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Research Article

Bifurcation of Positive Solutions for a Class of Boundary Value Problems of Fractional Differential Inclusions

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Using Krein-Rutman theorem, topological degree theory, and bifurcation techniques, this paper investigates the existence of positive solutions for a class of boundary value problems of fractional differential inclusions.

1. Introduction

Fractional differential equations have been of great interest recently. Engineers and scientists have developed new models that involve fractional differential equations. These models have been applied successfully, for example, in mechanics (theory of viscoelasticity and viscoplasticity), (bio)chemistry (modelling of polymers and proteins), electrical engineering (transmission of ultrasound waves), medicine (modelling of human tissue under mechanical loads), and so forth. For details, see [1–7] and references therein. For example, in [5], Qiu and Bai considered the existence of positive solutions to BVP of the nonlinear fractional differential equation

$$\begin{gathered} {}^{C}D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = 0, \end{gathered} \tag{1}$$

where $2 < \alpha \leq 3$, $f : (0,1] \times [0,+\infty) \to [0,+\infty)$, and ${}^{C}D_{0^+}^{\alpha}$ is the Caputo's fractional derivatives. They obtained the existence of at least one positive solution by using Krasnoselskii's fixed point theorem and nonlinear alternative of Leray-Schauder type in a cone.

In [8], Tian and Liu investigated the following singular fractional boundary value problem (BVP, for short) of the form

$$\begin{gathered} {}^{C}D_{0^+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\ u^{(j)}(0) = 0, \quad 0 \leq j \leq n-1, \ j \neq 2, \\ u''(1) = 0, \end{gathered} \tag{2}$$

where $n - 1 < \alpha \leq n$, $n \geq 4$, and $f : (0,1) \times (0,+\infty) \to [0,+\infty)$ is continuous; that is, $f(t,u)$ may be singular at $t = 0, 1$ and $u = 0$. By constructing a special cone, under some suitable assumptions, they obtained that there exist positive numbers λ^* and λ^{**} with $\lambda^* < \lambda^{**}$ such that the above system has at least two positive solutions for $\lambda \in (0, \lambda^*)$ and no solution for $\lambda > \lambda^{**}$.

In this paper, we consider the following boundary value problem of fractional differential inclusions of the form

$$\begin{gathered} {}^{C}D_{0^+}^{\alpha} u(t) \in -F(t, u(t)), \quad 0 < t < 1, \\ u^{(j)}(0) = 0, \quad 0 \leq j \leq n-1, \ j \neq 2, \\ u''(1) = 0, \end{gathered} \tag{3}$$

where $n - 1 < \alpha \leq n$, $n \geq 4$, ${}^{C}D_{0^+}^{\alpha}$ is the Caputo's fractional derivatives, and $F : J \times \mathbb{R}^+ \to 2^{\mathbb{R}^+}$.

As mentioned in [9], the field of differential inclusions is a versatile and general area of mathematics that provides a framework for modelling physical processes that feature discontinuities. Examples of such phenomena include mechanical systems with Coulomb friction modeled as a force proportional to the sign of a velocity and systems whose control laws have discontinuities [10]. In addition, differential inclusions are a useful format for treating differential equations where the right-hand side may be inaccurately known [11]. Differential inclusions are also employed in the dynamic modelling of economic processes and game theory [12], control theory, optimization, partial differential equations, and the study of general evolution processes [13]. The types of the aforementioned applications naturally motivate a deeper theoretical analysis of the subject.

Also there are some papers concerned with initial or boundary value problems of fractional differential inclusions (see, for instance, [9, 14–20] and references therein). The method used in these references is fixed point theorem. However, to the best of our knowledge, there is no paper studying such problems using bifurcation ideas. As we know, the bifurcation technique is widely used in solving boundary value problems (see, for instance, [21–24] and references therein). The purpose of present paper is to fill this gap. By using Krein-Rutman theorem, topological degree theory, and bifurcation techniques, the existence of positive solutions of BVP (3) is investigated.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3, by using bifurcation techniques, Krein-Rutman theorem, and topological degree theory, bifurcation results from infinity and trivial solution are established. Finally, in Section 4, the main results of the present paper are given and proved.

2. Preliminaries

For convenience, we present some necessary definitions and results from fractional calculus theory (see [6]).

Definition 1. The fractional (arbitrary) order integral of the function $h \in L^1([a,b])$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\,\mathrm{d}s, \tag{4}$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$ and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2. For a function h given on the interval $[a,b]$, the αth Caputo fractional-order derivative of h is defined by

$$\left({}^C D_{a+}^\alpha h\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)\,\mathrm{d}s. \tag{5}$$

Here n is the smallest integer greater than or equal α.

Lemma 3. *Let $\alpha > 0$, then the differential equation*

$${}^C D_{0^+}^\alpha u(t) = 0 \tag{6}$$

has solutions $u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where n is the smallest integer greater than or equal to α.

