

Research Article

Bifurcation Analysis of a Coupled Kuramoto-Sivashinsky- and Ginzburg-Landau-Type Model

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We study the bifurcation and stability of trivial stationary solution $(0, 0)$ of coupled Kuramoto-Sivashinsky- and Ginzburg-Landau-type equations (KS-GL) on a bounded domain $(0, L)$ with Neumann's boundary conditions. The asymptotic behavior of the trivial solution of the equations is considered. With the length L of the domain regarded as bifurcation parameter, branches of nontrivial solutions are shown by using the perturbation method. Moreover, local behavior of these branches is studied, and the stability of the bifurcated solutions is analyzed as well.

1. Introduction

The mathematical theory of pattern formation [1–3] has a wide range of applications. In the field of fluid mechanics, Rayleigh-Bénard convection is the most widely studied example. Like the thermally driven Bénard convection, the surface tension-driven Marangoni convection is also an interesting pattern formation of nonlinear system. With mass transfer from liquid phase to gas phase, a typical setup for the Marangoni convection is a liquid layer resting on a rigid surface, with a free deformable upper interface contacting an infinite layer of gas. Nevertheless, at present, we still find it hard to analyze the governing equations for the Marangoni convection, that is, Navier-Stokes equations and mass (surfactant) transfer equation. What we can only use is numerical simulation so far. There is little work that has been done on the nonlinear Marangoni convection.

In this paper, we consider a simplified model, which is proposed to capture important nonlinear features yet more amendable to analyse:

$$\begin{aligned} A_t &= A + A_{xx} - |A^2| A + Ah, \\ h_t &= -h_{xx} - h_{xxx} + \alpha |A^2|_{xx}, \end{aligned} \quad (1)$$

where $A(x, t)$ is the amplitude for the Marangoni convection mode and $h(x, t)$ is the interface deformation (real function). Equation (1) has been derived by Golovin et al. [4]. The constant $\alpha = k/(Ma - Ma_c)^{1/5}$ is positive, with Ma denoting the Marangoni number, and represents the gradient (derivative) of surface tension with respect to surfactant concentration. Ma_c is the critical Marangoni number at which the trivial stationary state becomes linearly unstable, and $k > 0$ is a constant related to other system parameters.

Equation (1) without the interaction term Ah is the well-known Ginzburg-Landau equation [5, 6], while the second equation without the term $|A^2|_{xx}$ is the linearized version of the so-called Kuramoto-Sivashinsky equation [7]. Both the G-L equation and K-S equation have been extensively investigated as model examples of infinite dimensional dynamical systems. In [8], Kazhdan et al. have done numerical simulation of this coupled system of Kuramoto-Sivashinsky- and Ginzburg-Landau-type equations (hereafter, KS-GL system). Duan et al. [9] have discussed the existence and uniqueness of global solutions of this coupled system, using the contraction mapping principle and energy estimates. Despite these publications on KS-GL equation, the static bifurcations of the equation have not been thoroughly studied. In this paper, we focus on studying bifurcations of the KS-GL system. In

[10], Xiao and Gao analyzed the bifurcations of the 1D Swift-Hohenberg equation with quintic nonlinearity. Two types of structures in the bifurcation diagrams are presented when the bifurcation points are closer, and their stabilities are analyzed. Li and Chen have applied singularity theory and the perturbation method to study the bifurcations of the 1D and 2D K-S equations and get the asymptotic expressions of the steady-state solution branches that have bifurcated from the equilibrium in [11, 12]. In this paper, we will use the methods in [10–12] to discuss the bifurcated solutions. In (1), A is complex. Namely, we can write $A = R \exp(i\theta)$. The additional phase θ makes analysis very complicated. Here, we restrict our attention to invariant subspace in which A is real. We hope to return to the general case in future study. In this paper, we discuss steady solutions of the parabolic partial differential equations

$$\begin{aligned} A_t &= A + A_{xx} - A^3 + Ah, \\ h_t &= -h_{xx} - h_{xxxx} + \alpha(A^2)_{xx}, \end{aligned} \quad (2)$$

on the cylindrical domain $Q = (0, L) \times R^+$ subject to the boundary conditions

