

Research Article

Properties of a Class of p -Harmonic Functions

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A p times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subseteq \mathbb{C}$ is p -harmonic if F satisfies the p -harmonic equation $\Delta \cdots \Delta F = 0$, where p is a positive integer. By using the generalized Salagean differential operator, we introduce a class of p -harmonic functions and investigate necessary and sufficient coefficient conditions, distortion bounds, extreme points, and convex combination of the class.

1. Introduction

A continuous complex-valued function $f = u + iv$ in a domain $D \subseteq \mathbb{C}$ is harmonic if both u and v are real harmonic in D ; that is, $\Delta u = 0$ and $\Delta v = 0$. Here Δ represents the complex Laplacian operator

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (1)$$

In any simply connected domain D we can write $f = h + \bar{g}$, where h and g are analytic in D . We call h the analytic part and g the coanalytic part of f . A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $J_f = |f_z|^2 - |f_{\bar{z}}|^2 > 0$ in D . See [1, 2].

Denote by SH the class of functions $f = h + \bar{g}$ that are harmonic, univalent, and sense preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in SH$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{j=2}^{\infty} a_j z^j, \quad g(z) = \sum_{j=1}^{\infty} b_j z^j, \quad |b_1| < 1. \quad (2)$$

The properties of the class SH and its geometric subclasses have been investigated by many authors; see ([1–6]). Note that SH reduces to the class S of normalized analytic univalent functions in U if the coanalytic part of f is identically zero.

A p times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subseteq \mathbb{C}$ is p -harmonic if

F satisfies the p -harmonic equation $\Delta \cdots \Delta F = 0$, where p is a positive integer.

A function F is p -harmonic in a simply connected domain D if and only if F has the following representation:

$$F(z) = \sum_{k=1}^p |z|^{2(k-1)} f_{p-k+1}(z), \quad (3)$$

where $\Delta f_{p-k+1}(z) = 0$ in D for each $k \in \{1, \dots, p\}$. f_{p-k+1} has the form

$$f_{p-k+1} = h_{p-k+1} + \bar{g}_{p-k+1}, \quad (4)$$

where

$$h_p(z) = z + \sum_{j=2}^{\infty} a_{j,p} z^j,$$

$$h_{p-k+1}(z) = \sum_{j=1}^{\infty} a_{j,p-k+1} z^j, \quad (k \geq 2), \quad (5)$$

$$g_{p-k+1}(z) = \sum_{j=1}^{\infty} b_{j,p-k+1} z^j, \quad (k \geq 1).$$

Denote by SH_p the class of functions F of the form (3) that are harmonic, univalent, and sense-preserving in the unit disk. Apparently, if $p = 1$ and $p = 2$, F is harmonic and biharmonic, respectively.

Biharmonic functions have been studied by several authors, such as, [7–9]. Also, biharmonic functions arise in many physical situations, particularly, in fluid dynamics and elasticity problems. They have many important applications in engineering, biology, and medicine, such as in [10, 11].

For a function f in S , differential operator D^n ($n \in \mathbb{N}_0$) was introduced by Sălăgean [12]. Al-Oboudi [13] generalized D^n as follows:

$$D_\lambda^0 f(z) = f(z),$$

$$D_\lambda^1 f(z) = (1 - \lambda) f(z) + \lambda z f'(z), \quad \lambda \geq 0, \quad (6)$$

$$D_\lambda^n f(z) = D_\lambda^1 (D_\lambda^{n-1} f(z)).$$

When $\lambda = 1$, we get the Salagean differential operator.