Lemma 4. *Assume that $u \in C(0,1) \cap L^1[0,1]$ with a derivative of order n that belongs to $C(0,1) \cap L^1[0,1]$. Then*

$$I_{0^+}^\alpha {}^C D_{0^+}^\alpha u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}. \tag{7}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, where n is the smallest integer greater than or equal α.

Lemma 5. *The relation*

$$I_{0^+}^\alpha I_{0^+}^\beta \varphi = I_{0^+}^{\alpha+\beta} \varphi \tag{8}$$

is valid in the following case:

$$\operatorname{Re}\beta > 0, \quad \operatorname{Re}(\alpha+\beta) > 0, \quad \varphi \in L^1[a,b]. \tag{9}$$

For more detailed results of fractional calculus, we refer the reader to [6]. In addition, we need the following preliminaries on multivalued operators.

Let $(X, \|\cdot\|)$ be a Banach space. Then a multivalued map $\Theta : X \to 2^X$ is convex (closed) valued if $\Theta(x)$ is convex (closed) for all $x \in X$. Θ is bounded if $\Theta(B) = \bigcup_{x \in B} \Theta(x)$ is bounded in X for any bounded set B of X.

$\Theta : D \to 2^X$ is said to be lower semicontinuous, l.s.c. for short, if $\Theta^{-1}(V)$ is open in D whenever $V \subset X$ is open.

Let $\Theta : D \to 2^X$ be a multivalued map and $\theta : D \to X$ a single-valued function; if for all $x \in D$, $\theta(x) \in \Theta(x)$, then θ is called a selection function of Θ. If in addition θ is continuous, then θ is called a continuous selection.

The following lemmas are crucial in the proof of our main result.

Lemma 6. *[25, Lemma 2.1, page 14]. Let $D \neq \emptyset$ be a subset of a Banach space X, and $\Theta : D \to 2^X$ a l.s.c. with closed convex values. Then, given $(w_0, x_0) \in graph(\Theta)$, Θ has a continuous selection θ such that $\theta(w_0) = x_0$.*

For more details on multivalued maps, see the books of Deimling [25].

Finally in this section, we list the following results on topological degree of completely operators.

Lemma 7 (Schmitt and Thompson [26]). *Let V be a real reflexive Banach space. Let $G : \mathbb{R} \times V$ to V be completely continuous such that $G(\lambda, 0) = 0$, forall $\lambda \in \mathbb{R}$. Let $a, b \in \mathbb{R}\,(a < b)$ be such that $u = 0$ is an isolated solution of the equation*

$$u - G(\lambda, u) = 0, \quad u \in V, \tag{10}$$

for $\lambda = a$ and $\lambda = b$, where $(a,0)$, $(b,0)$ are not bifurcation points of (10). Furthermore, assume that

$$\deg\left(I - G(a,\cdot), B_r(0), 0\right) \neq \deg\left(I - G(b,\cdot), B_r(0), 0\right), \tag{11}$$

where $B_r(0)$ is an isolating neighborhood of the trivial solution. Let

$$\mathcal{T} = \overline{\{(\lambda,u) : (\lambda,u) \text{ is a solution of } (2.1) \text{ with } u \neq 0\}} \cup ([a,b] \times 0). \tag{12}$$

Then there exists a connected component $\mathcal{C}$ of $\mathcal{T}$ containing $[a,b] \times 0$ in $\mathbb{R} \times V$, and either

(i) *$\mathcal{C}$ is unbounded in $\mathbb{R} \times V$ or*

(ii) *$\mathcal{C} \cap [(\mathbb{R} \setminus [a,b]) \times 0] \neq \emptyset$.*

Lemma 8 (Schmitt [27]). *Let V be a real reflexive Banach space. Let $G : \mathbb{R} \times V$ to V be completely continuous, and let $a, b \in \mathbb{R}$ $(a < b)$ be such that the solution of* (10) *is, a priori, bounded in V for $\lambda = a$ and $\lambda = b$; that is, there exists an $R > 0$ such that*

$$G(a, u) \neq u \neq G(b, u) \tag{13}$$

for all u with $\|u\| \geq R$. Furthermore, assume that

$$\deg\left(I - G(a, \cdot), B_R(0), 0\right) \neq \deg\left(I - G(b, \cdot), B_R(0), 0\right), \tag{14}$$

for sufficiently large $R > 0$. Then there exists a closed connected set $\mathscr{C}$ of solutions of (10) *that is unbounded in $[a, b] \times V$, and either*

(i) *$\mathscr{C}$ is unbounded in λ direction or*

(ii) *there exists an interval $[c, d]$ such that $(a, b) \cap (c, d) = \emptyset$ and $\mathscr{C}$ bifurcates from infinity in $[c, d] \times V$.*

Lemma 9 (Guo [28]). *Let Ω be a bounded open set of infinite-dimensional real Banach space E, and let $A : \overline{\Omega} \to E$ be completely continuous. Suppose that*

(i) $\inf_{x \in \partial\Omega} \|Ax\| > 0$;

(ii) $Ax = \mu x$, $x \in \partial\Omega \Rightarrow \mu \notin (0, 1]$.

