quad + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{R_1(s) + |g(s)|}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw \\ &\quad < \min\{(1-c_2)M - L, L - N - c_1 M\}.\end{aligned} \tag{26}$$

Define a mapping $S_L : A(N,M) \to CB([a,+\infty),\mathbb{R})$ by

$$S_L x(t) = \begin{cases} L - p(t)x(t-\tau) \\ \quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times h(u, x(h_1(u)),\ldots,x(h_k(u)))\,dudvdw \\ \quad - \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times [f(s, x(f_1(s)),\ldots,x(f_k(s))) \\ \qquad - g(s)]\,dsdudvdw, \\ \quad t \ge T,\ x \in A(N,M), \\ S_L x(T),\ a \le t < T,\ x \in A(N,M). \end{cases} \tag{27}$$

Obviously, $S_L x$ is continuous for each $x \in A(N,M)$. Combining (16), (17), (19), (20), and (25)–(27), we derive that for $x, y \in A(N,M)$ and $t \ge T$

$$\begin{aligned}&|S_L x(t) - S_L y(t)| \\ &\le |p(t)|\,|x(t-\tau) - y(t-\tau)| \\ &\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\ &\quad \times |h(u, x(h_1(u)),\ldots,x(h_k(u))) \\ &\qquad - h(u, y(h_1(u)),\ldots,y(h_k(u)))|\,dudvdw \\ &\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\ &\quad \times |f(s, x(f_1(s)),\ldots,x(f_k(s))) \\ &\qquad - f(s, y(f_1(s)),\ldots,y(f_k(s)))|\,dsdudvdw \\ &\le (c_1 + c_2)\|x - y\| + \|x - y\| \\ &\quad \times \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw \\ &\quad + \|x - y\| \\ &\quad \times \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw \\ &\le \Big[c_1 + c_2 + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw \\ &\quad + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw\Big] \\ &\quad \times \|x - y\| \\ &= \theta\|x - y\|,\end{aligned}$$

$$\begin{aligned}&S_L x(t) \\ &= L - p(t)x(t-\tau) \\ &\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ &\quad \times h(u, x(h_1(u)),\ldots,x(h_k(u)))\,dudvdw \\ &\quad - \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ &\quad \times [f(s, x(f_1(s)),\ldots,x(f_k(s))) - g(s)]\,dsdudvdw \\ &\le L + c_2 M \\ &\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\ &\quad \times |h(u, x(h_1(u)),\ldots,x(h_k(u)))|\,dudvdw \\ &\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\ &\quad \times [|f(s, x(f_1(s)),\ldots,x(f_k(s)))| + |g(s)|]\,dsdudvdw \\ &\le L + c_2 M \\ &\quad + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw \\ &\quad + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{R_1(s) + |g(s)|}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw \\ &< L + c_2 M + \min\{(1-c_2)M - L, L - N - c_1 M\} \\ &\le M,\end{aligned}$$

$$\begin{aligned}&S_L x(t) \\ &\ge L - c_1 M\end{aligned}$$

$$
\begin{aligned}
&- \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\
&\quad \times |h(u, x(h_1(u)), \ldots, x(h_k(u)))|\, dudvdw \\
&- \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\
&\quad \times |[f(s, x(f_1(s)), \ldots, x(f_k(s))) - g(s)]|\, dsdudvdw \\
&\geq L - c_1 M \\
&- \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw \\
&- \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{R_1(s) + |g(s)|}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw \\
&> L - c_1 M - \min\{(1 - c_2) M - L, L - N - c_1 M\} \\
&\geq N,
\end{aligned} \tag{28}
$$

which mean that

$$
\|S_L x - S_L y\| \leq \theta \|x - y\|, \quad x, y \in A(N, M), \qquad S_L(A(N, M)) \subseteq A(N, M). \tag{29}
$$

That is, S_L is a contraction mapping in $A(N, M)$ and has a unique fixed point $x \in A(N, M)$, which is a bounded positive solution of (11). By virtue of (21), (27), and (29), we get that for any $m \in \mathbb{N}_0$ and $t \geq T$

$$
\begin{aligned}
&|x_{m+1}(t) - x(t)| \\
&= \Big| (1 - \alpha_m - \beta_m) x_m(t) + \alpha_m \\
&\quad \times \{L - p(t) x_m(t - \tau) \\
&\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times h(u, x_m(h_1(u)), \ldots, x_m(h_k(u)))\, dudvdw \\
&\quad - \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times [f(s, x_m(f_1(s)), \ldots, x_m(f_k(s))) \\
&\quad - g(s)]\, dsdudvdw\} + \beta_m \gamma_m(t) - x(t)\Big| \\
&\leq (1 - \alpha_m - \beta_m) |x_m(t) - x(t)| \\
&\quad + \alpha_m |S_L x_m(t) - S_L x(t)| + \beta_m |\gamma_m(t) - x(t)| \\
&\leq (1 - \alpha_m - \beta_m) |x_m(t) - x(t)| \\
&\quad + \alpha_m \theta |x_m(t) - x(t)| + 2M\beta_m \\
&\leq (1 - (1 - \theta)\alpha_m) |x_m(t) - x(t)| + 2M\beta_m \\
&\leq (1 - (1 - \theta)\alpha_m) \|x_m - x\| + 2M\beta_m,
\end{aligned} \tag{30}
$$

which yielded that

$$
\|x_{m+1} - x\| \leq (1 - (1 - \theta)\alpha_m) \|x_m - x\| + 2M\beta_m, \quad m \in \mathbb{N}_0. \tag{31}
$$

That is, (22) holds. Thus Lemma 1, (23), and (24) ensure that $\lim_{m\to\infty} x_m = x$.

Next we show that (b) holds. Let $L_1, L_2 \in (N + c_1 M, (1 - c_2)M)$ with $L_1 \neq L_2$. As in the proof of (a), we conclude that for each $j \in \{1, 2\}$ there exist $\theta_j \in (0, 1)$, $T_j > t_0 + |a| + \tau + |T_0|$, and $S_{L_j} : A(N, M) \to A(N, M)$ satisfying (25)–(27), where θ, T, S_L are replaced by θ_j, T_j, and S_{L_j}, respectively, and the contraction mapping S_{L_j} has a unique fixed point $z_j \in A(N, M)$, which is also a bounded positive solution of (11). In order to prove (b), we need only to show that $z_1 \neq z_2$. Put $T^* = \max\{T_1, T_2\}$. Note that for $t \geq T^*$ and $j \in \{1, 2\}$

$$
\begin{aligned}
&z_j(t) \\
&= S_L z_j(t) \\
&= L_j - p(t) z_j(t - \tau) \\
&\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times h\left(u, z_j(h_1(u)), \ldots, z_j(h_k(u))\right) dudvdw \\
&\quad - \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times \left[f\left(s, z_j(f_1(s)), \ldots, z_j(f_k(s))\right) - g(s)\right] dsdudvdw,
\end{aligned} \tag{32}
$$

which together with (16) and (25) implies that for $t \geq T^*$

$$
\begin{aligned}
&|z_1(t) - z_2(t)| \\
&= |L_1 - L_2 - p(t)(z_1(t - \tau) - z_2(t - \tau)) \\
&\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times [h(u, z_1(h_1(u)), \ldots, z_1(h_k(u))) \\
&\quad - h(u, z_2(h_1(u)), \ldots, z_2(h_k(u)))]\, dudvdw \\
&\quad - \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)}
\end{aligned}
$$

$$\begin{aligned}
&\times [f(s, z_1(f_1(s)), \ldots, z_1(f_k(s))) \\
&\quad - f(s, z_2(f_1(s)), \ldots, z_2(f_k(s)))]\, ds\,du\,dv\,dw\Big| \\
&\geq |L_1 - L_2| - (c_1 + c_2)\|z_1 - z_2\| - \|z_1 - z_2\| \\
&\quad \times \int_{T^*}^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} du\,dv\,dw \\
&\quad - \|z_1 - z_2\| \\
&\quad \times \int_{T^*}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} ds\,du\,dv\,dw \\
&\geq |L_1 - L_2| - \|z_1 - z_2\| \max\{\theta_1, \theta_2\},
\end{aligned} \tag{33}$$

which yields that

$$\|z_1 - z_2\| \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0. \tag{34}$$

That is, $z_1 \neq z_2$. This completes the proof. □

Theorem 3. *Assume that there exist constants M, N, c_1, c_2, and $T_0 \geq t_0$ and functions Q_1, Q_2, R_1, and $R_2 \in C([t_0, +\infty), R^+)$ satisfying* (16)–(18) *and*

$$\begin{aligned}
&0 < (1 - c_2) N < (1 - c_1) M, \\
&0 \leq c_2 \leq p(t) \leq c_1 < 1, \quad t \geq T_0.
\end{aligned} \tag{35}$$

Then

(a) for any $L \in (N + c_1 M, M + c_2 N)$, there exist $\theta \in (0, 1)$ and $T > t_0 + |a| + \tau + |T_0|$ such that, for each $x_0 \in A(N, M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}$ generated by the scheme (21) *with* (23) *and* (24) *converges to a bounded positive solution $x \in A(N, M)$ of* (11) *and has the error estimate* (22);

(b) equation (11) *has uncountably many bounded positive solutions in $A(N, M)$.*

Proof. Let $L \in (N + c_1 M, M + c_2 N)$. It follows from (18) and (35) that there exist $\theta \in (0, 1)$ and $T > t_0 + |a| + \tau + |T_0|$ satisfying

$$\begin{aligned}
\theta &= c_1 + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} du\,dv\,dw \\
&\quad + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} ds\,du\,dv\,dw,
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} du\,dv\,dw \\
&\quad + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{R_1(s) + |g(s)|}{|\alpha(u)\beta(v)\gamma(w)|} ds\,du\,dv\,dw \\
&< \min\{M + c_2 N - L, L - N - c_1 M\}.
\end{aligned} \tag{37}$$