For $f(z) = h(z) + \overline{g(z)}$ given by (2), Li and Liu [14] defined the following generalized Salagean operator D_λ^n in SH :

$$D_\lambda^n f(z) = D_\lambda^n h(z) + \overline{D_\lambda^n g(z)}, \quad \lambda \geq 0, \quad (7)$$

where

$$D_\lambda^n h(z) = z + \sum_{j=2}^{\infty} [1 + (k-1)\lambda]^n a_j z^j, \quad (8)$$

$$D_\lambda^n g(z) = \sum_{j=1}^{\infty} [1 + (k-1)\lambda]^n b_j z^j.$$

For a p -harmonic function F given by (3), we define the following operator:

$$D_\lambda^0 F(z) = F(z), \quad (9)$$

$$D_\lambda^1 F(z) = (1 - \lambda) D_\lambda^0 F(z) + \lambda \left[z(D_\lambda^0 F(z))_z + \overline{z(D_\lambda^0 F(z))_{\overline{z}}} \right], \quad \lambda \geq 0,$$

$$D_\lambda^n F(z) = D_\lambda^1 (D_\lambda^{n-1} F(z)), \quad (n \in \mathbb{N}). \quad (10)$$

If F is given by (3), then from (10) we see that

$$D_\lambda^n F(z) = \sum_{k=1}^p |z|^{2(k-1)} \sum_{j=1}^{\infty} [1 + (j-1)\lambda + 2(k-1)\lambda]^n \times a_{j,p-k+1} z^j + \sum_{k=1}^p |z|^{2(k-1)} \sum_{j=1}^{\infty} [1 + (j-1)\lambda + 2(k-1)\lambda]^n \times \overline{b_{j,p-k+1} \overline{z}^j}, \quad (a_{1,p} = 1, |b_{1,p}| < 1). \quad (11)$$

When $p = 1$, we get the generalized Salagean operator for harmonic univalent functions defined by Li and Liu [14].

Denote by $SH_p(n, \lambda, \alpha)$ the class of functions F of the form (3) which satisfy the condition

$$\operatorname{Re} \left\{ \frac{D_\lambda^{n+1} F(z)}{D_\lambda^n F(z)} \right\} \geq \alpha, \quad 0 \leq \alpha < 1, \quad (12)$$

where $D_\lambda^n F(z)$ is defined by (11).

We let the subclass \overline{SH}_p of SH_p consist of functions F of the form (3) which include $f_{p-k+1} = h_{p-k+1} + \overline{g_{p-k+1}}$, where

$$h_p(z) = z - \sum_{j=2}^{\infty} |a_{j,p}| z^j, \quad (13)$$

$$h_{p-k+1}(z) = - \sum_{j=1}^{\infty} |a_{j,p-k+1}| z^j, \quad (k \geq 2),$$

$$g_{p-k+1}(z) = - \sum_{j=1}^{\infty} |b_{j,p-k+1}| z^j, \quad (k \geq 1). \quad (14)$$

Define $\overline{SH}_p(n, \lambda, \alpha) := SH_p(n, \lambda, \alpha) \cap \overline{SH}_p$.

The main object of the paper is to introduce a class of p -harmonic functions by using the generalized Salagean operator which was defined by Li and Liu [14]. We investigate necessary and sufficient coefficient conditions, extreme points, distortion bounds, and convex combination of the class.

2. Main Results

Theorem 1. Let F be a p -harmonic function given by (3). Furthermore, let

$$\sum_{k=1}^p \sum_{j=1}^{\infty} [1 + (j-1)\lambda + 2(k-1)\lambda - \alpha] \times [1 + (j-1)\lambda + 2(k-1)\lambda]^n \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] \leq 2(1 - \alpha), \quad (15)$$

where $\lambda \geq 1$, $n \in \mathbb{N}$, $0 \leq \alpha < 1$, and $a_{1,p} = 1$. Then F is sense preserving, p -harmonic, and univalent in U and $F \in SH_p(n, \lambda, \alpha)$.