Then

$$\deg(I - A, \Omega, \theta) = 0. \tag{15}$$

3. Bifurcation Results

3.1. Assumptions and Conversion of BVP (3). Suppose that the following two assumptions hold throughout the paper.

(H1) Let $F : J \times \mathbb{R}^+ \to 2^{\mathbb{R}^+}$ be a nonempty, closed and convex multivalued map such that F is l.s.c., where $J = [0, 1]$.

(H2) There exist functions a_0, a^0, b_∞, $b^\infty \in C(J, \mathbb{R}^+)$ with $a_0(t)$, $a^0(t)$, $b_\infty(t)$, $b^\infty(t) \not\equiv 0$ in any subinterval of $[0, 1]$ such that

$$\begin{aligned} F(t, u) & \\ &\subset \left[a_0(t)u - \xi_1(t, u), a^0(t)u + \xi_2(t, u)\right] \\ &\cap \left[b_\infty(t)u - \zeta_1(t, u), b^\infty(t)u + \zeta_2(t, u)\right], \end{aligned} \tag{16}$$

for all $(t, u) \in J \times \mathbb{R}^+$, where ξ_i, $\zeta_i \in C(J \times \mathbb{R}^+)$ with $\xi_i(t, u) = o(u)$ as $u \to 0$ uniformly with respect to $t \in [0, 1]$, $(i = 1, 2)$, and $\zeta_i(t, u) = o(u)$ as $u \to +\infty$ uniformly with respect to $t \in [0, 1]$, $(i = 1, 2)$.

The basic space used in this paper is $C[0, 1]$. Obviously, $C[0, 1]$ is a Banach space with norm $\|u\| = \max_{t \in J}|u(t)|$ (for all $u \in C[0, 1]$). Let

$$Q := \left\{u \in C\left[J, \mathbb{R}^+\right] : u(t) \geq t^2 u(s), \forall t, s \in J\right\}. \tag{17}$$

It is easy to see that Q is a cone of E. Moreover, from (17), we have for all $u \in Q$,

$$u(t) \geq t^2 \|u\|, \quad \forall t \in J. \tag{18}$$

We first consider the following linear boundary problem of fractional differential equation:

$$\begin{gathered} {}^{C}D_{0^+}^{\alpha} u(t) + g(t) = 0, \quad 0 < t < 1, \\ u^{(j)}(0) = 0, \quad 0 \leq j \leq n - 1, \ j \neq 2, \\ u''(1) = 0, \end{gathered} \tag{19}$$

where $g \in C[0, 1]$.

Lemma 10 (Tian and Liu [8]). *Given $g \in C[0, 1]$, the unique solution of* (19) *is*

$$u(t) = \int_0^1 G(t, s) g(s) \, \mathrm{d}s, \tag{20}$$

where

$$\begin{aligned} &G(t, s) \\ &= \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(\alpha - 1)(\alpha - 2)}{2} t^2 (1 - s)^{\alpha - 3} - (t - s)^{\alpha - 1}, & s \leq t; \\ \dfrac{(\alpha - 1)(\alpha - 2)}{2} t^2 (1 - s)^{\alpha - 3}, & t \leq s. \end{cases} \end{aligned} \tag{21}$$

Lemma 11 (Tian and Liu [8]). *The function $G(t, s)$ defined by* (21) *has the following properties:*

$$\begin{gathered} \text{(i) } G(t, s) > 0, \quad \forall t, s \in [0, 1]; \\ \text{(ii) } G(t, s) \leq H(s) \leq \frac{(1 - s)^{\alpha - 3}}{2\Gamma(\alpha - 2)}, \end{gathered} \tag{22}$$

where

$$\begin{aligned} &H(s) \\ &= \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(\alpha - 1)(\alpha - 2)}{2} s^2 (1 - s)^{\alpha - 3} - (1 - s)^{\alpha - 1}, & s \leq t, \\ \dfrac{(\alpha - 1)(\alpha - 2)}{2} s^2 (1 - s)^{\alpha - 3}, & t \leq s; \end{cases} \\ &\text{(iii) } G(t, s) \geq t^2 G(\tau, s), \quad \forall t, s, \tau \in [0, 1]. \end{aligned} \tag{23}$$

For the sake of using bifurcation technique to investigate BVP (3), we study the following fractional boundary value problem with parameters:

$$\begin{gathered} {}^{C}D_{0^+}^{\alpha} u(t) \in -\lambda F(t, u(t)), \quad 0 < t < 1, \\ u^{(j)}(0) = 0, \quad 0 \leq j \leq n - 1, \ j \neq 2, \\ u''(1) = 0. \end{gathered} \tag{24}$$

A function (λ, u) is said to be a solution of BVP (24) if (λ, u) satisfies (24). In addition, if $\lambda > 0$, $u(t) > 0$ for $t \in (0, 1)$, then (λ, u) is said to be a positive solution of BVP (24). Obviously, if $\lambda > 0$, $u \in Q \setminus \{\theta\}$ is a solution of BVP (24), then by (18) we know that (λ, u) is a positive solution of BVP (24), where θ denotes the zero element of Banach space E.