Define a mapping $S_L : A(N, M) \to CB([a, +\infty), \mathbb{R})$ by (27). Clearly $S_L x$ is continuous for each $x \in A(N, M)$. On account of (16), (17), (27), (35), (36), and (37), we infer that for $x, y \in A(N, M)$ and $t \geq T$

$$\begin{aligned}
&|S_L x(t) - S_L y(t)| \\
&\leq |p(t)| |x(t - \tau) - y(t - \tau)| \\
&\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\
&\quad \times |h(u, x(h_1(u)), \ldots, x(h_k(u))) \\
&\qquad - h(u, y(h_1(u)), \ldots, y(h_k(u)))|\, du\,dv\,dw \\
&\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\
&\quad \times |f(s, x(f_1(s)), \ldots, x(f_k(s))) \\
&\qquad - f(s, y(f_1(s)), \ldots, y(f_k(s)))|\, ds\,du\,dv\,dw \\
&\leq c_1 \|x - y\| + \|x - y\| \\
&\quad \times \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} du\,dv\,dw \\
&\quad + \|x - y\| \\
&\quad \times \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} ds\,du\,dv\,dw \\
&\leq \Bigg(c_1 + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} du\,dv\,dw \\
&\qquad + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} ds\,du\,dv\,dw \Bigg) \\
&\quad \times \|x - y\| \\
&= \theta \|x - y\|,
\end{aligned}$$

$$\begin{aligned}
&S_L x(t) \\
&= L - p(t) x(t - \tau) \\
&\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times h(u, x(h_1(u)), \ldots, x(h_k(u)))\, du\,dv\,dw \\
&\quad - \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times [f(s, x(f_1(s)), \ldots, x(f_k(s))) - g(s)]\, ds\,du\,dv\,dw \\
&\leq L - c_2 N \\
&\quad + \int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\
&\quad \times |h(u, x(h_1(u)), \ldots, x(h_k(u)))|\, du\,dv\,dw
\end{aligned}$$

$$\begin{aligned}
&+\int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{|\alpha(u)\beta(v)\gamma(w)|}\\
&\quad\times\left[\left|f\left(s,x\left(f_1(s)\right),\ldots,x\left(f_k(s)\right)\right)\right|+|g(s)|\right]dsdudvdw\\
&\le L-c_2N\\
&\quad+\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw\\
&\quad+\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw\\
&< L-c_2N+\min\{M+c_2N-L,L-N-c_1M\}\\
&\le M,\\
&S_Lx(t)\\
&\ge L-c_1M\\
&\quad-\int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{|\alpha(u)\beta(v)\gamma(w)|}\\
&\quad\times\left|h\left(u,x\left(h_1(u)\right),\ldots,x\left(h_k(u)\right)\right)\right|dudvdw\\
&\quad-\int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{|\alpha(u)\beta(v)\gamma(w)|}\\
&\qquad\times\left|\left[f\left(s,x\left(f_1(s)\right),\ldots,x\left(f_k(s)\right)\right)-g(s)\right]\right|dsdudvdw\\
&\ge L-c_1M\\
&\quad-\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw\\
&\quad-\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw\\
&> L-c_1M-\min\{M+c_2N-L,L-N-c_1M\}\\
&\ge N,
\end{aligned}\tag{38}$$

which imply that (29) holds. The rest of the proof is similar to that of Theorem 2 and is omitted. This completes the proof. □

Theorem 4. *Assume that there exist constants* M, N, c_1, c_2 *and* $T_0 \ge t_0$ *and functions* Q_1, Q_2, R_1 *and* $R_2 \in C([t_0,+\infty),\mathbb{R}^+)$ *satisfying* (17), (18), *and*

$$\begin{gathered}
0<(1-c_2)N<(1-c_1)M;\\
-1<-c_1\le p(t)\le -c_2\le 0,\quad t\ge T_0.
\end{gathered}\tag{39}$$

Then

(a) for any $L \in ((1-c_2)N,(1-c_1)M)$, *there exist* $\theta \in (0,1)$ *and* $T > t_0+|a|+\tau+|T_0|$ *such that, for each* $x_0 \in A(N,M)$, *the Mann iterative sequence* $\{x_m\}_{m\in\mathbb{N}_0}$ *generated by the scheme* (21) *with* (23) *and* (24) *converges to a bounded positive solution* $x \in A(N,M)$ *of* (11) *and has the error estimate* (22);

(b) equation (11) *has uncountably many bounded positive solutions in* $A(N,M)$.

Proof. Let $L \in ((1-c_2)N,(1-c_1)M)$. Equations (18) and (39) guarantee that there exist $\theta \in (0,1)$ and $T > t_0+|a|+\tau+|T_0|$ satisfying (36) and

$$\begin{aligned}
&\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw\\
&\quad+\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw\\
&<\min\{(1-c_1)M-L,(c_2-1)N+L\}.
\end{aligned}\tag{40}$$

Let the mapping $S_L : A(N,M) \to CB([a,+\infty),\mathbb{R})$ be defined by (27). Using (17), (27), (39), and (40), we deduce that for any $x \in A(N,M)$ and $t \ge T$

$$\begin{aligned}
&S_Lx(t)\\
&=L-p(t)x(t-\tau)\\
&\quad+\int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)}\\
&\quad\times h\left(u,x\left(h_1(u)\right),\ldots,x\left(h_k(u)\right)\right)dudvdw\\
&\quad-\int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)}\\
&\quad\times\left[f\left(s,x\left(f_1(s)\right),\ldots,x\left(f_k(s)\right)\right)-g(s)\right]dsdudvdw\\
&\le L+c_1M\\
&\quad+\int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{|\alpha(u)\beta(v)\gamma(w)|}\\
&\quad\times\left|h\left(u,x\left(h_1(u)\right),\ldots,x\left(h_k(u)\right)\right)\right|dudvdw\\
&\quad+\int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{|\alpha(u)\beta(v)\gamma(w)|}\\
&\quad\times\left[\left|f\left(s,x\left(f_1(s)\right),\ldots,x\left(f_k(s)\right)\right)\right|\right.\\
&\qquad\left.+|g(s)|\right]dsdudvdw\\
&\le L+c_1M\\
&\quad+\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw\\
&\quad+\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw
\end{aligned}$$

$$
\begin{aligned}
&< L + c_1 M + \min\{(1-c_1)M - L, (c_2-1)N + L\} \\
&\le M, \\
&S_L x(t) \\
&\ge L + c_2 N \\
&\quad - \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} du\,dv\,dw \\
&\quad - \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{R_1(s) + |g(s)|}{|\alpha(u)\beta(v)\gamma(w)|} ds\,du\,dv\,dw \\
&> L + c_2 N - \min\{(1-c_1)M - L, (c_2-1)N + L\} \\
&\ge N,
\end{aligned}
\tag{41}
$$

which mean that (29) holds. That is, S_L is a contraction mapping and possesses a unique fixed point $x \in A(N, M)$, which is a bounded positive solution of (11). The rest of the proof is similar to that of Theorem 2 and is omitted. This completes the proof. □

Theorem 5. *Assume that there exist constants M, N, and $T_0 \ge t_0$ and functions Q_1, Q_2, R_1, and $R_2 \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (16)–(18) *and*

$$0 < N < M, \quad p(t) = 1, \quad t \ge T_0. \tag{42}$$

Then

(a) for any $L \in (N, M)$, there exist $\theta \in (0,1)$ and $T > t_0 + |a| + \tau + |T_0|$ such that, for each $x_0 \in A(N, M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}$ generated by the scheme

$$
x_{m+1}(t) = \begin{cases}
(1-\alpha_m-\beta_m)x_m(t) + \alpha_m \\
\quad \times \Big\{ L + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
\quad \times h(u, x_m(h_1(u)), \dots, x_m(h_k(u)))\, du\,dv\,dw \\
\quad - \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
\quad \times [f(s, x_m(f_1(s)), \dots, x_m(f_k(s))) \\
\qquad - g(s)]\, ds\,du\,dv\,dw \Big\} \\
\quad + \beta_m \gamma_m(t), \quad t \ge T,\ m \in \mathbb{N}_0, \\
(1-\alpha_m-\beta_m)x_m(T) + \alpha_m \\
\quad \times \Big\{ L + \sum_{i=1}^{\infty} \int_{T+(2i-1)\tau}^{T+2i\tau} \int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
\quad \times h(u, x_m(h_1(u)), \dots, x_m(h_k(u)))\, du\,dv\,dw \\
\quad - \sum_{i=1}^{\infty} \int_{T+(2i-1)\tau}^{T+2i\tau} \int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
\quad \times [f(s, x_m(f_1(s)), \dots, x_m(f_k(s))) \\
\qquad - g(s)]\, ds\,du\,dv\,dw \Big\} \\
\quad + \beta_m \gamma_m(T), \quad a \le t < T,\ m \in \mathbb{N}_0
\end{cases}
\tag{43}
$$

converges to a bounded positive solution $x \in A(N, M)$ of (11) *and has the error estimate* (22)*, where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $A(N, M)$ and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying* (23) *and* (24)*;*

(b) equation (11) *possesses uncountably many bounded positive solutions in $A(N, M)$.*

Proof. First of all, we show that (a) holds. Let $L \in (N, M)$. It follows from (18) that there exist $\theta \in (0,1)$ and $T > t_0 + |a| + \tau + |T_0|$ satisfying

$$
\begin{aligned}
\theta = &\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} du\,dv\,dw \\
&+ \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} ds\,du\,dv\,dw,
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
&\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} du\,dv\,dw \\
&\quad + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{R_1(s) + |g(s)|}{|\alpha(u)\beta(v)\gamma(w)|} ds\,du\,dv\,dw \\
&\quad < \min\{M - L, L - N\}.
\end{aligned}
\tag{45}
$$

Define a mapping $S_L : A(N, M) \to CB([a, +\infty), \mathbb{R})$ by

$$
S_L x(t) = \begin{cases}
L + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
\quad \times h(u, x(h_1(u)), \dots, x(h_k(u)))\, du\,dv\,dw \\
- \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
\quad \times [f(s, x(f_1(s)), \dots, x(f_k(s))) \\
\qquad - g(s)]\, ds\,du\,dv\,dw, \\
t \ge T,\ x \in A(N, M), \\
S_L x(T), \quad a \le t < T,\ x \in A(N, M).
\end{cases}
\tag{46}
$$