Proof. Suppose $z_1, z_2 \in U$ and $z_1 \neq z_2$, so that $|z_1| \leq |z_2| < 1$:

$$\begin{aligned} & |F(z_1) - F(z_2)| \\ & \geq |f_p(z_1) - f_p(z_2)| \\ & - \left| \sum_{k=2}^p \left[|z_1|^{2(k-1)} f_{p-k+1}(z_1) - |z_2|^{2(k-1)} f_{p-k+1}(z_2) \right] \right| \\ & \geq |z_1 - z_2| \left[1 - \sum_{j=2}^{\infty} \frac{|z_1^j - z_2^j|}{|z_1 - z_2|} |a_{j,p}| - \sum_{j=1}^{\infty} \frac{|z_1^j - z_2^j|}{|z_1 - z_2|} |b_{j,p}| \right] \end{aligned}$$

$$\begin{aligned}
 & - \sum_{k=2}^p \sum_{j=1}^{\infty} |z_2|^{2(k-1)} \frac{|z_1^j - z_2^j|}{|z_1 - z_2|} \\
 & \times \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] \\
 & > |z_1 - z_2| \left[1 - \sum_{j=2}^{\infty} j |a_{j,p}| - \sum_{j=1}^{\infty} j |b_{j,p}| \right. \\
 & \quad \left. - \sum_{k=2}^p \sum_{j=1}^{\infty} j \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] \right] \\
 & \geq |z_1 - z_2| \left[2 - \sum_{k=1}^p \sum_{j=1}^{\infty} \left([1 + (j-1)\lambda + 2(k-1)\lambda - \alpha] \right. \right. \\
 & \quad \left. \left. \times [1 + (j-1)\lambda + 2(k-1)\lambda]^n \right) \right. \\
 & \quad \left. \times (1-\alpha)^{-1} \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] \right] \\
 & \geq 0,
 \end{aligned} \tag{16}$$

which proves univalence.

In order to prove that F is sense preserving, we need to show that $|F_z(z)| - |F_{\bar{z}}(z)| > 0$:

$$\begin{aligned}
 & |F_z(z)| - |F_{\bar{z}}(z)| \\
 & = \left| 1 + \sum_{j=2}^{\infty} j a_{j,p} z^{j-1} + \sum_{k=2}^p |z|^{2(k-1)} \sum_{j=1}^{\infty} (j+k-1) a_{j,p-k+1} z^{j-1} \right. \\
 & \quad \left. + \sum_{k=1}^p \frac{|z|^{2(k-1)}}{z} \sum_{j=1}^{\infty} (k-1) \bar{b}_{j,p-k+1} \bar{z}^j \right| \\
 & - \left| \sum_{k=1}^p \frac{|z|^{2(k-1)}}{\bar{z}} \sum_{j=1}^{\infty} (k-1) a_{j,p-k+1} z^j \right. \\
 & \quad \left. + \sum_{k=1}^p |z|^{2(k-1)} \sum_{j=1}^{\infty} (j+k-1) \bar{b}_{j,p-k+1} \bar{z}^{j-1} \right| \\
 & > 2 - \sum_{k=1}^p \sum_{j=1}^{\infty} [j + 2(k-1)] \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] \\
 & \geq 2 - \sum_{k=1}^p \sum_{j=1}^{\infty} \left([1 + (j-1)\lambda + 2(k-1)\lambda - \alpha] \right. \\
 & \quad \left. \times [1 + (j-1)\lambda + 2(k-1)\lambda]^n \right) \\
 & \quad \left. \times (1-\alpha)^{-1} \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] \right) \\
 & \geq 0,
 \end{aligned} \tag{17}$$

for all $z \in U$.

Using the fact that $\operatorname{Re} w \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that

$$\begin{aligned}
 & |(1-\alpha) D_{\lambda}^n F(z) + D_{\lambda}^{n+1} F(z)| \\
 & - |(1+\alpha) D_{\lambda}^n F(z) - D_{\lambda}^{n+1} F(z)| \geq 0.
 \end{aligned} \tag{18}$$