For $a \in C(J, \mathbb{R}^+)$ with $a(t) \not\equiv 0$ in any subinterval of J, define the linear operator $L_a : C(J) \to C(J)$ by

$$L_a u(t) = \int_0^1 G(t,s) a(s) u(s) \,ds, \quad (25)$$

where $G(t,s)$ is defined by (21).

From Lemmas 10, 11, and the well-known Krein-Rutman Theorem, one can obtain the following lemma.

Lemma 12. *The operator defined by* (25) *has a unique characteristic value $\lambda_1(a)$, which is positive, real, and simple and the corresponding eigenfunction $\phi(t)$ is of one sign in $(0,1)$; that is, we have $\phi(t) = \lambda_1(a) L_a \phi(t)$.*

Notice that the operator L_a can be regarded as $L_a : L^2[0,1] \to L^2[0,1]$. This together with Lemma 12 guarantees that $\lambda_1(a)$ is also the characteristic value of L_a^*, where L_a^* is the conjugate operator of L_a. Let φ^* denote the nonnegative eigenfunction of L_a^* corresponding to $\lambda_1(a)$. Then we have

$$\varphi^*(t) = \lambda_1(a) L_a^* \varphi^*(t), \quad \forall t \in J. \quad (26)$$

Note that condition (H1) implies that $F(t,u)$ is lower semicontinuous. Then, from Lemma 6, there exists a continuous function $f : J \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(t,u) \in F(t,u)$ for all $(t,u) \in J \times \mathbb{R}^+$. Therefore, to solve BVP (24), we consider the problem

$$\begin{gathered} {}^C D_{0^+}^{\alpha} u(t) + \lambda f(t, u(t)) = 0, \quad 0 < t < 1, \\ u^{(j)}(0) = 0, \quad 0 \le j \le n-1, \ j \ne 2, \\ u''(1) = 0. \end{gathered} \quad (27)$$

Define

$$\overline{f}(t,u) = \begin{cases} f(t,u), & (t,u) \in J \times \mathbb{R}^+, \\ f(t,0), & (t,u) \in J \times (-\infty, 0). \end{cases} \quad (28)$$

Then $\overline{f}(t,u) \ge 0$ on $J \times \mathbb{R}$. From Lemma 10, the solution of

$$\begin{gathered} {}^C D_{0^+}^{\alpha} u(t) + \lambda \overline{f}(t, u(t)) = 0, \quad 0 < t < 1, \\ u^{(j)}(0) = 0, \quad 0 \le j \le n-1, \ j \ne 2, \\ u''(1) = 0 \end{gathered} \quad (29)$$

is equivalent to the fixed point of operator

$$A_\lambda u(t) = \lambda \int_0^1 G(t,s) \overline{f}(s, u(s)) \,ds, \quad \forall u \in C[0,1]. \quad (30)$$

Let $\Sigma \subset \mathbb{R}^+ \times C[0,1]$ be the closure of the set of positive solutions of BVP (27). From Lemma 11 and the definitions of $\overline{f}$ and the cone Q, it is easy to see $\Sigma \subset Q$ and $A_\lambda : C[0,1] \to Q$. Moreover, we have the following conclusion.

Lemma 13. *For $\lambda > 0$, (λ, u) is a positive solution of BVP* (27) *if and only if (λ, u) is a nontrivial solution of BVP* (29)*; that is, u is a nontrivial fixed point of operator A_λ in Q. Therefore, the closure of the set of nontrivial solutions (λ, u) of BVP* (29) *in $\mathbb{R}^+ \times Q$ is exactly Σ.*

3.2. Bifurcation from Infinity and Trivial Solution

Lemma 14. *Let $[c,d] \subset \mathbb{R}^+$ be a compact interval with $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \cap [c,d] = \emptyset$. Then there exists $R_1 > 0$ such that*

$$u \ne A_\lambda u, \quad \forall \lambda \in [c,d], \quad \forall u \in C[0,1] \quad \text{with } \|u\| \ge R_1. \quad (31)$$

Proof. Suppose, on the contrary, that there exist $\{(\mu_n, u_n)\} \subset [c,d] \times C[0,1]$ with $\|u_n\| \to \infty (n \to +\infty)$ such that $u_n = A_{\mu_n} u_n$. Without loss of generality, assume $\mu_n \to \mu \in [c,d]$. Notice that $u_n \in Q$. By Lemma 13, (17), and (18), we have $u_n(t) > 0$ in $(0,1]$. Set $v_n = u_n/\|u_n\|$. Then $v_n = A_{\mu_n} u_n/\|u_n\|$. From the continuity of $\overline{f}(t,u)$, it is easy to see that $\{v_n\}$ is relatively compact in $C[0,1]$. Taking a subsequence and relabeling if necessary, suppose $v_n \to v$ in $C[0,1]$. Then $\|v\| = 1$ and $v \in Q$.