Clearly, $S_L x$ is continuous for each $x \in A(N, M)$. Notice that (16), (17), (42), and (44)–(46) ensure that for $x, y \in A(N, M)$ and $t \ge T$

$$
\begin{aligned}
&|S_L x(t) - S_L y(t)| \\
&\le \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty}\int_v^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\
&\quad \times |h(u, x(h_1(u)), \dots, x(h_k(u))) \\
&\qquad - h(u, y(h_1(u)), \dots, y(h_k(u)))|\, du\,dv\,dw \\
&\quad + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left|f\left(s, x\left(f_1(s)\right), \ldots, x\left(f_k(s)\right)\right)\right. \\
&\quad \left.-f\left(s, y\left(f_1(s)\right), \ldots, y\left(f_k(s)\right)\right)\right| dsdudvdw2 \le \|x-y\| \\
&\times \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty} \int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw \\
&+ \|x-y\| \\
&\times \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw \\
&\le \left(\int_T^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw \right. \\
&\quad \left. + \int_T^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw \right) \\
&\times \|x-y\| \\
&= \theta \|x-y\|, \\
&S_L x(t) \\
&= L + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty} \int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times h\left(u, x\left(h_1(u)\right), \ldots, x\left(h_k(u)\right)\right) dudvdw \\
&\quad - \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times \left[f\left(s, x\left(f_1(s)\right), \ldots, x\left(f_k(s)\right)\right) - g(s)\right] dsdudvdw \\
&\le L + \int_t^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw \\
&\quad + \int_t^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw \\
&< L + \min\{M-L, L-N\} \\
&\le M, \\
&S_L x(t) \\
&\ge L - \int_t^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw \\
&\quad - \int_t^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw \\
&> L - \min\{M-L, L-N\} \\
&\ge N,
\end{aligned} \tag{47}
$$

which imply that (29) holds. That is, S_L is a contraction mapping and has a unique fixed point $x \in A(N, M)$. It follows that

$$
\begin{aligned}
&x(t) \\
&= L + \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty} \int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times h\left(u, x\left(h_1(u)\right), \ldots, x\left(h_k(u)\right)\right) dudvdw \\
&\quad - \sum_{i=1}^{\infty} \int_{t+(2i-1)\tau}^{t+2i\tau} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times \left[f\left(s, x\left(f_1(s)\right), \ldots, x\left(f_k(s)\right)\right)\right. \\
&\quad \left. - g(s)\right] dsdudvdw, \quad t \ge T,
\end{aligned} \tag{48}
$$

$$
\begin{aligned}
&x(t-\tau) \\
&= L + \sum_{i=1}^{\infty} \int_{t+(2i-2)\tau}^{t+(2i-1)\tau} \int_w^{+\infty} \int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times h\left(u, x\left(h_1(u)\right), \ldots, x\left(h_k(u)\right)\right) dudvdw \\
&\quad - \sum_{i=1}^{\infty} \int_{t+(2i-2)\tau}^{t+(2i-1)\tau} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times \left[f\left(s, x\left(f_1(s)\right), \ldots, x\left(f_k(s)\right)\right)\right. \\
&\quad \left. - g(s)\right] dsdudvdw, \quad t \ge T+\tau.
\end{aligned} \tag{49}
$$

Adding (48) and (49), we infer that

$$
\begin{aligned}
&x(t) + x(t-\tau) \\
&= 2L + \sum_{i=1}^{\infty} \left[\int_{t+(2i-1)\tau}^{t+2i\tau} + \int_{t+(2i-2)\tau}^{t+(2i-1)\tau} \right] \\
&\quad \times \int_w^{+\infty} \int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times h\left(u, x\left(h_1(u)\right), \ldots, x\left(h_k(u)\right)\right) dudvdw \\
&\quad - \sum_{i=1}^{\infty} \left[\int_{t+(2i-1)\tau}^{t+2i\tau} + \int_{t+(2i-2)\tau}^{t+(2i-1)\tau} \right] \\
&\quad \times \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times \left[f\left(s, x\left(f_1(s)\right), \ldots, x\left(f_k(s)\right)\right) - g(s)\right] dsdudvdw \\
&= 2L + \int_t^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times h\left(u, x\left(h_1(u)\right), \ldots, x\left(h_k(u)\right)\right) dudvdw
\end{aligned}
$$

$$-\int_t^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \times [f(s,x(f_1(s)),\ldots,x(f_k(s))) - g(s)]\,dsdudvdw,\quad t\geq T+\tau, \tag{50}$$

which yields that x is a bounded positive solution of (11). By means of (29), (43), and (46), we know that for any $t \geq T$ and $m \in \mathbb{N}_0$

$$\begin{aligned}
&|x_{m+1}(t)-x(t)| \\
&= \Bigg|(1-\alpha_m-\beta_m)x_m(t)+\alpha_m \\
&\quad\times\Bigg\{L+\sum_{i=1}^{\infty}\int_{t+(2i-1)\tau}^{t+2i\tau}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times h(u,x_m(h_1(u)),\ldots,x_m(h_k(u)))\,dudvdw \\
&\quad-\sum_{i=1}^{\infty}\int_{t+(2i-1)\tau}^{t+2i\tau}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times[f(s,x_m(f_1(s)),\ldots,x_m(f_k(s))) \\
&\quad-g(s)]\,dsdudvdw\Bigg\}+\beta_m\gamma_m(t)-x(t)\Bigg| \\
&\leq(1-\alpha_m-\beta_m)|x_m(t)-x(t)| \\
&\quad+\alpha_m|S_Lx_m(t)-S_Lx(t)|+\beta_m|\gamma_m(t)-x(t)| \\
&\leq(1-\alpha_m-\beta_m)|x_m(t)-x(t)| \\
&\quad+\alpha_m\theta|x_m(t)-x(t)|+2M\beta_m \\
&\leq(1-(1-\theta)\alpha_m)|x_m(t)-x(t)|+2M\beta_m,
\end{aligned} \tag{51}$$

which gives (22). Thus Lemma 1, (23), and (24) ensure that $\lim_{m\to\infty}x_m = x$.

Now we show that (b) holds. Let $L_1, L_2 \in (N, M)$ with $L_1 \neq L_2$. As in the proof of (a), for each $\in \{1, 2\}$, we infer that there exists $\theta_j \in (0, 1)$, $T_j > t_0 + |a| + \tau + |T_0|$, and $S_{L_j} : A(N, M) \to A(N, M)$ satisfying (44)–(46), where θ, T, S_L are replaced by θ_j, T_j, S_{L_j}, respectively, and the contraction mapping S_{L_j} possesses a unique fixed point $z_j \in A(N, M)$, and z_j is a bounded positive solution of (11); that is,

$$\begin{aligned}
&z_j(t) \\
&= S_Lz_j(t) \\
&= L_j+\sum_{i=1}^{\infty}\int_{t+(2i-1)\tau}^{t+2i\tau}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times h(u,z_j(h_1(u)),\ldots,z_j(h_k(u)))\,dudvdw \\
&\quad-\sum_{i=1}^{\infty}\int_{t+(2i-1)\tau}^{t+2i\tau}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times\left[f(s,z_j(f_1(s)),\ldots,z_j(f_k(s)))\right. \\
&\quad\left.-g(s)\right]dsdudvdw,\quad t\geq T_j.
\end{aligned} \tag{52}$$

Put $T^* = \max\{T_1, T_2\}$. Using (16), (44), and (52), we conclude that for $t > T^*$

$$\begin{aligned}
&|z_1(t)-z_2(t)| \\
&= \Bigg|L_1-L_2 \\
&\quad+\sum_{i=1}^{\infty}\int_{t+(2i-1)\tau}^{t+2i\tau}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times[h(u,z_1(h_1(u)),\ldots,z_1(h_k(u))) \\
&\quad-h(u,z_2(h_1(u)),\ldots,z_2(h_k(u)))]\,dudvdw \\
&\quad-\sum_{i=1}^{\infty}\int_{t+(2i-1)\tau}^{t+2i\tau}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times[f(s,z_1(f_1(s)),\ldots,z_1(f_k(s))) \\
&\quad-f(s,z_2(f_1(s)),\ldots,z_2(f_k(s)))]\,dsdudvdw\Bigg| \\
&\geq|L_1-L_2|-\|z_1-z_2\| \\
&\quad\times\sum_{i=1}^{\infty}\int_{t+(2i-1)\tau}^{t+2i\tau}\int_w^{+\infty}\int_v^{+\infty}\frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw \\
&\quad-\|z_1-z_2\| \\
&\quad\times\sum_{i=1}^{\infty}\int_{t+(2i-1)\tau}^{t+2i\tau}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw \\
&\geq|L_1-L_2|-\|z_1-z_2\| \\
&\quad\times\Bigg(\int_{T^*}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw \\
&\quad+\int_{T^*}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw\Bigg) \\
&\geq|L_1-L_2|-\|z_1-z_2\|\max\{\theta_1,\theta_2\},
\end{aligned} \tag{53}$$

which yields that

$$\|z_1-z_2\|\geq\frac{|L_1-L_2|}{1+\max\{\theta_1,\theta_2\}}>0; \tag{54}$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 6. *Assume that there exist constants M, N and $T_0 \geq t_0$ and functions Q_1, Q_2, R_1 and $R_2 \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (16), (17), *and*

$$0 < N < M, \quad p(t) = -1, \quad t \geq T_0, \tag{55}$$

$$\sum_{i=1}^{\infty} \int_{t_0+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \frac{\max\{Q_2(u), R_2(u)\}}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw < +\infty;$$
$$\sum_{i=1}^{\infty} \int_{t_0+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \int_{u}^{+\infty} \frac{\max\{Q_1(s), R_1(s), |g(s)|\}}{|\alpha(u)\beta(v)\gamma(w)|} < +\infty. \tag{56}$$

Then

(a) for any $L \in (N, M)$, there exist $\theta \in (0,1)$ and $T > t_0 + |a| + \tau + |T_0|$ such that, for each $x_0 \in A(N, M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}$ generated by the scheme

$$x_{m+1}(t) = \begin{cases} (1-\alpha_m-\beta_m)x_m(t) + \alpha_m \\ \quad \times \Big\{ L - \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times h(u, x_m(h_1(u)), \ldots, x_m(h_k(u))) dudvdw \\ \quad + \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \int_{u}^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times [f(s, x_m(f_1(s)), \ldots, x_m(f_k(s))) \\ \quad - g(s)] dsdudvdw \Big\} + \beta_m \gamma_m(t), \\ \quad t \geq T, \ m \in \mathbb{N}_0, \\ (1-\alpha_m-\beta_m)x_m(T) + \alpha_m \\ \quad \times \Big\{ L - \sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times h(u, x_m(h_1(u)), \ldots, x_m(h_k(u))) dudvdw \\ \quad + \sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \int_{u}^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times [f(s, x_m(f_1(s)), \ldots, x_m(f_k(s))) \\ \quad - g(s)] dsdudvdw \Big\} + \beta_m \gamma_m(T), \\ \quad a \leq t < T, \ m \in \mathbb{N}_0 \end{cases} \tag{57}$$

converges to a bounded positive solution $x \in A(N, M)$ of (11) *and has the error estimate* (22), *where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $A(N, M)$, and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying* (23) *and* (24);