$$A_x = h_x = A_{xxx} = h_{xxx} = 0 \quad \text{at } x = 0, L. \quad (3)$$

The steady-state equation of (2) reads as

$$\begin{aligned} A + A_{xx} - A^3 + Ah &= 0, \quad 0 < x < L, \\ -h_{xx} - h_{xxxx} + \alpha(A^2)_{xx} &= 0, \quad 0 < x < L, \\ \frac{\partial A}{\partial x} = 0, \quad \frac{\partial^3 A}{\partial x^3} = 0, \quad x = 0, L, t > 0, \\ \frac{\partial h}{\partial x} = 0, \quad \frac{\partial^3 h}{\partial x^3} = 0, \quad x = 0, L, t > 0. \end{aligned} \quad (4)$$

We will discuss the bifurcation of the trivial steady state of the equations.

2. Analysis of the Trivial Steady State (0, 0)

In this section, we study some properties of the linear problem associated with problem (4). Let $X = \{(A, h) \in H^2[0, L] \times H^2[0, L] \mid A'(0) = A'(L) = A'''(0) = A'''(L) = h'(0) = h'(L) = h'''(0) = h'''(L) = 0\}$. We linearize the problem at the trivial solution (0, 0), and then we have the corresponding differential operator matrix

$$\Gamma_1 = \begin{pmatrix} \frac{\partial^2}{\partial x^2} + I & 0 \\ 0 & -\frac{\partial^2}{\partial x^2} - \frac{\partial^4}{\partial x^4} \end{pmatrix}, \quad (5)$$

and the eigenvalue problem is

$$\begin{aligned} \varphi_{xx} + \varphi &= \lambda\varphi, \quad 0 < x < L, \\ -\psi_{xx} - \psi_{xxxx} &= \lambda\psi, \quad 0 < x < L, \\ \varphi_x(0) = \varphi_x(L) = \varphi_{xxx}(0) = \varphi_{xxx}(L) &= 0, \\ \psi_x(0) = \psi_x(L) = \psi_{xxx}(0) = \psi_{xxx}(L) &= 0. \end{aligned} \quad (6)$$

The eigenvalues λ are given by

$$\lambda = \lambda_m = -\left(\frac{m\pi}{L}\right)^2 + 1, \quad \lambda = \delta_n = \left(\frac{n\pi}{L}\right)^2 - \left(\frac{n\pi}{L}\right)^4, \quad n, m \in N. \quad (7)$$

If $\lambda_m = \delta_n$, then the corresponding eigenvectors are

$$\begin{pmatrix} \cos \frac{m\pi x}{L} \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \cos \frac{n\pi x}{L} \end{pmatrix}. \quad (8)$$

If $\lambda_m \neq \delta_n$, then for $\lambda_m = -(m\pi/L)^2 + 1$, the corresponding eigenvector is

$$\begin{pmatrix} \cos \frac{m\pi x}{L} \\ 0 \end{pmatrix}, \quad (9)$$

and for $\delta_n = (n\pi/L)^2 - (n\pi/L)^4$, the corresponding eigenvector is

$$\begin{pmatrix} 0 \\ \cos \frac{n\pi x}{L} \end{pmatrix}. \quad (10)$$

It will be convenient to rescale the spatial variable so that the domain $(0, L)$ maps onto the fixed domain $(0, 1)$. Thus, we introduce the variables \tilde{x} and \tilde{u} , $L\tilde{x} = x$, $\tilde{u}(\tilde{x}) = u(x)$, $\tilde{v}(\tilde{x}) = v(x)$, then omit the tildes, and we find that problem (4) becomes

$$\begin{aligned} A + L^2 A_{xx} - L^2 A^3 + L^2 Ah &= 0, \quad 0 < x < 1, \\ -L^2 h_{xx} - h_{xxxx} + \alpha L^2 (A^2)_{xx} &= 0, \quad 0 < x < 1, \\ \frac{\partial A}{\partial x} = 0, \quad \frac{\partial^3 A}{\partial x^3} = 0, \quad x = 0, 1, t > 0, \\ \frac{\partial h}{\partial x} = 0, \quad \frac{\partial^3 h}{\partial x^3} = 0, \quad x = 0, 1, t > 0. \end{aligned} \quad (11)$$