On the other hand, from (H2) we know

$$f(t,u) \in [b_\infty(t) u - \zeta_1(t,u), b^\infty(t) u + \zeta_2(t,u)], \quad \forall (t,u) \in J \times \mathbb{R}^+. \quad (32)$$

Therefore, by virtue of (30), we know

$$v_n(t) \le \mu_n \int_0^1 G(t,s) \left(b^\infty(s) v_n(s) + \frac{\zeta_2(s, u_n(s))}{\|u_n\|} \right) ds, \quad (33)$$

$$v_n(t) \ge \mu_n \int_0^1 G(t,s) \left(b_\infty(s) v_n(s) - \frac{\zeta_1(s, u_n(s))}{\|u_n\|} \right) ds. \quad (34)$$

Let ψ^* and ψ_* be the positive eigenfunctions of $L_{b^\infty}^*$, $L_{b_\infty}^*$ corresponding to $\lambda_1(b^\infty)$ and $\lambda_1(b_\infty)$, respectively. Then from (33), it follows that

$$\begin{aligned} &\langle v_n, \psi^* \rangle \\ &\quad \le \mu_n \langle L_{b^\infty} v_n, \psi^* \rangle \\ &\qquad + \mu_n \int_0^1 \psi^*(t) \int_0^1 G(t,s) \frac{\zeta_2(s, u_n(s))}{\|u_n\|} ds\,dt. \end{aligned} \quad (35)$$

Letting $n \to +\infty$ and using condition (H2), we have

$$\langle v, \psi^* \rangle \le \mu \langle L_{b^\infty} v, \psi^* \rangle = \mu \langle v, L_{b^\infty}^* \psi^* \rangle = \mu \left\langle v, \frac{\psi^*}{\lambda_1(b^\infty)} \right\rangle, \quad (36)$$

which implies $\mu \ge \lambda_1(b^\infty)$. Similarly, one can deduce from (34) that $\mu \le \lambda_1(b_\infty)$.

To sum up, $\lambda_1(b^\infty) \le \mu \le \lambda_1(b_\infty)$, which contradicts with $\mu \in [c,d]$. The conclusion of this lemma follows. □

Lemma 15. *For $\mu \in (0, \lambda_1(b^\infty))$, there exists $R_1 > 0$ such that*

$$\deg\left(I - A_\mu, B_R, 0\right) = 1, \quad \forall R \ge R_1. \quad (37)$$

Proof. Notice that $[0,\mu] \cap [\lambda_1(b^{\infty}), \lambda_1(b_{\infty})] = \emptyset$. From Lemma 14, there exists $R_1 > 0$ such that

$$u \neq A_\lambda u, \quad \forall \lambda \in [0,\mu], \quad \forall u \in C[0,1] \quad \text{with } \|u\| \geq R_1, \tag{38}$$

which means

$$u \neq \tau A_\mu u, \quad \forall \tau \in [0,1], \quad \forall u \in C[0,1] \quad \text{with } \|u\| \geq R_1. \tag{39}$$

Therefore, by the homotopy invariance of topological degree, we have

$$\deg(I - A_\mu, B_R, 0) = \deg(I, B_R, 0) = 1, \quad \forall R \geq R_1. \tag{40}$$

□

Lemma 16. *For $\lambda > \lambda_1(b_\infty)$, there exists $R_2 > 0$ such that*

$$\deg(I - A_\lambda, B_R, 0) = 0, \quad \forall R \geq R_2. \tag{41}$$

Proof. We first prove that for $\lambda > \lambda_1(b_\infty)$, there exists $R_2 > 0$ such that

$$A_\lambda u \neq \mu u, \quad \forall \mu \in (0,1], \quad \forall u \in C[0,1] \quad \text{with } \|u\| \geq R_2. \tag{42}$$

Suppose, on the contrary, that there exist $\{(\mu_n, u_n)\} \subset (0,1] \times C[0,1]$ with $\|u_n\| \to \infty\,(n \to +\infty)$ such that $A_\lambda u_n = \mu_n u_n$.

By Lemma 13, $u_n(t) > 0$ in $(0,1]$. Set $v_n = u_n/\|u_n\|$; that is, $\mu_n v_n = A_\lambda u_n/\|u_n\|$. Without loss of generality, assume $\mu_n \to \overline{\mu} \in [0,1]$. First we show $\overline{\mu} \neq 0$. From (32) and the continuity of $f(t,u)$, it is easy to see that $A_\lambda u_n/\|u_n\|$ is relatively compact in $C[0,1]$. Suppose $(A_\lambda u_n/\|u_n\|) \to y$. Notice that $v_n \in Q$ and $\|v_n\| = 1$. Therefore, $v_n(t) \geq t^2$ for $t \in (0,1]$. Consequently,

$$\begin{aligned} &\mu_n v_n(t) \\ &\quad = \frac{A_\lambda u_n(t)}{\|u_n\|} \\ &\quad \geq \lambda \int_0^1 G(t,s)\left(b_\infty(s) v_n(s) - \frac{\zeta_1(s, u_n(s))}{\|u_n\|}\right) \mathrm{d}s \\ &\quad \geq \lambda t^2 \max_{\tau\in J} \int_0^1 s^2 G(\tau,s) b_\infty(s)\,\mathrm{d}s \\ &\qquad - \lambda \int_0^1 G(t,s) \frac{\zeta_1(s,u_n(s))}{\|u_n\|}\mathrm{d}s. \end{aligned} \tag{43}$$