(b) equation (11) *possesses uncountably many bounded positive solutions in $A(N, M)$.*

Proof. Firstly, we show that (a) holds. Let $L \in (N, M)$. It follows from (56) that there exist $\theta \in (0,1)$ and $T > t_0 + |a| + \tau + |T_0|$ satisfying

$$\theta = \sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw + \sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \int_{u}^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw, \tag{58}$$

$$\sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} dudvdw + \sum_{i=1}^{\infty} \int_{T+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \int_{u}^{+\infty} \frac{R_1(s) + |g(s)|}{|\alpha(u)\beta(v)\gamma(w)|} dsdudvdw < \min\{M - L, L - N\}. \tag{59}$$

Define a mapping $S_L : A(N, M) \to CB([a, +\infty), \mathbb{R})$ by

$$S_L x(t) = \begin{cases} L - \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times h(u, x(h_1(u)), \ldots, x(h_k(u))) dudvdw \\ \quad + \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \int_{u}^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times [f(s, x(f_1(s)), \ldots, x(f_k(s))) \\ \quad - g(s)] dsdudvdw, \\ \quad t \geq T, \ x \in A(N, M), \\ S_L x(T), \quad a \leq t < T, \ x \in A(N, M). \end{cases} \tag{60}$$

Clearly, $S_L x$ is continuous for each $x \in A(N, M)$. In light of (16), (17), and (58)–(60), we conclude that for $x, y \in A(N, M)$ and $t \geq T$

$$\begin{aligned} &|S_L x(t) - S_L y(t)| \\ &\leq \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\ &\quad \times |h(u, x(h_1(u)), \ldots, x(h_k(u))) \\ &\qquad - h(u, y(h_1(u)), \ldots, y(h_k(u)))| dudvdw \\ &\quad + \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_{w}^{+\infty} \int_{v}^{+\infty} \int_{u}^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\ &\quad \times |f(s, x(f_1(s)), \ldots, x(f_k(s))) \\ &\qquad - f(s, y(f_1(s)), \ldots, y(f_k(s)))| dsdudvdw \end{aligned}$$

$$\begin{aligned}
&\leq \|x-y\| \\
&\quad\times \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw \\
&\quad+\|x-y\| \\
&\quad\times \sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\int_{u}^{+\infty}\frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw \\
&\leq \Bigg(\sum_{i=1}^{\infty}\int_{T+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw \\
&\quad+\sum_{i=1}^{\infty}\int_{T+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\int_{u}^{+\infty}\frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw\Bigg) \\
&\quad\times\|x-y\| \\
&=\theta\|x-y\|, \\
&S_L x(t) \\
&=L-\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times h(u,x(h_1(u)),\dots,x(h_k(u)))\,dudvdw \\
&\quad+\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\int_{u}^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times[f(s,x(f_1(s)),\dots,x(f_k(s)))-g(s)]\,dsdudvdw \\
&\leq L+\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw \\
&\quad+\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\int_{u}^{+\infty}\frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw \\
&< L+\min\{M-L,L-N\} \\
&\leq M, \\
&S_L x(t) \\
&\geq L-\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw \\
&\quad-\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\int_{u}^{+\infty}\frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw \\
&> L-\min\{M-L,L-N\} \\
&\geq N,
\end{aligned} \tag{61}$$

which yield that (29) holds. That is, S_L is a contraction mapping in $A(N,M)$ and has a unique fixed point $x \in A(N,M)$; that is,

$$\begin{aligned}
&x(t) \\
&=L-\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times h(u,x(h_1(u)),\dots,x(h_k(u)))\,dudvdw \\
&\quad+\sum_{i=1}^{\infty}\int_{t+i\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\int_{u}^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times[f(s,x(f_1(s)),\dots,x(f_k(s))) \\
&\quad\quad-g(s)]\,dsdudvdw,\quad t\geq T,
\end{aligned} \tag{62}$$

$$\begin{aligned}
&x(t-\tau) \\
&=L-\sum_{i=1}^{\infty}\int_{t+(i-1)\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times h(u,x(h_1(u)),\dots,x(h_k(u)))\,dudvdw \\
&\quad+\sum_{i=1}^{\infty}\int_{t+(i-1)\tau}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\int_{u}^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times[f(s,x(f_1(s)),\dots,x(f_k(s))) \\
&\quad\quad-g(s)]\,dsdudvdw,\quad t\geq T+\tau.
\end{aligned} \tag{63}$$

By virtue of (62) and (63), we get that

$$\begin{aligned}
&x(t)-x(t-\tau) \\
&=\int_{t}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times h(u,x(h_1(u)),\dots,x(h_k(u)))\,dudvdw \\
&\quad-\int_{t}^{+\infty}\int_{w}^{+\infty}\int_{v}^{+\infty}\int_{u}^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad\times[f(s,x(f_1(s)),\dots,x(f_k(s))) \\
&\quad\quad-g(s)]\,dsdudvdw,\quad t\geq T+\tau,
\end{aligned} \tag{64}$$

which yields that

$$\begin{aligned}
&(x(t)-x(t-\tau))' \\
&=-\int_{t}^{+\infty}\int_{v}^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(t)} \\
&\quad\times h(u,x(h_1(u)),\dots,x(h_k(u)))\,dudv \\
&\quad+\int_{t}^{+\infty}\int_{v}^{+\infty}\int_{u}^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(t)} \\
&\quad\times[f(s,x(f_1(s)),\dots,x(f_k(s))) \\
&\quad\quad-g(s)]\,dsdudv,\quad t\geq T+\tau,
\end{aligned}$$

$$
\begin{aligned}
&\left(\gamma(t)\left(x(t)-x(t-\tau)\right)'\right)' \\
&= \int_t^{+\infty} \frac{1}{\alpha(u)\beta(t)} \\
&\quad \times h\left(u, x\left(h_1(u)\right), \ldots, x\left(h_k(u)\right)\right) du \\
&\quad - \int_t^{+\infty} \int_u^{+\infty} \frac{1}{\alpha(u)\beta(t)} \\
&\quad \times \left[f\left(s, x\left(f_1(s)\right), \ldots, x\left(f_k(s)\right)\right)\right. \\
&\qquad \left. - g(s)\right] ds\, du, \quad t \geq T+\tau,
\end{aligned} \tag{65}
$$

which ensures that

$$
\begin{aligned}
&\left[\beta(t)\left(\gamma(t)\left(x(t)-x(t-\tau)\right)'\right)'\right]' \\
&= -\frac{1}{\alpha(t)} h\left(u, x\left(h_1(u)\right), \ldots, x\left(h_k(u)\right)\right) \\
&\quad + \int_t^{+\infty} \frac{1}{\alpha(t)} \left[f\left(s, x\left(f_1(s)\right), \ldots, x\left(f_k(s)\right)\right) - g(s)\right] ds, \\
&\qquad t \geq T+\tau, \\
&\left[\alpha(t)\left[\beta(t)\left(\gamma(t)\left(x(t)-x(t-\tau)\right)'\right)'\right]'\right]' \\
&= -\left[h\left(t, x\left(h_1(t)\right), \ldots, x\left(h_k(t)\right)\right)\right]' \\
&\quad - \left[f\left(t, x\left(f_1(t)\right), \ldots, x\left(f_k(t)\right)\right) - g(t)\right], \\
&\qquad t \geq T+\tau,
\end{aligned} \tag{66}
$$

which gives that x is a bounded positive solution of (11). By means of (29), (57), (58), and (60), we deduce that, for any $m \in \mathbb{N}_0$ and $t \geq T$,

$$
\begin{aligned}
&\left|x_{m+1}(t) - x(t)\right| \\
&= \Bigg| \left(1-\alpha_m-\beta_m\right) x_m(t) + \alpha_m \\
&\quad \times \Bigg\{ L - \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times h\left(u, x_m\left(h_1(u)\right), \ldots, x_m\left(h_k(u)\right)\right) du\,dv\,dw \\
&\quad + \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times \left[f\left(s, x_m\left(f_1(s)\right), \ldots, x_m\left(f_k(s)\right)\right)\right. \\
&\qquad \left. - g(s)\right] ds\,du\,dv\,dw \Bigg\} + \beta_m \gamma_m(t) - x(t) \Bigg| \\
&\leq \left(1-\alpha_m-\beta_m\right)\left|x_m(t)-x(t)\right| \\
&\quad + \alpha_m \left|S_L x_m(t) - S_L x(t)\right| + \beta_m \left|\gamma_m(t) - x(t)\right| \\
&\leq \left(1-\alpha_m-\beta_m\right)\left|x_m(t)-x(t)\right| \\
&\quad + \alpha_m \theta \left|x_m(t)-x(t)\right| + 2M\beta_m \\
&\leq \left(1-(1-\theta)\alpha_m\right)\left|x_m(t)-x(t)\right| + 2M\beta_m,
\end{aligned} \tag{67}
$$

which implies (22). Thus Lemma 1, (23), and (24) ensure that $\lim_{m\to\infty} x_m = x$.