Substituting for $D_{\lambda}^n F$ in (18), we obtain

$$\begin{aligned}
 & |(1-\alpha) D_{\lambda}^n F(z) + D_{\lambda}^{n+1} F(z)| \\
 & - |(1+\alpha) D_{\lambda}^n F(z) - D_{\lambda}^{n+1} F(z)| \\
 & \geq 2(1-\alpha)|z| \\
 & - 2 \sum_{j=2}^{\infty} [1 + (j-1)\lambda - \alpha] [1 + (j-1)\lambda]^n |a_{j,p}| |z|^j \\
 & - 2 \sum_{j=1}^{\infty} [1 + (j-1)\lambda - \alpha] [1 + (j-1)\lambda]^n |b_{j,p}| |z|^j \\
 & - 2 \sum_{k=2}^p |z|^{2(k-1)} \sum_{j=1}^{\infty} [1 + (j-1)\lambda + 2(k-1)\lambda - \alpha] \\
 & \quad \times [1 + (j-1)\lambda + 2(k-1)\lambda]^n \\
 & \quad \times \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] |z|^j \\
 & > 2(1-\alpha)|z| \left[2 - \sum_{k=1}^p \sum_{j=1}^{\infty} \left([1 + (j-1)\lambda + 2(k-1)\lambda - \alpha] \right. \right. \\
 & \quad \left. \left. \times [1 + (j-1)\lambda + 2(k-1)\lambda]^n \right) \right. \\
 & \quad \left. \times (1-\alpha)^{-1} \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] \right].
 \end{aligned} \tag{19}$$

This last expression is nonnegative by (15), and so the proof is complete. \square

Theorem 2. Let F be given by (13) and (14). Then $F \in \overline{SH}_p(n, \lambda, \alpha)$ if and only if

$$\begin{aligned}
 & \sum_{k=1}^p \sum_{j=1}^{\infty} [1 + (j-1)\lambda + 2(k-1)\lambda - \alpha] \\
 & \quad \times [1 + (j-1)\lambda + 2(k-1)\lambda]^n \\
 & \quad \times \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] \\
 & \leq 2(1-\alpha),
 \end{aligned} \tag{20}$$

where $\lambda \geq 1$, $n \in \mathbb{N}$, $0 \leq \alpha < 1$, and $a_{1,p} = 1$.

Proof. The “if” part follows from Theorem 1 upon noting that $\overline{SH}_p(n, \lambda, \alpha) \subset SH_p(n, \lambda, \alpha)$. For the “only if” part, we show that $F \notin \overline{SH}_p(n, \lambda, \alpha)$ if the condition (20) does not hold.

Note that a necessary and sufficient condition for F given by (13) and (14) to be in $\overline{SH}_p(n, \lambda, \alpha)$ is that the condition (12) should be satisfied.

This is equivalent to $\operatorname{Re}\{A(z)/B(z)\} \geq 0$, where

$$\begin{aligned}
 A(z) &= (1 - \alpha)z \\
 &\quad - \sum_{j=2}^{\infty} (1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n |a_{j,p}| z^j \\
 &\quad - \sum_{j=1}^{\infty} (1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n |b_{j,p}| \bar{z}^j \\
 &\quad - \sum_{k=2}^p |z|^{2(k-1)} \sum_{j=1}^{\infty} [1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \\
 &\quad \quad \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n \\
 &\quad \quad \times \left[|a_{j,p-k+1}| z^j + |b_{j,p-k+1}| \bar{z}^j \right], \\
 B(z) &= z - \sum_{j=2}^{\infty} [1 + (j - 1)\lambda]^n |a_{j,p}| z^j \\
 &\quad - \sum_{j=1}^{\infty} [1 + (j - 1)\lambda]^n |b_{j,p}| \bar{z}^j \\
 &\quad - \sum_{k=2}^p |z|^{2(k-1)} \sum_{j=1}^{\infty} [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n \\
 &\quad \quad \times \left[|a_{j,p-k+1}| z^j + |b_{j,p-k+1}| \bar{z}^j \right]. \tag{21}
 \end{aligned}$$