The corresponding eigenvalue problem at (0, 0) is

$$\begin{aligned} \varphi_{xx} + L^2 \varphi &= \lambda' \varphi, \quad 0 < x < 1, \\ -L^2 \psi_{xx} - \psi_{xxxx} &= \lambda' \psi, \quad 0 < x < 1, \\ \varphi_x(0) = \varphi_x(1) = \varphi_{xxx}(0) = \varphi_{xxx}(1) &= 0, \\ \psi_x(0) = \psi_x(1) = \psi_{xxx}(0) = \psi_{xxx}(1) &= 0. \end{aligned} \quad (12)$$

It is easy to find that $\lambda' = L^2 \lambda$.

3. Bifurcation and Stability Analysis in Different Cases

Let L be a bifurcation parameter; $L = \pi$ is a bifurcation point. In this section, we discuss how many nontrivial solution branches will be bifurcated from the trivial solution

near $L = \pi$ and their asymptotic expression. Moreover, we will discuss the stability of the solution branches.

When $L = \pi$, the eigenvector at $(0, 0)$ of (12) is

$$e_1 = \begin{pmatrix} \cos \pi x \\ c \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ \cos \pi x \end{pmatrix}, \quad c \in R. \quad (13)$$

We set

$$L = \pi + \gamma(\varepsilon) \cdot \varepsilon, \quad (14)$$

$$\begin{pmatrix} A \\ h \end{pmatrix} = \sum_{i=1}^n \varepsilon \alpha_i e_i + \varepsilon^2 \begin{pmatrix} a(x, \varepsilon) \\ b(x, \varepsilon) \end{pmatrix}, \quad (15)$$

where ε is a small parameter, $\begin{pmatrix} a(x, \varepsilon) \\ b(x, \varepsilon) \end{pmatrix} \in \{e_1, e_2\}^\perp, \alpha_1^2 + \alpha_2^2 = 1$.

Substituting (14), (15) into (11) leads to

$$\begin{aligned} & (\varepsilon \alpha_1 \cos \pi x + \varepsilon^2 a(x, \varepsilon))'' + (\pi + \gamma(\varepsilon) \cdot \varepsilon)^2 \\ & \times (\varepsilon \alpha_1 \cos \pi x + \varepsilon^2 a(x, \varepsilon)) - (\pi + \gamma(\varepsilon) \cdot \varepsilon)^2 \\ & \times (\varepsilon \alpha_1 \cos \pi x + \varepsilon^2 a(x, \varepsilon))^3 + (\pi + \gamma(\varepsilon) \cdot \varepsilon)^2 \\ & \times (\varepsilon \alpha_1 \cos \pi x + \varepsilon^2 a(x, \varepsilon)) \\ & \times (\varepsilon \alpha_1 c + \varepsilon \alpha_2 \cos \pi x + \varepsilon^2 b(x, \varepsilon)) = 0, \\ & - (\pi + \gamma(\varepsilon) \cdot \varepsilon)^2 (\varepsilon \alpha_1 c + \varepsilon \alpha_2 \cos \pi x + \varepsilon^2 b(x, \varepsilon))'' \\ & - (\varepsilon \alpha_1 c + \varepsilon \alpha_2 \cos \pi x + \varepsilon^2 b(x, \varepsilon))'''' \\ & + \alpha (\pi + \gamma(\varepsilon) \cdot \varepsilon)^2 \left((\varepsilon \alpha_1 \cos \pi x + \varepsilon^2 a(x, \varepsilon))^2 \right)'' = 0, \end{aligned} \quad (16)$$

$$(17)$$

where ' means $\partial/\partial x$ and '' means $\partial^2/\partial x^2$.