Next we show that (b) holds. Let $L_1, L_2 \in (N, M)$ with $L_1 \neq L_2$. As in the proof of (a), for each $j \in \{1,2\}$, we infer that there exists $\theta_j \in (0,1)$, $T_j > t_0 + |a| + \tau + |T_0|$, and $S_{L_j} : A(N,M) \to A(N,M)$ satisfying (58)–(60), where θ, T, S_L are replaced by θ_j, T_j, S_{L_j}, respectively, and the contraction mapping S_{L_j} possesses a unique fixed point $z_j \in A(N,M)$, and z_j is a bounded positive solution of (11); that is,

$$
\begin{aligned}
&z_j(t) \\
&= L_j - \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times h\left(u, z_j\left(h_1(u)\right), \ldots, z_j\left(h_k(u)\right)\right) du\,dv\,dw \\
&\quad + \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times \left[f\left(s, z_j\left(f_1(s)\right), \ldots, z_j\left(f_k(s)\right)\right)\right. \\
&\qquad \left. - g(s)\right] ds\,du\,dv\,dw, \quad t \geq T_j.
\end{aligned} \tag{68}
$$

Put $T^* = \max\{T_1, T_2\}$. Using (16), (58), and (68), we conclude that for $t > T^*$

$$
\begin{aligned}
&\left|z_1(t) - z_2(t)\right| \\
&= \Bigg| L_1 - L_2 \\
&\quad - \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times \left[h\left(u, z_1\left(h_1(u)\right), \ldots, z_1\left(h_k(u)\right)\right)\right. \\
&\qquad \left. - h\left(u, z_2\left(h_1(u)\right), \ldots, z_2\left(h_k(u)\right)\right)\right] du\,dv\,dw \\
&\quad + \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\
&\quad \times \left[f\left(s, z_1\left(f_1(s)\right), \ldots, z_1\left(f_k(s)\right)\right)\right. \\
&\qquad \left. - f\left(s, z_2\left(f_1(s)\right), \ldots, z_2\left(f_k(s)\right)\right)\right] ds\,du\,dv\,dw \Bigg| \\
&\geq \left|L_1 - L_2\right| - \left\|z_1 - z_2\right\| \\
&\quad \times \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \frac{Q_2(u)}{\left|\alpha(u)\beta(v)\gamma(w)\right|} du\,dv\,dw \\
&\quad - \left\|z_1 - z_2\right\| \\
&\quad \times \sum_{i=1}^{\infty} \int_{t+i\tau}^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{Q_1(s)}{\left|\alpha(u)\beta(v)\gamma(w)\right|} ds\,du\,dv\,dw \\
&\geq \left|L_1 - L_2\right| - \left\|z_1 - z_2\right\| \\
&\quad \times \Bigg(\sum_{i=1}^{\infty} \int_{\max\{T_1,T_2\}+i\tau}^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \frac{Q_2(u)}{\left|\alpha(u)\beta(v)\gamma(w)\right|} du\,dv\,dw \\
&\quad + \sum_{i=1}^{\infty} \int_{\max\{T_1,T_2\}+i\tau}^{+\infty} \int_w^{+\infty} \int_v^{+\infty} \int_u^{+\infty} \frac{Q_1(s)}{\left|\alpha(u)\beta(v)\gamma(w)\right|} ds\,du\,dv\,dw \Bigg) \\
&\geq \left|L_1 - L_2\right| - \left\|z_1 - z_2\right\| \max\left\{\theta_1, \theta_2\right\},
\end{aligned} \tag{69}
$$

which yields that

$$\|z_1 - z_2\| \geq \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0; \tag{70}$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 7. *Assume that there exist constants M, N, c_1, c_2 and $T_0 \geq t_0$ and functions Q_1, Q_2, R_1 and $R_2 \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (16)–(18) *and*

$$1 < c_2 < c_1 < c_2^2, \quad 0 < \frac{c_1^2 - c_2}{c_1} N < \frac{c_2^2 - c_1}{c_2} M; \tag{71}$$

$$c_2 \leq p(t) \leq c_1, \quad t \geq T_0. \tag{72}$$

Then

(a) for any $L \in ((c_1/c_2)M + c_1 N, (c_2/c_1)N + c_2 M)$, there exist $\theta \in (0,1)$ and $T > t_0 + |a| + \tau + |T_0|$ such that, for each $x_0 \in A(N, M)$, the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}$ with mixed errors generated by the scheme

$$x_{m+1}(t) = \begin{cases} (1-\alpha_m-\beta_m)x_m(t) + \alpha_m \\ \quad \times \left\{ \frac{L}{p(t+\tau)} - \frac{x_m(t+\tau)}{p(t+\tau)} + \frac{1}{p(t+\tau)} \right. \\ \quad \times \int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times h(u, x_m(h_1(u)), \ldots, x_m(h_k(u)))\,dudvdw \\ \quad - \frac{1}{p(t+\tau)} \\ \quad \times \int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times [f(s, x_m(f_1(s)), \ldots, x_m(f_k(s))) \\ \quad \left. - g(s)]\,dsdudvdw \right\} + \beta_m\gamma_m(t), \\ \quad t \geq T,\ m \in \mathbb{N}_0, \\ (1-\alpha_m-\beta_m)x_m(T) + \alpha_m \\ \quad \times \left\{ \frac{L}{p(T+\tau)} - \frac{x_m(T+\tau)}{p(T+\tau)} + \frac{1}{p(T+\tau)} \right. \\ \quad \times \int_{T+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times h(u, x_m(h_1(u)), \ldots, x_m(h_k(u)))\,dudvdw \\ \quad - \frac{1}{p(T+\tau)} \int_{T+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times [f(s, x_m(f_1(s)), \ldots, x_m(f_k(s))) \\ \quad \left. - g(s)]\,dsdudvdw \right\} + \beta_m\gamma_m(T), \\ \quad a \leq t < T,\ m \in \mathbb{N}_0 \end{cases} \tag{73}$$

with (23) *and* (24) *converges to a bounded positive solution $x \in A(N, M)$ of* (11) *and has the error estimate* (22);

(b) equation (11) *possesses uncountably many bounded positive solutions in $A(N, M)$.*

Proof. In the first place, we prove that (a) holds. Let $L \in ((c_1/c_2)M + c_1 N, (c_2/c_1)N + c_2 M)$. It follows from (18) and (71) that there exist $\theta \in (0,1)$ and $T > t_0 + |a| + \tau + |T_0|$ satisfying

$$\begin{aligned} \theta &= \frac{1}{c_2}\left(1 + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw \right. \\ &\quad \left. + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw\right), \end{aligned} \tag{74}$$

$$\begin{aligned} &\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw \\ &\quad + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{R_1(s) + |g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw \\ &\quad < \min\left\{c_2 M - L + \frac{c_2}{c_1}N, -c_2 N + \frac{c_2}{c_1}L - M\right\}. \end{aligned} \tag{75}$$

Define a mapping $S_L : A(N, M) \to CB([a, +\infty), \mathbb{R})$ by

$$S_L x(t) = \begin{cases} \frac{L}{p(t+\tau)} - \frac{x(t+\tau)}{p(t+\tau)} + \frac{1}{p(t+\tau)} \\ \quad \times \int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times h(u, x(h_1(u)), \ldots, x(h_k(u)))\,dudvdw \\ \quad - \frac{1}{p(t+\tau)} \\ \quad \times \int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad \times [f(s, x(f_1(s)), \ldots, x(f_k(s))) \\ \quad - g(s)]\,dsdudvdw, \\ \quad t \geq T,\ x \in A(N, M), \\ S_L x(T), \quad a \leq t < T,\ x \in A(N, M). \end{cases} \tag{76}$$

Obviously, $S_L x$ is continuous for each $x \in A(N, M)$. In view of (16), (17), (71), (72), and (74)–(76), we conclude that for $x, y \in A(N, M)$ and $t \geq T$

$$\begin{aligned} &|S_L x(t) - S_L y(t)| \\ &\leq \frac{1}{|p(t+\tau)|}|x(t+\tau) - y(t+\tau)| + \frac{1}{|p(t+\tau)|} \\ &\quad \times \int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\ &\quad \times |h(u, x(h_1(u)), \ldots, x(h_k(u))) \\ &\qquad - h(u, y(h_1(u)), \ldots, y(h_k(u)))|\,dudvdw \\ &\quad + \frac{1}{|p(t+\tau)|}\int_{t+t}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{|\alpha(u)\beta(v)\gamma(w)|} \\ &\quad \times |f(s, x(f_1(s)), \ldots, x(f_k(s))) \\ &\qquad - f(s, y(f_1(s)), \ldots, y(f_k(s)))|\,dsdudvdw \end{aligned}$$

$$\leq \frac{1}{c_2}\|x-y\|$$
$$\times\left(1+\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw\right.$$
$$\left.+\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw\right)$$
$$=\theta\|x-y\|,$$
$$S_Lx(t)$$
$$=\frac{L}{p(t+\tau)}-\frac{x(t+\tau)}{p(t+\tau)}$$
$$+\frac{1}{p(t+\tau)}\int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)}$$
$$\times h(u,x(h_1(u)),\ldots,x(h_k(u)))dudvdw$$
$$-\frac{1}{p(t+\tau)}\int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)}$$
$$\times[f(s,x(f_1(s)),\ldots,x(f_k(s)))$$
$$-g(s)]dsdudvdw$$
$$\leq\frac{L}{c_2}-\frac{N}{c_1}+\frac{1}{c_2}$$
$$\times\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw$$
$$+\frac{1}{c_2}\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw$$
$$<\frac{L}{c_2}-\frac{N}{c_1}+\frac{1}{c_2}$$
$$\times\min\left\{c_2M-L+\frac{c_2}{c_1}N,-c_2N+\frac{c_2}{c_1}L-M\right\}$$
$$\leq M,$$
$$S_Lx(t)$$
$$\geq\frac{L}{c_1}-\frac{M}{c_2}-\frac{1}{c_2}$$
$$\times\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw$$
$$-\frac{1}{c_2}\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw$$
$$>\frac{L}{c_1}-\frac{M}{c_2}-\frac{1}{c_2}$$
$$\times\min\left\{c_2M-L+\frac{c_2}{c_1}N,-c_2N+\frac{c_2}{c_1}L-M\right\}$$
$$\geq N, \tag{77}$$

which imply that (29) holds. That is, S_L is a contraction mapping in $A(N,M)$ and has a unique fixed point $x \in A(N,M)$, which is a bounded positive solution of (11). By means of (29), (73), and (76), we obtain that for any $m\in\mathbb{N}_0$ and $t\geq T$

$$|x_{m+1}(t)-x(t)|$$
$$=\left|(1-\alpha_m-\beta_m)x_m(t)+\alpha_m\right.$$
$$\times\left\{\frac{L}{p(t+\tau)}-\frac{x(t+\tau)}{p(t+\tau)}+\frac{1}{p(t+\tau)}\right.$$
$$\times\int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)}$$
$$\times h(u,x_m(h_1(u)),\ldots,x_m(h_k(u)))dudvdw$$
$$-\frac{1}{p(t+\tau)}\int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)}$$
$$\times[f(s,x_m(f_1(s)),\ldots,x_m(f_k(s)))$$
$$\left.\left.-g(s)]dsdudvdw\right\}+\beta_m\gamma_m(t)-x(t)\right|$$
$$\leq(1-\alpha_m-\beta_m)|x_m(t)-x(t)|$$
$$+\alpha_m|S_Lx_m(t)-S_Lx(t)|+\beta_m|\gamma_m(t)-x(t)|$$
$$\leq(1-\alpha_m-\beta_m)|x_m(t)-x(t)|$$
$$+\alpha_m\theta|x_m(t)-x(t)|+2M\beta_m$$
$$\leq(1-(1-\theta)\alpha_m)|x_m(t)-x(t)|+2M\beta_m$$
$$\leq(1-(1-\theta)\alpha_m)\|x_m-x\|+2M\beta_m, \tag{78}$$

which gives (22). Thus Lemma 1, (23), and (24) mean that $\lim_{m\to\infty}x_m=x$.