The above condition must hold for all values of z , $|z| = r < 1$. Upon choosing the values of z on the positive real axis, where $0 \leq z = r < 1$ we must have

$$\begin{aligned}
 &\left((1 - \alpha) - \sum_{j=2}^{\infty} (1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n |a_{j,p}| r^{j-1} \right. \\
 &\quad \left. - \sum_{j=1}^{\infty} (1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n |b_{j,p}| r^{j-1} \right) \\
 &\quad \times \left(1 - \sum_{j=2}^{\infty} [1 + (j - 1)\lambda]^n |a_{j,p}| - \sum_{j=1}^{\infty} [1 + (j - 1)\lambda]^n |b_{j,p}| \right. \\
 &\quad \left. - \sum_{k=2}^p \sum_{j=1}^{\infty} [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n \right. \\
 &\quad \left. \times \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] \right)^{-1}
 \end{aligned}$$

$$\begin{aligned}
 &- \left(\sum_{k=2}^p \sum_{j=1}^{\infty} [1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \right. \\
 &\quad \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n \\
 &\quad \left. \times \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] r^{j+2k-3} \right) \\
 &\quad \times \left(1 - \sum_{j=2}^{\infty} [1 + (j - 1)\lambda]^n |a_{j,p}| \right. \\
 &\quad \left. - \sum_{j=1}^{\infty} [1 + (j - 1)\lambda]^n |b_{j,p}| \right. \\
 &\quad \left. - \sum_{k=2}^p \sum_{j=1}^{\infty} [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n \right. \\
 &\quad \left. \times \left[|a_{j,p-k+1}| + |b_{j,p-k+1}| \right] \right)^{-1} \\
 &\geq 0. \tag{22}
 \end{aligned}$$

If the condition (20) does not hold, then the numerator in (22) is negative for r is sufficiently close to 1. Hence there exist $z_0 = r_0$ in $(0, 1)$ for which the quotient in (22) is negative. This contradicts the required condition for $F \in \overline{SH}_p(n, \lambda, \alpha)$ and so the proof is complete. \square

Theorem 3. Let F be given by (13) and (14). Then $F \in \overline{SH}_p(n, \lambda, \alpha)$ if and only if

$$F(z) = \sum_{k=1}^p \sum_{j=1}^{\infty} \left(X_{j,p-k+1} h_{j,p-k+1}(z) + Y_{j,p-k+1} g_{j,p-k+1}(z) \right), \tag{23}$$

where

$$\begin{aligned}
 h_{1,p}(z) &= z, \\
 h_{j,p}(z) &= z - \frac{2(1 - \alpha)}{(1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n} z^j, \quad (j \geq 2), \\
 g_{j,p}(z) &= z - \frac{2(1 - \alpha)}{(1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n} \bar{z}^j, \quad (j \geq 1), \\
 h_{j,p-k+1}(z) &= z - |z|^{2(k-1)} (2(1 - \alpha)) \\
 &\quad \times \left([1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \right)
 \end{aligned}$$

$$\begin{aligned}
 & \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n)^{-1} z^j, \\
 & \quad (j \geq 1, 2 \leq k \leq p), \\
 g_{j,p-k+1}(z) & \\
 & = z - |z|^{2(k-1)} (2(1 - \alpha)) \\
 & \times ([1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \\
 & \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n)^{-1} \bar{z}^j \\
 & \quad (j \geq 1, 2 \leq k \leq p),
 \end{aligned} \tag{24}$$

and $\sum_{k=1}^p \sum_{j=1}^{\infty} (X_{j,p-k+1} + Y_{j,p-k+1}) = 1$, $X_{j,p-k+1} \geq 0$, $Y_{j,p-k+1} \geq 0$.

In particular, the extreme points of $\overline{SH}_p(n, \lambda, \alpha)$ are $\{h_{j,p-k+1}(z)\}$ and $\{g_{j,p-k+1}(z)\}$, where $j \geq 1$ and $1 \leq k \leq p$.