From (H2) and Lemma 11, it is easy to see $\max_{\tau\in J} \int_0^1 s^2 G(\tau,s) b_\infty(s)\mathrm{d}s > 0$. If $\mu = 0$, letting $n \to +\infty$ in the above inequality, we can obtain a contradiction. So $\mu \in (0,1]$ and it is reasonable to suppose $v_n \to v$ (relabeling if necessary) in $C[0,1]$. By virtue of (32), we know

$$\begin{aligned} \langle v_n, \psi_* \rangle &\geq \langle \mu_n v_n, \psi_* \rangle \\ &= \frac{1}{\|u_n\|} \langle A_\lambda u_n, \psi_* \rangle \\ &\geq \lambda \langle L_{b_\infty} v_n, \psi_* \rangle \\ &\quad - \lambda \int_0^1 \psi_*(t) \int_0^1 G(t,s) \frac{\zeta_1(s,u_n(s))}{\|u_n\|} \mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{44}$$

Letting $n \to +\infty$ and using condition (H2), we obtain that

$$\begin{aligned} \langle v, \psi_* \rangle &\geq \lambda \langle L_{b_\infty} v, \psi_* \rangle \\ &= \lambda \langle v, L^*_{b_\infty} \psi_* \rangle = \lambda \left\langle v, \frac{\psi_*}{\lambda_1(b_\infty)} \right\rangle, \end{aligned} \tag{45}$$

which implies $\lambda \leq \lambda_1(b_\infty)$. This is a contradiction. Therefore, (42) holds. By Lemma 9, for each $\lambda > \lambda_1(b_\infty)$, there exists $R_2 > 0$ such that

$$\deg(I - A_\lambda, B_R, 0) = 0, \quad \forall R \geq R_2. \tag{46}$$

The conclusion of this lemma follows. □

Theorem 17. *$[\lambda_1(b^\infty), \lambda_1(b_\infty)]$ is a bifurcation interval of positive solutions from infinity for BVP (27), and there exists no bifurcation interval of positive solutions from infinity which is disjoint with $[\lambda_1(b^\infty), \lambda_1(b_\infty)]$. More precisely, there exists an unbounded component $\mathscr{C}^\infty$ of solutions of BVP (27) which meets $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \times \infty$ and is unbounded in λ direction.*

Proof. From Lemma 13, we need only to prove that the conclusion holds for (29).

For fixed $n \in \mathbb{N}$ with $\lambda_1(b^\infty) - 1/n > 0$, by Lemmas 15, 16, and their proof, there exists $R > 0$ such that all of the conditions of Lemma 8 are satisfied with $G(\lambda, u) = A_\lambda u$, $a = \lambda_1(b^\infty) - 1/n$, and $b = \lambda_1(b_\infty) + 1/n$. So, there exists a closed connected set $\mathscr{C}_n$ of solutions of (29), which is unbounded in $[\lambda_1(b^\infty) - 1/n, \lambda_1(b_\infty) + 1/n] \times C[0,1]$. From Lemma 14, the case (ii) of Lemma 8 cannot occur. Thus, $\mathscr{C}_n$ bifurcates from infinity in $[\lambda_1(b^\infty) - 1/n, \lambda_1(b_\infty) + 1/n] \times C[0,1]$ and is unbounded in λ direction. In addition, for any closed interval $[c,d] \subset [\lambda_1(b^\infty) - 1/n, \lambda_1(b_\infty) + 1/n] \setminus [\lambda_1(b^\infty), \lambda_1(b_\infty)]$, by Lemma 14, the set $\{u \in C[0,1] : (\lambda, u) \in \Sigma, \lambda \in [c,d]\}$ is bounded in $C[0,1]$. Therefore, $\mathscr{C}_n$ must be bifurcated from infinity in $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \times C[0,1]$, which implies that $\mathscr{C}_n$ can be regarded as $\mathscr{C}^\infty$. Consequently, $\mathscr{C}^\infty$ is unbounded in λ direction. □

By a process similar to the above, one can obtain the following conclusions.