Next we show that (b) holds. Let $L_1,L_2\in((c_1/c_2)M+c_1N,(c_2/c_1)N+c_2M)$ with $L_1\neq L_2$. As in the proof of (a), we conclude that, for each $\in\{1,2\}$, there exist $\theta_j\in(0,1)$, $T_j>t_0+|a|+\tau+|T_0|$ and $S_{L_j}:A(N,M)\to A(N,M)$ satisfying (74)–(76), where θ,T,S_L are replaced by θ_j,T_j, and S_{L_j}, respectively, and the contraction mapping S_{L_j} has a unique fixed point $z_j \in A(N, M)$, which is also a bounded positive solution of (11). In order to prove (b), we need only to show that $z_1 \neq z_2$. Put $T^* = \max\{T_1, T_2\}$. Note that for $t \geq T^*$ and $j \in \{1, 2\}$

$$\begin{aligned} z_j(t) &= \frac{L_j}{p(t+\tau)} - \frac{z_j(t+\tau)}{p(t+\tau)} + \frac{1}{p(t+\tau)} \\ &\quad \times \int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ &\quad \times h\left(u, z_j(h_1(u)), \ldots, z_j(h_k(u))\right) du\,dv\,dw \\ &\quad - \frac{1}{p(t+\tau)} \\ &\quad \times \int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ &\quad \times \left[f\left(s, z_j(f_1(s)), \ldots, z_j(f_k(s))\right) \right. \\ &\qquad \left. - g(s)\right] ds\,du\,dv\,dw, \end{aligned} \tag{79}$$

which together with (16) and (74) implies that for $t \geq T^*$

$$\begin{aligned} &\left|z_1(t) - z_2(t)\right| \\ &= \left|\frac{L_1 - L_2}{p(t+\tau)} - \frac{z_1(t+\tau) - z_2(t+\tau)}{p(t+\tau)} + \frac{1}{p(t+\tau)} \right. \\ &\quad \times \int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ &\quad \times \left[h(u, z_1(h_1(u)), \ldots, z_1(h_k(u)))\right. \\ &\qquad \left. - h(u, z_2(h_1(u)), \ldots, z_2(h_k(u)))\right] du\,dv\,dw \\ &\quad - \frac{1}{p(t+\tau)} \\ &\quad \times \int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ &\quad \times \left[f(s, z_1(f_1(s)), \ldots, z_1(f_k(s)))\right. \\ &\qquad \left.\left. - f(s, z_2(f_1(s)), \ldots, z_2(f_k(s)))\right] ds\,du\,dv\,dw\right| \\ &\geq \frac{|L_1 - L_2|}{c_1} - \frac{\|z_1 - z_2\|}{c_2} - \frac{\|z_1 - z_2\|}{c_2} \\ &\quad \times \int_{T^*}^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{Q_2(u)}{|\alpha(u)\beta(v)\gamma(w)|} du\,dv\,dw \\ &\quad - \frac{\|z_1 - z_2\|}{c_2} \\ &\quad \times \int_{T^*}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{Q_1(s)}{|\alpha(u)\beta(v)\gamma(w)|} ds\,du\,dv\,dw \\ &\geq \frac{|L_1 - L_2|}{c_1} - \|z_1 - z_2\| \max\{\theta_1, \theta_2\}, \end{aligned} \tag{80}$$

which yields that

$$\|z_1 - z_2\| \geq \frac{|L_1 - L_2|}{c_1(1 + \max\{\theta_1, \theta_2\})} > 0; \tag{81}$$

that is, $z_1 \neq z_2$. This completes the proof. □

Theorem 8. *Assume that there exist constants* M, N, c_1, c_2 *and* $T_0 \geq t_0$ *and functions* Q_1, Q_2, R_1 *and* $R_2 \in C([t_0, +\infty), \mathbb{R}^+)$ *satisfying* (16)–(18) *and*

$$\begin{gathered} 0 < (c_1 - 1)N < (c_2 - 1)M; \\ -\infty < -c_1 \leq p(t) \leq -c_2 < -1, \quad t \geq T_0. \end{gathered} \tag{82}$$

Then

(a) for any $L \in ((c_1 - 1)N, (c_2 - 1)M)$, *there exist* $\theta \in (0, 1)$ *and* $T > t_0 + |a| + \tau + |T_0|$ *such that, for each* $x_0 \in A(N, M)$, *the Mann iterative sequence* $\{x_m\}_{m\in\mathbb{N}_0}$ *with mixed errors generated by the scheme*

$$x_{m+1}(t) = \begin{cases} (1 - \alpha_m - \beta_m)x_m(t) + \alpha_m \\ \quad \times \left\{ \dfrac{-L}{p(t+\tau)} - \dfrac{x_m(t+\tau)}{p(t+\tau)} + \dfrac{1}{p(t+\tau)} \right. \\ \qquad \times \displaystyle\int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \qquad \times h(u, x_m(h_1(u)), \ldots, x_m(h_k(u)))\, du\,dv\,dw \\ \qquad - \dfrac{1}{p(t+\tau)} \displaystyle\int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \qquad \times [f(s, x_m(f_1(s)), \ldots, x_m(f_k(s))) \\ \qquad\quad \left. - g(s)]\, ds\,du\,dv\,dw \right\} + \beta_m\gamma_m(t), \\ \qquad\qquad t \geq T,\ m \in \mathbb{N}_0, \\ (1 - \alpha_m - \beta_m)x_m(T) + \alpha_m \\ \quad \times \left\{ \dfrac{-L}{p(T+\tau)} - \dfrac{x_m(T+\tau)}{p(T+\tau)} + \dfrac{1}{p(T+\tau)} \right. \\ \qquad \times \displaystyle\int_{T+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \qquad \times h(u, x_m(h_1(u)), \ldots, x_m(h_k(u)))\, du\,dv\,dw \\ \qquad - \dfrac{1}{p(T+\tau)} \\ \qquad \times \displaystyle\int_{T+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty} \frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \qquad \times [f(s, x_m(f_1(s)), \ldots, x_m(f_k(s))) \\ \qquad\quad \left. - g(s)]\, ds\,du\,dv\,dw \right\} + \beta_m\gamma_m(T), \\ \qquad\qquad a \leq t < T,\ m \in \mathbb{N}_0 \end{cases} \tag{83}$$

with (23) *and* (24) *converges to a bounded positive solution* $x \in A(N, M)$ *of* (11) *and has the error estimate* (22);

(b) equation (11) *possesses uncountably many bounded positive solutions in* $A(N, M)$.

Proof. First of all, we prove that (a) holds. Let $L \in ((c_1 - 1)N, (c_2 - 1)M)$. It follows from (18) and (82) that there exist $\theta \in (0,1)$ and $T > t_0 + |a| + \tau + |T_0|$ satisfying (74) and

$$\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw + \int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw < \min\left\{(c_2-1)M-L, \left(\frac{c_2}{c_1}-c_2\right)N+\frac{c_2}{c_1}L\right\}. \tag{84}$$

Define a mapping $S_L : A(N,M) \to CB([a,+\infty),\mathbb{R})$ by

$$S_Lx(t) = \begin{cases} \dfrac{-L}{p(t+\tau)} - \dfrac{x(t+\tau)}{p(t+\tau)} + \dfrac{1}{p(t+\tau)} \\ \quad\times\displaystyle\int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad\times h(u,x(h_1(u)),\ldots,x(h_k(u)))\,dudvdw \\ \quad-\dfrac{1}{p(t+\tau)} \\ \times\displaystyle\int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ \quad\times[f(s,x(f_1(s)),\ldots,x(f_k(s))) \\ \qquad - g(s)]\,dsdudvdw, \\ t\ge T,\ x\in A(N,M), \\ S_Lx(T), \quad a\le t<T,\ x\in A(N,M). \end{cases} \tag{85}$$

Obviously S_Lx is continuous for each $x \in A(N,M)$. On account of (17), (82), (84), and (85), we get that for any $x \in A(N,M)$ and $t \ge T$

$$\begin{aligned} S_Lx(t) &= \frac{-L}{p(t+\tau)} - \frac{x(t+\tau)}{p(t+\tau)} + \frac{1}{p(t+\tau)} \\ &\quad\times\int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ &\quad\times h(u,x(h_1(u)),\ldots,x(h_k(u)))\,dudvdw \\ &\quad-\frac{1}{p(t+\tau)} \\ &\quad\times\int_{t+\tau}^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{1}{\alpha(u)\beta(v)\gamma(w)} \\ &\quad\times[f(s,x(f_1(s)),\ldots,x(f_k(s)))-g(s)]\,dsdudvdw \\ &\le \frac{L}{c_2}+\frac{M}{c_2}+\frac{1}{c_2} \\ &\quad\times\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw \\ &\quad+\frac{1}{c_2} \\ &\quad\times\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw \\ &< \frac{L}{c_2}+\frac{M}{c_2}+\frac{1}{c_2} \\ &\quad\times\min\left\{(c_2-1)M-L,\left(\frac{c_2}{c_1}-c_2\right)N+\frac{c_2}{c_1}L\right\} \\ &\le M, \end{aligned} \tag{86}$$

$$\begin{aligned} S_Lx(t) &\ge \frac{L}{c_1}+\frac{N}{c_1}-\frac{1}{c_2} \\ &\quad\times\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\frac{R_2(u)}{|\alpha(u)\beta(v)\gamma(w)|}dudvdw \\ &\quad-\frac{1}{c_2} \\ &\quad\times\int_T^{+\infty}\int_w^{+\infty}\int_v^{+\infty}\int_u^{+\infty}\frac{R_1(s)+|g(s)|}{|\alpha(u)\beta(v)\gamma(w)|}dsdudvdw \\ &> \frac{L}{c_1}+\frac{N}{c_1}-\frac{1}{c_2} \\ &\quad\times\min\left\{(c_2-1)M-L,\left(\frac{c_2}{c_1}-c_2\right)N+\frac{c_2}{c_1}L\right\} \\ &\ge N, \end{aligned} \tag{87}$$

which imply (29). That is, S_L is a contraction mapping in $A(N,M)$ and has a unique fixed point $x \in A(N,M)$, which is a bounded positive solution of (11). The rest of the proof is similar to that of Theorem 7 and is omitted. This completes the proof. □

3. Examples

Now we construct seven examples as applications of the results presented in Section 2.