Proof. For functions F of the form (13) and (14) we have

$$\begin{aligned}
 F(z) & \\
 & = \sum_{k=1}^p \sum_{j=1}^{\infty} (X_{j,p-k+1} h_{j,p-k+1}(z) + Y_{j,p-k+1} g_{j,p-k+1}(z)) \\
 & = z - \sum_{j=2}^{\infty} (2(1 - \alpha)) ((1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n)^{-1} \\
 & \quad \times X_{j,p} z^j \\
 & - \sum_{j=1}^{\infty} (2(1 - \alpha)) ((1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n)^{-1} \\
 & \quad \times Y_{j,p} \bar{z}^j \\
 & - \sum_{k=2}^p |z|^{2(k-1)} \\
 & \quad \times \sum_{j=1}^{\infty} (2(1 - \alpha)) ([1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \\
 & \quad \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n)^{-1} \\
 & \quad \times [X_{j,p-k+1} z^j + Y_{j,p-k+1} \bar{z}^j].
 \end{aligned} \tag{25}$$

Then

$$\begin{aligned}
 & \sum_{j=2}^{\infty} \frac{(1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n}{2(1 - \alpha)} \\
 & \times \left(\frac{2(1 - \alpha)}{(1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n} X_{j,p} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^{\infty} \frac{(1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n}{2(1 - \alpha)} \\
 & \times \left(\frac{2(1 - \alpha)}{(1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n} Y_{j,p} \right) \\
 & + \sum_{k=2}^p \sum_{j=1}^{\infty} ([1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \\
 & \quad \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n) (2(1 - \alpha))^{-1} \\
 & \quad \times ((2(1 - \alpha)) ([1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \\
 & \quad \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n)^{-1} \\
 & \quad \times [X_{j,p-k+1} + Y_{j,p-k+1}]) \\
 & = \sum_{j=2}^{\infty} X_{j,p} + \sum_{j=1}^{\infty} Y_{j,p} \\
 & + \sum_{k=2}^p \sum_{j=1}^{\infty} [X_{j,p-k+1} + Y_{j,p-k+1}] = 1 - X_{1,p} \leq 1,
 \end{aligned} \tag{26}$$

and so $F \in \overline{SH}_p(n, \lambda, \alpha)$. Conversely, if $F \in \overline{SH}_p(n, \lambda, \alpha)$, then

$$\begin{aligned}
 |a_{j,p}| & \\
 & \leq \frac{2(1 - \alpha)}{(1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n}, \quad (j \geq 2), \\
 |a_{j,p-k+1}| & \\
 & \leq (2(1 - \alpha)) ([1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \\
 & \quad \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n)^{-1}, \\
 & \quad (j \geq 1, 2 \leq k \leq p),
 \end{aligned}$$

$$\begin{aligned}
 |b_{j,p}| & \\
 & \leq \frac{2(1 - \alpha)}{(1 + (j - 1)\lambda - \alpha) [1 + (j - 1)\lambda]^n}, \quad (j \geq 2),
 \end{aligned}$$

$$\begin{aligned}
 |b_{j,p-k+1}| & \\
 & \leq (2(1 - \alpha)) ([1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \\
 & \quad \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n)^{-1}, \\
 & \quad (j \geq 1, 2 \leq k \leq p).
 \end{aligned} \tag{27}$$

Set

$$\begin{aligned}
 X_{j,p} &= \left((1 + (j - 1) \lambda - \alpha) \right. \\
 &\quad \times [1 + (j - 1) \lambda]^n \left. \right) (2(1 - \alpha))^{-1} |a_{j,p}|, \\
 &\quad (j \geq 2), \\
 Y_{j,p} &= \left((1 + (j - 1) \lambda - \alpha) [1 + (j - 1) \lambda]^n \right) \\
 &\quad \times (2(1 - \alpha))^{-1} |b_{j,p}|, \quad (j \geq 1), \\
 X_{j,p-k+1} &= \left([1 + (j - 1) \lambda + 2(k - 1) \lambda - \alpha] \right. \\
 &\quad \times [1 + (j - 1) \lambda + 2(k - 1) \lambda]^n \\
 &\quad \times (2(1 - \alpha))^{-1} |a_{j,p-k+1}|, \quad (j \geq 1, 2 \leq k \leq p), \\
 Y_{j,p-k+1} &= \left([1 + (j - 1) \lambda + 2(k - 1) \lambda - \alpha] \right. \\
 &\quad \times [1 + (j - 1) \lambda + 2(k - 1) \lambda]^n \\
 &\quad \times (2(1 - \alpha))^{-1} |b_{j,p-k+1}|, \quad (j \geq 1, 2 \leq k \leq p), \\
 X_{1,p} &= 1 - \sum_{j=2}^{\infty} X_{j,p} - \sum_{j=1}^{\infty} Y_{j,p} \\
 &\quad - \sum_{k=2}^p \sum_{j=1}^{\infty} [X_{j,p-k+1} + Y_{j,p-k+1}],
 \end{aligned} \tag{28}$$