Lemma 18. *Let $[c,d] \subset \mathbb{R}^+$ be a compact interval with $[\lambda_1(a^0), \lambda_1(a_0)] \cap [c,d] = \emptyset$. Then there exists $\delta_1 > 0$ such that*

$$u \neq A_\lambda u, \quad \forall \lambda \in [c,d], \quad \forall u \in C[0,1] \quad \text{with } 0 < \|u\| \leq \delta_1. \tag{47}$$

Lemma 19. *For $\mu \in (0, \lambda_1(a^0))$, there exists $\delta_1 > 0$ such that*

$$\deg\left(I - A_\mu, B_\delta, 0\right) = 1, \quad \forall\delta \in \left(0, \delta_1\right]. \tag{48}$$

Lemma 20. *For $\lambda > \lambda_1(a_0)$, there exists $\delta_2 > 0$ such that*

$$\deg\left(I - A_\lambda, B_\delta, 0\right) = 0, \quad \forall\delta \in \left(0, \delta_2\right]. \tag{49}$$

Finally, using Lemmas 18–20, Lemma 7, and the similar method used in the proof of Theorem 17, the following conclusion can be proved.

Theorem 21. *$[\lambda_1(a^0), \lambda_1(a_0)]$ is a bifurcation interval of positive solutions from the trivial solution for BVP (27); that is, there exists an unbounded component $\mathscr{C}_0$ of positive solutions of BVP (27), which meets $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$. Moreover, there exists no bifurcation interval of positive solutions from the trivial solution which is disjointed with $[\lambda_1(a^0), \lambda_1(a_0)]$.*

4. Main Results

The main results of this paper are the following two conclusions.

Theorem 22. *Suppose that (H1) and (H2) hold. In addition, suppose either*

(i) $\lambda_1(b_\infty) < 1 < \lambda_1(a^0)$ *or*

(ii) $\lambda_1(a_0) < 1 < \lambda_1(b^\infty)$.

Then BVP (3) has at least one positive solution.

Proof. We need only to prove that there is a component of Σ that crosses the hyperplane $\{1\} \times C(J)$, where $\Sigma \subset \mathbb{R}^+ \times C[0, 1]$ is the closure of the set of positive solutions of BVP (27). Notice that $(0, 0)$ is the only solution of (27) with $\lambda = 0$. By Lemmas 14 and 18, for any component $\mathscr{C}$ of Σ, we have $\mathscr{C} \cap (\{0\} \times C(J)) = \emptyset$.

Case (i). Consider $\lambda_1(b_\infty) < 1 < \lambda_1(a^0)$.

From Theorem 17, there exists an unbounded component $\mathscr{C}^\infty$ of solutions of (27), which meets $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \times \infty$ and is unbounded in λ direction.

If $\mathscr{C}^\infty \cap (\mathbb{R}^+ \times \{0\}) = \emptyset$, by $\mathscr{C}^\infty \cap (\{0\} \times C(J)) = \emptyset$ and Theorem 17, we know that $\mathscr{C}^\infty$ must cross the hyperplane $\{1\} \times C(J)$.

If $\mathscr{C}^\infty \cap (\mathbb{R}^+ \times \{0\}) \neq \emptyset$, by Theorem 21, we know $\mathscr{C}^\infty \cap (\mathbb{R}^+ \times \{0\}) \in [\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$. Therefore, $\mathscr{C}^\infty$ joins $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ to $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \times \infty$. This together with $\lambda_1(b_\infty) < 1 < \lambda_1(a^0)$ guarantees that $\mathscr{C}^\infty$ crosses the hyperplane $\{1\} \times C(J)$.

Case (ii). Consider $\lambda_1(a_0) < 1 < \lambda_1(b^\infty)$.

From Theorem 21, there exists an unbounded component $\mathscr{C}_0$ of positive solutions of BVP (27), which meets $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$. Moreover, there exists no bifurcation interval of positive solutions from the trivial solution, which is disjointed with $[\lambda_1(a^0), \lambda_1(a_0)]$.

We show that $\mathscr{C}_0$ must cross the hyperplane $\{1\} \times C(J)$. Suppose, on the contrary, $\mathscr{C}_0 \cap \{1\} \times C(J) = \emptyset$. From $\lambda_1(a_0) < 1$, we know $\mathscr{C}_0 \subset [0, 1] \times C(J)$. Notice that $\mathscr{C}_0$ is unbounded. Then $\mathscr{C}_0$ must joint $[0, 1] \times \{\infty\}$. By Theorem 17, it is a contradiction with $\lambda_1(b^\infty) > 1$. Thus the result follows. □

Theorem 23. *Suppose that (H1), (H2), and the following assumption holds.*

(H3) *There exist $R > 0$ and $h \in L[0, 1]$ such that for $t \in J$,*

$$\sup_{t^2 R \le u \le R} F(t, u) \le h(t), \qquad \max_{t\in J} \int_0^1 G(t, s) h(s)\, ds < R. \tag{50}$$

In addition, suppose

$$\lambda_1(a_0) < 1, \qquad \lambda_1(b_\infty) < 1. \tag{51}$$

Then BVP (3) has at least two positive solutions.