Example 9. Consider the following fourth order nonlinear neutral delay differential equation:

$$\begin{aligned}&\left(\left(t^2+1\right)\right.\\&\times\left(\frac{2t^4+1}{\ln(t+1)}\right.\\&\times\left.\left.\left(-t^3\left(x(t)+\frac{3t^2\cos t-1/2}{1+10t^2}x(t-\tau)\right)'\right)'\right)'\right)'\\&+\left(\frac{t\sin^2\left(tx^3(t-1)\right)}{(t+3)^2(t+1)^3}\right)'+\frac{\cos(t)\,x^3\left(t^2\right)}{t^3}\\&=\frac{\sin\left(2-t^2\right)+1}{3t^4},\quad t\geq 3,\end{aligned}\tag{88}$$

where $\tau>0$, $t_0=T_0=3$, $k=1$, $N=2$, $M=20$, $c_1=3/10$, $c_2=3/5$, $a=\min\{3-\tau,2\}$, and

$$\begin{gathered}\alpha(t)=t^2+1,\qquad \beta(t)=\frac{2t^4+1}{\ln(t+1)},\\ \gamma(t)=-t^3,\qquad p(t)=\frac{3t^2\cos t-1/2}{1+10t^2},\\ g(t)=\frac{\sin\left(2-t^2\right)+1}{3t^4},\qquad h_1(t)=t-1,\\ f_1(t)=t^2,\qquad f(t,u)=\frac{u^3\cos(t)}{t^3},\\ h(t,u)=\frac{t\sin^2\left(tu^3\right)}{(t+3)^2(t+1)^3},\qquad Q_1(t)=\frac{3M^2}{t^3},\\ Q_2(t)=\frac{6t^2M^2}{(t+3)^2(t+1)^3},\qquad R_1(t)=\frac{M^3}{t^3},\\ R_2(t)=\frac{t}{(t+3)^2(t+1)^3},\qquad (t,u)\in\left[t_0,+\infty\right)\times\mathbb{R}.\end{gathered}\tag{89}$$

It is easy to verify that (16)–(20) are satisfied. It follows from Theorem 2 that (11) possesses uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L\in(N+c_1M,(1-c_2)M)$, there exist $\theta\in(0,1)$ and $T>t_0+|a|+\tau+|T_0|$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}$ with mixed errors generated by (21) converges to a bounded positive solution $x\in A(N,M)$ of (11) and has the error estimate (22), where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $A(N,M)$, and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying (23) and (24).

Example 10. Consider the following fourth order nonlinear neutral delay differential equation:

$$\begin{aligned}&\left(\frac{t^6\ln\left(1+t^2\right)}{1+t}\right.\\&\times\left.\left(t^4\left(-\sqrt{t^7+1}\left(x(t)+\frac{t^3}{3t^3+2}x(t-\tau)\right)'\right)'\right)'\right)'\\&+\left(\frac{x^2(t-2)+x^2\left(\sqrt{t+1}\right)-x\left(\sqrt{t+1}\right)}{1+t^3}\right)'\\&+\frac{2x^2(t-1)+x^2\left(t-\sqrt{t}\right)}{2t^4+1}=\frac{1-t}{t^4+\cos^2t},\quad t\geq 10,\end{aligned}\tag{90}$$

where $\tau>0$, $t_0=T_0=10$, $k=2$, $N=5$, $M=21$, $c_1=1/3$, $c_2=1/4$, $a=\min\{10-\tau,\sqrt{11}\}$, and

$$\begin{gathered}\alpha(t)=\frac{t^6\ln\left(1+t^2\right)}{1+t},\qquad \beta(t)=t^4,\\ \gamma(t)=-\sqrt{t^7+1},\qquad p(t)=\frac{t^3}{3t^3+2},\\ g(t)=\frac{1-t}{t^4+\cos^2t},\qquad h_1(t)=t-2,\\ h_2(t)=\sqrt{t+1},\qquad f_1(t)=t-1,\\ f_2(t)=t-\sqrt{t},\qquad f(t,u,v)=\frac{2u^2+v^2}{2t^4+1},\\ h(t,u,v)=\frac{u^2+v^2-v}{1+t^3},\\ Q_1(t)=\frac{6M}{2t^4+1},\qquad Q_2(t)=\frac{4M+1}{1+t^3},\\ R_1(t)=\frac{3M^2}{2t^4+1},\\ R_2(t)=\frac{2M^2+M}{1+t^3},\qquad (t,u,v)\in\left[t_0,+\infty\right)\times\mathbb{R}^2.\end{gathered}\tag{91}$$

It is easy to verify that (16)–(18) and (35) are satisfied. It follows from Theorem 3 that (11) possesses uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L\in(N+c_1M,M+c_2N)$, there exist $\theta\in(0,1)$ and $T>t_0+|a|+\tau+|T_0|$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}$ with mixed errors generated by (21) converges to a bounded positive solution $x\in A(N,M)$ of (11) and has the error estimate (22), where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $A(N,M)$, and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying (23) and (24).

Example 11. Consider the following fourth order nonlinear neutral delay differential equation:

$$\begin{aligned}&\Bigg(-t^5\Bigg(\left(3t^2+2\right)\\&\quad\times\left(\frac{(2t+1)^3}{t}\left(x(t)+\frac{1-t}{3+5t}x(t-\tau)\right)'\right)'\Bigg)'\Bigg)'\\&+\left(\frac{tx^2(2t-3)x^2(e^t)}{(1+t^2)(2+\sin(x(2t-3)x(t^2)))}\right)'\\&+\frac{t^2}{1+t^7}x^2\left((t+1)^2\right)\\&+\frac{\sqrt{\ln(t+1)}}{e^t+1}\sin^2\left(tx(3t+1)x^3(2t)\right)\\&=\frac{\sqrt{1+t^2}}{t^4+1},\quad t\ge 2,\end{aligned}\tag{92}$$

where $\tau>0$, $t_0=T_0=2$, $k=3$, $N=100$, $M=500$, $c_1=4/5$, $c_2=1/13$, $a=\min\{2-\tau,1\}$, and

$$\begin{gathered}\alpha(t)=-t^5,\qquad\beta(t)=3t^2+2,\\\gamma(t)=\frac{(2t+1)^3}{t},\qquad p(t)=\frac{1-t}{3+5t},\\g(t)=\frac{\sqrt{1+t^2}}{t^4+1},\qquad h_1(t)=2t-3,\\h_2(t)=t^2,\qquad h_3(t)=e^t,\\f_1(t)=(t+1)^2,\qquad f_2(t)=3t+1,\quad f_3(t)=2t,\\f(t,u,v,w)=\frac{t^2u^2}{1+t^7}+\frac{\sqrt{\ln(t+1)}}{e^t+1}\sin^2\left(tvw^3\right),\\h(t,u,v,w)=\frac{tu^2w^2}{(1+t^2)(2+\sin(uv))},\\Q_1(t)=\frac{2Mt^2}{1+t^7}+\frac{4tM^3\sqrt{\ln(t+1)}}{e^t+1},\\Q_2(t)=\frac{t\left(2M^5+12M^3\right)}{1+t^2},\\R_1(t)=\frac{t^2M^2}{1+t^7}+\frac{\sqrt{\ln(t+1)}}{e^t+1},\\R_2(t)=\frac{tM^4}{1+t^2},\quad(t,u,v,w)\in[t_0,+\infty)\times\mathbb{R}^3.\end{gathered}\tag{93}$$

It is easy to verify that (17), (18), and (39) are satisfied. It follows from Theorem 4 that (11) possesses uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L\in((1-c_2)N,(1-c_1)M)$, there exist $\theta\in(0,1)$ and $T>t_0+|a|+\tau+|T_0|$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}$ with mixed errors generated by (21) converges to a bounded positive solution $x\in A(N,M)$ of (11) and has the error estimate (22), where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $A(N,M)$, and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying (23) and (24).

Example 12. Consider the following fourth order nonlinear neutral delay differential equation:

$$\begin{aligned}&\left(\frac{2t^4}{1+t}\left(\left(t^2+1\right)\left(-t^8(x(t)+x(t-\tau))'\right)'\right)'\right)'\\&\quad+\left(\frac{t^3+1}{t^{10}\ln(t+2)}x^2(t-2)x(t-1)\right)'\\&\quad+\frac{x^2(t-3)}{(1+t^3)(1+x^2(t-4))}\\&\quad=\frac{\sqrt{1+t^2}-\sin\left(2t^2+1\right)}{t^8},\quad t\ge 3,\end{aligned}\tag{94}$$

where $\tau>0$, $t_0=T_0=3$, $k=2$, $N=10$, $M=20$, $a=\min\{3-\tau,-1\}$, and

$$\begin{gathered}\alpha(t)=\frac{2t^4}{1+t},\qquad\beta(t)=t^2+1,\\\gamma(t)=-t^8,\qquad p(t)=1,\\g(t)=\frac{\sqrt{1+t^2}-\sin\left(2t^2+1\right)}{t^8},\qquad h_1(t)=t-2,\\h_2(t)=t-1,\\f_1(t)=t-3,\qquad f_2(t)=t-4,\\f(t,u,v)=\frac{u^2}{(1+t^3)(1+v^2)},\\h(t,u,v)=\frac{\left(t^3+1\right)u^2v}{t^{10}\ln(t+2)},\qquad Q_1(t)=\frac{2M+4M^3}{(1+t^3)(1+N^2)^2},\\Q_2(t)=\frac{3M^2\left(1+t^3\right)}{t^{10}\ln(2+t)},\\R_1(t)=\frac{M^2}{(1+t^3)(1+N^2)},\\R_2(t)=\frac{M^3\left(1+t^3\right)}{t^{10}\ln(t+2)},\quad(t,u,v)\in[t_0,+\infty)\times\mathbb{R}^2.\end{gathered}\tag{95}$$

It is easy to verify that (16)–(18) and (42) are satisfied. It follows from Theorem 5 that (11) possesses uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L\in(N,M)$, there exist $\theta\in(0,1)$ and $T>t_0+|a|+\tau+|T_0|$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}$ with mixed errors generated by (43) converges to a bounded positive solution $x\in A(N,M)$ of (11) and has the error estimate (22), where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in

$A(N, M)$, and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0, 1]$ satisfying (23) and (24).