where $X_{1,p} \geq 0$. Then, as required, we obtain

$$F(z) = \sum_{k=1}^p \sum_{j=1}^{\infty} \left(X_{j,p-k+1} h_{j,p-k+1}(z) + Y_{j,p-k+1} g_{j,p-k+1}(z) \right). \tag{29}$$

□

Theorem 4. Let $F \in \overline{SH}_p(n, \lambda, \alpha)$. Then for $|z| = r < 1$ we have

$$\begin{aligned}
 |F(z)| &\leq \left(1 + |b_{1,p}| + \sum_{k=2}^p (|a_{1,p-k+1}| + |b_{1,p-k+1}|) \right) r \\
 &\quad + \left(\frac{2(1 - \alpha)}{(1 + \lambda - \alpha) [1 + \lambda]^n} \right. \\
 &\quad \left. - \sum_{k=1}^p \frac{[1 + 2(k - 1) \lambda - \alpha] [1 + 2(k - 1) \lambda]^n}{(1 + \lambda - \alpha) [1 + \lambda]^n} \right)
 \end{aligned}$$

$$\times \left[|a_{1,p-k+1}| + |b_{1,p-k+1}| \right] r^2,$$

$$\begin{aligned}
 |F(z)| &\geq \left(1 - |b_{1,p}| - \sum_{k=2}^p (|a_{1,p-k+1}| + |b_{1,p-k+1}|) \right) r \\
 &\quad - \left(\frac{2(1 - \alpha)}{(1 + \lambda - \alpha) [1 + \lambda]^n} \right. \\
 &\quad \left. - \sum_{k=1}^p \frac{[1 + 2(k - 1) \lambda - \alpha] [1 + 2(k - 1) \lambda]^n}{(1 + \lambda - \alpha) [1 + \lambda]^n} \right) \\
 &\quad \times \left[|a_{1,p-k+1}| + |b_{1,p-k+1}| \right] r^2.
 \end{aligned} \tag{30}$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $F \in \overline{SH}_p(n, \lambda, \alpha)$. Taking the absolute value of F we have

$$\begin{aligned}
 |F(z)| &\leq \left(\sum_{k=1}^p (|a_{1,p-k+1}| + |b_{1,p-k+1}|) \right) r \\
 &\quad + \left(\sum_{k=1}^p \sum_{j=2}^{\infty} (|a_{j,p-k+1}| + |b_{j,p-k+1}|) \right) r^2 \\
 &\leq \left(\sum_{k=1}^p (|a_{1,p-k+1}| + |b_{1,p-k+1}|) \right) r \\
 &\quad + \frac{2(1 - \alpha) r^2}{(1 + \lambda - \alpha) [1 + \lambda]^n} \\
 &\quad \times \left(\sum_{k=1}^p \sum_{j=2}^{\infty} ([1 + (j - 1) \lambda + 2(k - 1) \lambda - \alpha] \right. \\
 &\quad \times [1 + (j - 1) \lambda + 2(k - 1) \lambda]^n) \\
 &\quad \times (2(1 - \alpha))^{-1} \left. [|a_{j,p-k+1}| + |b_{j,p-k+1}|] \right) \\
 &\leq \left(1 + |b_{1,p}| + \sum_{k=2}^p (|a_{1,p-k+1}| + |b_{1,p-k+1}|) \right) r \\
 &\quad + \frac{2(1 - \alpha) r^2}{(1 + \lambda - \alpha) [1 + \lambda]^n} \\
 &\quad \times \left(1 - \sum_{k=1}^p \frac{[1 + 2(k - 1) \lambda - \alpha] [1 + 2(k - 1) \lambda]^n}{2(1 - \alpha)} \right. \\
 &\quad \left. \times [|a_{1,p-k+1}| + |b_{1,p-k+1}|] \right).
 \end{aligned} \tag{31}$$