Letting the coefficient of ε^2 vanish in (16), (17) gives

$$\begin{aligned} & a_0''(x) + \pi^2 a_0(x) + 2\pi\gamma(0) \alpha_1 \cos \pi x \\ & + \pi^2 \alpha_1 \cos \pi x (\alpha_1 c + \alpha_2 \cos \pi x) = 0, \end{aligned} \quad (18)$$

$$\begin{aligned} & b_0''''(x) + \pi^2 b_0''(x) + 2\pi\gamma(0) \alpha_1 \cos \pi x \\ & + \pi^2 \alpha_1 \cos \pi x (\alpha_1 c + \alpha_2 \cos \pi x) = 0, \end{aligned} \quad (19)$$

where $\begin{pmatrix} a_0(x) \\ b_0(x) \end{pmatrix} = \begin{pmatrix} a(x, 0) \\ b(x, 0) \end{pmatrix} \in \{e_1, e_2\}^\perp$.

Taking the inner product of (16) with $\cos \pi x$, we obtain

$$\begin{aligned} & \int_0^1 a_0''(x) \cos \pi x \, dx + \pi^2 \int_0^1 a_0(x) \cos \pi x \, dx \\ & + 2\pi\gamma(0) \alpha_1 \int_0^1 \cos^2 \pi x \, dx \\ & + \pi^2 \alpha_1 \int_0^1 \cos^2 \pi x (\alpha_1 c + \alpha_2 \cos \pi x) \, dx = 0. \end{aligned} \quad (20)$$

For the first term of (20), integrate it by part and from the boundary condition $a'(0) = a'(1) = 0$, and we have

$$\int_0^1 a_0''(x) \cos \pi x \, dx = -\pi^2 \int_0^1 a_0(x) \cos \pi x \, dx. \quad (21)$$

Substituting (21) to (20), we get

$$\gamma(0) = -\frac{1}{2} c \alpha_1 \pi, \quad (\alpha_1 \neq 0), \quad (22)$$

$$\gamma(0) \text{ is an arbitrary constant, } (\alpha_1 = 0).$$

From the aforementioned orthogonality condition, we know that

$$\int_0^1 a_0(x) \cos \pi x + c b_0(x) \, dx = 0, \quad (23)$$

$$\int_0^1 b_0(x) \cos \pi x \, dx = 0. \quad (24)$$

Taking the inner product of (19) with $\cos \pi x$, we obtain

$$\begin{aligned} & \int_0^1 b_0''''(x) \cos \pi x \, dx + \pi^2 \int_0^1 b_0''(x) \cos \pi x \, dx \\ & - 2\alpha_2 \pi^3 \gamma(0) \int_0^1 \cos^2 \pi x \\ & + 2\alpha \alpha_1^2 \pi^4 \int_0^1 \cos 2\pi x \cos \pi x \, dx = 0. \end{aligned} \quad (25)$$

Because of the boundary condition $b'(0) = b'(1) = b''(0) = b''(1) = 0$, orthogonality condition (24), we get

$$\gamma(0) = 0, \quad (\alpha_2 \neq 0), \quad (26)$$

$$\gamma(0) \text{ is an arbitrary constant, } (\alpha_2 = 0),$$

from (25).

Comparing (22) and (26), for $\alpha_1^2 + \alpha_2^2 = 1$, we deduce the following important relation:

$$\gamma(0) = -\frac{1}{2} c \alpha_1 \pi. \quad (27)$$

Substituting (27) to (18), we get

$$a_0''(x) + \pi^2 a_0(x) + \pi^2 \alpha_1 \alpha_2 \cos^2 \pi x = 0. \quad (28)$$

For $a_0(x)$ satisfies the boundary condition $a_0'(0) = a_0'(1) = a_0''(0) = a_0''(1)$, we calculate from (28) that

$$a_0(x) = k_1 \cos \pi x + \frac{\alpha_1 \alpha_2}{6} \cos 2\pi x - \frac{\alpha_1 \alpha_2}{2}, \quad k_1 \in R. \quad (29)$$

Next, we want to calculate $b_0(x)$ and substitute (27) to (19); we get the following ODE:

$$\begin{aligned} & b_0''''(x) + \pi^2 b_0''(x) + c \alpha_1 \alpha_2 \pi^4 \cos \pi x \\ & + 2\alpha \alpha_1^2 \pi^4 \cos 2\pi x = 0. \end{aligned} \quad (30)$$