Example 13. Consider the following fourth order nonlinear neutral delay differential equation:

$$\begin{aligned}
&\left(-e^t\left(t^5\left(\left(1-t^7\right)(x(t)-x(t-\tau))'\right)'\right)'\right)' \\
&\quad+\left(\frac{1}{1+t^6}\right. \\
&\quad\times\left.\left(x^2(t-3)\,x^2(t-1)-\frac{2x^3(t+1)}{1+\sqrt{1+x^2(t-3)}}\right)\right)' \\
&\quad+\frac{1}{3t^4+2t}\left(x^3(t-4)(1+\sin x(2t-3))\right. \\
&\qquad\left.+\frac{x(2t-3)\cos^2(t-1)}{1+x^2(3t-2)}\right) \\
&\quad=\frac{\ln(1+t)}{1+t^{10}},\quad t\geq 4,
\end{aligned} \tag{96}$$

where $\tau > 0$, $t_0 = T_0 = 4$, $k = 3$, $N = 100$, $M = 200$, $a = \min\{4-\tau, 0\}$, and

$$\alpha(t)=-e^t,\qquad \beta(t)=t^5,\qquad \gamma(t)=1-t^7, \tag{97}$$

$$p(t)=-1,\qquad g(t)=\frac{\ln(1+t)}{1+t^{10}},$$

$$h_1(t)=t+1,\qquad h_2(t)=t-3,\qquad h_3(t)=t-1,$$

$$f_1(t)=t-4,\qquad f_2(t)=2t-3,$$

$$f_3(t)=3t-2,$$

$$f(t,u,v,w)=\frac{1}{3t^4+2t}\left(u^3(1+\sin v)+\frac{v\cos^2(t-1)}{1+w^2}\right),$$

$$h(t,u,v,w)=\frac{1}{1+t^6}\left(v^2w^2-\frac{2u^3}{1+\sqrt{1+v^2}}\right),$$

$$Q_1(t)=\frac{1}{3t^4+2t}\left(M^2(6+M)+\frac{1+3M^2}{\left(1+N^2\right)^2}\cos^2(t-1)\right),$$

$$\begin{aligned}
Q_2(t)=&\frac{1}{1+t^6} \\
&\times\left(4M^3+\frac{6M^2\left(1+\sqrt{1+M^2}\right)}{\left(1+\sqrt{1+N^2}\right)^2}\right. \\
&\quad\left.+\frac{2M^4}{\left(1+\sqrt{1+N^2}\right)^2\sqrt{1+N^2}}\right),
\end{aligned}$$

$$R_1(t)=\frac{1}{3t^4+2t}\left(2M^3+\frac{M}{1+N^2}\right),$$

$$\begin{aligned}
R_2(t)&=\frac{1}{1+t^6}\left(M^4+\frac{2M^3}{1+\sqrt{1+N^2}}\right), \\
&(t,u,v,w)\in\left[t_0,+\infty\right)\times\mathbb{R}^3.
\end{aligned} \tag{98}$$

It is easy to verify that (16), (17), (55), and (56) are satisfied. It follows from Theorem 6 that (11) possesses uncountably many bounded positive solutions in $A(N, M)$. On the other hand, for any $L \in (N, M)$, there exist $\theta \in (0, 1)$ and $T > t_0 + |a| + \tau + |T_0|$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}$ with mixed errors generated by (57) converges to a bounded positive solution $x \in A(N, M)$ of (11) and has the error estimate (22), where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $A(N, M)$, and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0, 1]$ satisfying (23) and (24).

Example 14. Consider the following fourth order nonlinear neutral delay differential equation:

$$\begin{aligned}
&\left(\frac{3+\sin t}{t^2}\right. \\
&\quad\times\left(t\ln^3 t\right. \\
&\quad\quad\left.\left.\times\left(\frac{t^5}{1+\sqrt{t}}\left(x(t)+\frac{4+5t}{3+2t}x(t-\tau)\right)'\right)'\right)'\right)' \\
&\quad+\left(\frac{\left(1-t^2\right)x^3(3t-1)\,x(t-5)}{(1+t)^6+t\,|x(t-2)|}\right)' \\
&\quad+\frac{t^3x^3(t-\sin t)-t^5x^2(t-\cos t)}{1+t^{11}+x^2(t-22)}=\frac{t^3}{1+t^8}, \\
&\qquad t\geq 20,
\end{aligned} \tag{99}$$

where $\tau > 0$, $t_0 = T_0 = 20$, $k = 3$, $N = 3$, $M = 20$, $c_1 = 3$, $c_2 = 2$, $a = \min\{20-\tau, -2\}$, and

$$\alpha(t)=\frac{3+\sin t}{t^2},\qquad \beta(t)=t\ln^3 t,\qquad \gamma(t)=\frac{t^5}{1+\sqrt{t}},$$

$$p(t)=\frac{4+5t}{3+2t},\qquad g(t)=\frac{t^3}{1+t^8},$$

$$h_1(t)=3t-1,\qquad h_2(t)=t-5,\qquad h_3(t)=t-2,$$

$$f_1(t)=t-\sin t,\qquad f_2(t)=t-\cos t,$$

$$f_3(t) = t - 22, \qquad f(t,u,v,w) = \frac{t^3u^3 - t^5v^2}{1 + t^{11} + w^2},$$

$$h(t,u,v,w) = \frac{\left(1 - t^2\right)u^3 v}{(1+t)^6 + t|w|},$$

$$Q_1(t) = \frac{3\left(t^3 + t^{14}\right)M^2 + 5t^3M^4 + 2\left(t^5 + t^{16}\right)M + 4t^5M^3}{\left(1 + t^{11} + N^2\right)^2},$$

$$Q_2(t) = \frac{4\left(1+t^2\right)(1+t)^6M^3 + 5t\left(1+t^2\right)M^4}{\left[(1+t)^6 + tN\right]^2},$$

$$R_1(t) = \frac{t^3M^3 + t^5M^2}{1 + t^{11} + N^2},$$

$$R_2(t) = \frac{\left(1+t^2\right)M^4}{(1+t)^6 + tN}, \quad (t,u,v,w) \in \left[t_0, +\infty\right) \times \mathbb{R}^3. \tag{100}$$

It is easy to verify that (16)–(18), (71), and (72) are satisfied. It follows from Theorem 7 that (11) possesses uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L \in ((c_1/c_2)M + c_1N, (c_2/c_1)\,N + c_2M)$, there exist $\theta \in (0,1)$ and $T > t_0 + |a| + \tau + |T_0|$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}$ with mixed errors generated by (73) converges to a bounded positive solution $x \in A(N,M)$ of (11) and has the error estimate (22), where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $A(N,M)$, and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying (23) and (24).

Example 15. Consider the following fourth order nonlinear neutral delay differential equation:

$$\left(\frac{\ln t}{t}\left(\frac{t^2}{2+\cos t}\left(\frac{t\ln^2(1+t)}{1+\sin^2 t} \times \left(x(t) + (-3 - \sin t)\,x(t-\tau)\right)'\right)'\right)'\right)'$$
$$+ \left(\frac{\left(1-\sqrt{t}\right)x^2(t-5)\,x^2\left(\sqrt{t}-7\right)}{1 + t^3 + \sin^2(tx(t-3))}\right)'$$
$$+ \frac{2 - x^2\left(\sqrt{t}\right) - x^2(t-1)\,x^3(t-10)}{1 + t^5 + \ln\left(t^2+3\right)}$$
$$= \frac{3 + \cos\left(t^2+1\right)}{t^5 + 2t^3}, \quad t \ge 9, \tag{101}$$

where $\tau > 0$, $t_0 = T_0 = 9$, $k = 3$, $N = 10$, $M = 50$, $c_1 = 4$, $c_2 = 2$, $a = \min\{9 - \tau, -4\}$, and

$$\alpha(t) = \frac{\ln t}{t}, \qquad \beta(t) = \frac{t^2}{2 + \cos t},$$

$$\gamma(t) = \frac{t\ln^2(1+t)}{1 + \sin^2 t}, \qquad p(t) = -3 - \sin t,$$

$$g(t) = \frac{3 + \cos\left(t^2+1\right)}{t^5 + 2t^3}, \qquad h_1(t) = t - 5,$$

$$h_2(t) = \sqrt{t} - 7, \qquad h_3(t) = t - 3,$$

$$f_1(t) = \sqrt{t}, \qquad f_2(t) = t - 1, \qquad f_3(t) = t - 10,$$

$$h(t,u,v,w) = \frac{\left(1 - \sqrt{t}\right)u^2v^2}{1 + t^3 + \sin^2(tw)},$$

$$f(t,u,v,w) = \frac{2 - u^2 - v^2w^3}{1 + t^5 + \ln\left(t^2+3\right)},$$

$$Q_1(t) = \frac{2M + 5M^4}{1 + t^5 + \ln\left(t^2+3\right)},$$

$$Q_2(t) = \frac{2M^3\left(1+\sqrt{t}\right)\left(2t^3 + 4 + tM\right)}{\left(1+t^3\right)^2},$$

$$R_1(t) = \frac{2 + M^2 + M^5}{1 + t^5 + \ln\left(t^2+3\right)},$$

$$R_2(t) = \frac{\left(1+\sqrt{t}\right)M^4}{1 + t^3}, \quad (t,u,v,w) \in \left[t_0, +\infty\right) \times \mathbb{R}^3. \tag{102}$$

It is easy to verify that (16)–(18) and (82) are satisfied. It follows from Theorem 8 that (11) possesses uncountably many bounded positive solutions in $A(N,M)$. On the other hand, for any $L \in ((c_1 - 1)N, (c_2 - 1)M)$, there exist $\theta \in (0,1)$ and $T > t_0 + |a| + \tau + |T_0|$ such that the Mann iterative sequence $\{x_m\}_{m\in\mathbb{N}_0}$ with mixed errors generated by (83) converges to a bounded positive solution $x \in A(N,M)$ of (11) and has the error estimate (22), where $\{\gamma_m\}_{m\in\mathbb{N}_0}$ is an arbitrary sequence in $A(N,M)$, and $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ are any sequences in $[0,1]$ satisfying (23) and (24).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The first author was supported by the Science Research Foundation of Educational Department of Liaoning Province (L2012380). The third author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (2013R1A1A2057665).

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