□

The following covering result follows from the left-hand inequality in Theorem 4.

Corollary 5. *Let F of the form (13) and (14) be so that $F \in \overline{SH}_p(n, \lambda, \alpha)$. Then*

$$\left\{ w : |w| < \frac{(1 + \lambda - \alpha) [1 + \lambda]^n - (1 - \alpha)}{(1 + \lambda - \alpha) [1 + \lambda]^n} + \frac{(1 - \alpha) - (1 + \lambda - \alpha) [1 + \lambda]^n}{(1 + \lambda - \alpha) [1 + \lambda]^n} |b_{1,p}| + \sum_{k=2}^p ([1 + 2(k - 1)\lambda - \alpha] [1 + 2(k - 1)\lambda]^n - (1 + \lambda - \alpha) [1 + \lambda]^n) ((1 + \lambda - \alpha) [1 + \lambda]^n)^{-1} \times (|a_{1,p-k+1}| + |b_{1,p-k+1}|) \right\} \subset F(U). \tag{32}$$

Theorem 6. *The class $\overline{SH}_p(n, \lambda, \alpha)$ is closed under convex combinations.*

Proof. Let $F_i \in \overline{SH}_p(n, \lambda, \alpha)$ for $i = 1, 2, \dots$, where F_i is given by

$$F_i(z) = z - \sum_{j=2}^{\infty} |a_{ij,p}| z^j - \sum_{j=1}^{\infty} |b_{ij,p}| \bar{z}^j - \sum_{k=2}^p |z|^{2(k-1)} \sum_{j=1}^{\infty} [|a_{ij,p-k+1}| z^j + |b_{ij,p-k+1}| \bar{z}^j]. \tag{33}$$

Then by (20),

$$\sum_{k=1}^p \sum_{j=1}^{\infty} ([1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n) \times (2(1 - \alpha))^{-1} [|a_{ij,p-k+1}| + |b_{ij,p-k+1}|] \leq 1. \tag{34}$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of F_i may be written as

$$\sum_{i=1}^{\infty} t_i F_i(z) = z - \sum_{j=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i [|a_{ij,p}| z^j + |b_{ij,p}| \bar{z}^j] \right) - \sum_{k=2}^p |z|^{2(k-1)} \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i [|a_{ij,p-k+1}| z^j + |b_{ij,p-k+1}| \bar{z}^j] \right). \tag{35}$$

Then by (34),

$$\sum_{k=1}^p \sum_{j=1}^{\infty} ([1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n) (2(1 - \alpha))^{-1} \times \left(\sum_{i=1}^{\infty} t_i [|a_{ij,p-k+1}| + |b_{ij,p-k+1}|] \right) = \sum_{i=1}^{\infty} t_i \left[\sum_{k=1}^p \sum_{j=1}^{\infty} ([1 + (j - 1)\lambda + 2(k - 1)\lambda - \alpha] \times [1 + (j - 1)\lambda + 2(k - 1)\lambda]^n) \times (2(1 - \alpha))^{-1} [|a_{ij,p-k+1}| + |b_{ij,p-k+1}|] \right] \leq \sum_{i=1}^{\infty} t_i = 1. \tag{36}$$

This is the condition required by (20) and so $\sum_{i=1}^{\infty} t_i F_i(z) \in \overline{SH}_p(n, \lambda, \alpha)$. \square

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