Proof. From Theorems 17 and 21, there exist two unbounded components $\mathscr{C}_0$ and $\mathscr{C}^\infty$ of solutions of (27), which meet $[\lambda_1(a^0), \lambda_1(a_0)] \times \{0\}$ and $[\lambda_1(b^\infty), \lambda_1(b_\infty)] \times \infty$, respectively. It is sufficient to show that $\mathscr{C}_0$ and $\mathscr{C}^\infty$ are disjoint in $[0, 1] \times C(J)$ and both cross the hyperplane $\{1\} \times C(J)$.

For this sake, from assumption (H3), there exists $\varepsilon > 0$ such that

$$(1 + \varepsilon) \max_{t\in J} \int_0^1 G(t, s) h(s)\, ds < R. \tag{52}$$

Now we show $\Sigma \cap ([0, 1 + \varepsilon] \times \partial B_R) = \emptyset$, where $B_R = \{u \in C(J) : \|u\| < R\}$. Suppose that, on the contrary, (λ, u) is a solution of (27) such that $0 \le \lambda \le 1 + \varepsilon$ and $\|u\| = R$. Then by Lemma 13, we know $u \in Q$. Therefore, $u(t) \in [t^2 R, R]$ for $t \in J$. From (H3), (30), and Lemma 13, it follows that

$$\begin{aligned} R = \|u\| &= \max_{t\in J} \lambda \int_0^1 G(t, s) \overline{f}(s, u(s))\, ds \\ &\le (1 + \varepsilon) \max_{t\in J} \int_0^1 G(t, s) h(s)\, ds < R, \end{aligned} \tag{53}$$

which is a contradiction. Thus, $\Sigma \cap ([0, 1 + \varepsilon] \times \partial B_R) = \emptyset$, which implies

$$\begin{aligned} &\mathscr{C}_0 \cap \left([0, 1 + \varepsilon] \times \partial B_R\right) = \emptyset, \\ &\mathscr{C}^\infty \cap \left([0, 1 + \varepsilon] \times \partial B_R\right) = \emptyset. \end{aligned} \tag{54}$$

Immediately, $\mathscr{C}_0$ and $\mathscr{C}^\infty$ are disjoint in $[0, 1] \times C(J)$.

Notice that $\mathscr{C}_0$ and $\mathscr{C}^\infty$ are both unbounded. Moreover, $\mathscr{C}_0 \cap (\{0\} \times C(J)) = \emptyset$, $\mathscr{C}^\infty \cap (\{0\} \times C(J)) = \emptyset$, and $\mathscr{C}^\infty$ is unbounded in λ direction. So $\mathscr{C}_0$ and $\mathscr{C}^\infty$ both cross the hyperplane $\{1\} \times C(J)$. This means that there exist $(1, u_1) \in \mathscr{C}_0$ and $(1, u_2) \in \mathscr{C}^\infty$ with $\|u_1\| < R$ and $\|u_2\| > R$.

Consequently, BVP (3) has at least two positive solutions. □

5. An Example

Let ρ be the unique characteristic value of L_1 corresponding to positive eigenfunctions with $a(t) \equiv 1$ in (25). From Lemma 12 it follows that ρ exists.

Example 24. Consider the following boundary value problem of fractional differential inclusions

$$\begin{gathered} {}^{C}D_{0^+}^{3.5} u(t) \in -F(t, u(t)), \quad 0 < t < 1, \\ u^{(j)}(0) = 0, \quad 0 \le j \le 3, \ j \ne 2, \\ u''(1) = 0, \end{gathered} \tag{55}$$

where

$$\begin{gathered} F(t,u) = \left[\frac{\rho}{4}u - \xi(t,u), \frac{\rho}{2}u + \xi_2(t,u)\right] \\ \cap \left[2\rho u - \zeta_1(t,u), 3\rho u + \xi(t,u)\right], \\ \xi(t,u) = \begin{cases} \dfrac{\rho}{4}t^2u^3, & t \in J,\ u \in [0,1], \\ \dfrac{\rho}{4}t^2\sqrt{u}, & t \in J,\ u \in [1,+\infty), \end{cases} \\ \zeta_1(t,u) = \begin{cases} 2\rho u, & t \in J,\ u \in [0,1], \\ 2\rho\sqrt{u}, & t \in J,\ u \in [1,+\infty), \end{cases} \\ \xi_2(t,u) = \frac{5\rho}{2}u + \xi(t,u). \end{gathered} \tag{56}$$

Then BVP (55) has at least one positive solution.

Proof. BVP (55) can be regarded as the form (3). From (56), one can see that (H1) and (H2) are satisfied with $a_0(t) = \rho/4$, $b_\infty(t) = 2\rho$, $a^0(t) = \rho/2$, $b^\infty(t) = 3\rho$, and $\xi_1(t,u) = \zeta_2(t,u) = \xi(t,u)$.

By the definition of ρ, it is easy to see $\lambda_1(b_\infty) = 1/2 < 1 < 2 = \lambda_1(a^0)$.

Therefore, by Theorem 22, BVP (55) has at least one positive solution. □

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