By lengthy computations and the boundary conditions $b_0'(0) = b_0'(1) = b_0''(0) = b_0''(1)$, we deduce that

$$b_0(x) = c_1 + c_2 \cos \pi x - \frac{\alpha \alpha_1^2}{6} \cos 2\pi x, \quad (31)$$

$$c \alpha_1 \alpha_2 = 0. \quad (32)$$

Taking the inner product with $\cos \pi x$ of (31), from (24), we know that

$$\int_0^1 \left(c_1 + c_2 \cos \pi x - \frac{\alpha \alpha_1^2}{6} \cos 2\pi x \right) \cos \pi x \, dx = 0, \quad (33)$$

which gives us

$$c_2 = 0, \quad (34)$$

$$\begin{aligned} & \left(\frac{\pi^2}{96} \alpha_1 - \frac{\pi^2}{96} \alpha \alpha_1^3 - \frac{\pi^2}{24} \alpha_1^3 \right) \cos 3\pi x b_0(x) \\ & = c_1 - \frac{\alpha \alpha_1^2}{6} \cos 2\pi x. \end{aligned} \quad (35)$$

Substituting (29), (35) to (23), we have

$$\begin{aligned} & \int_0^1 \cos \pi x \left(k_1 \cos \pi x + \frac{\alpha_1 \alpha_2}{6} \cos 2\pi x - \frac{\alpha_1 \alpha_2}{2} \right) \\ & + c \left(c_1 - \frac{\alpha \alpha_1^2}{6} \cos 2\pi x \right) dx = 0, \end{aligned} \quad (36)$$

from which we get

$$\begin{aligned} & k_1 = -2cc_1, \\ & a_0(x) = -2cc_1 \cos \pi x + \frac{\alpha_1 \alpha_2}{6} \cos 2\pi x - \frac{\alpha_1 \alpha_2}{2}. \end{aligned} \quad (37)$$

From the previous discussion, we obtain

$$c \alpha_1 \alpha_2 = 0, \quad (38)$$

with the normal condition

$$\alpha_1^2 + \alpha_2^2 - 1 = 0. \quad (39)$$

When $c = 0$, then we have

$$\gamma_0 = 0; \quad (40)$$

at this time,

$$\begin{aligned} & a_0(x) = \frac{\alpha_1 \alpha_2}{6} \cos 2\pi x - \frac{\alpha_1 \alpha_2}{2}, \\ & b_0(x) = c_1 - \frac{\alpha \alpha_1^2}{6} \cos 2\pi x. \end{aligned} \quad (41)$$

Letting the coefficient of ε^3 of (16), (17) vanish, we get

$$\begin{aligned} & a_1''(x) + \pi^2 a_1(x) + 2\pi \alpha_1 \gamma_1(0) \cos \pi x \\ & - \pi^2 \alpha_1^3 \cos^3 \pi x + \pi^2 \alpha_1 \cos \pi x b_0(x) \end{aligned} \quad (42)$$

$$+ \pi^2 \alpha_2 a_0(x) \cos \pi x = 0,$$

$$\begin{aligned} & b_1''''(x) + \pi^2 b_1''(x) - 2\alpha_1 \alpha \pi^2 \cos \pi x a_0''(x) \\ & + 4\alpha_1 \alpha \pi^3 \sin \pi x a_0'(x) \\ & + 2\alpha_1 \alpha \pi^4 \cos \pi x a_0(x) \\ & - 2\pi^3 \gamma_1(0) \alpha_2 \cos \pi x = 0, \end{aligned} \quad (43)$$

where $a_1(x) = (\partial a(x, \varepsilon) / \partial x)|_{\varepsilon=0}$, $b_1(x) = (\partial b(x, \varepsilon) / \partial x)|_{\varepsilon=0}$ and $\gamma_1(0) = \gamma'(0)$.

Taking the inner product of (42) with $\cos \pi x$, we obtain

$$\begin{aligned} & \int_0^1 a_1''(x) \cos \pi x \, dx + \pi^2 \int_0^1 a_1(x) \cos \pi x \, dx \\ & + 2\pi \alpha_1 \gamma_1(0) \int_0^1 \cos^2 \pi x \, dx - \pi^2 \alpha_1^3 \int_0^1 \cos^4 \pi x \, dx \\ & + \pi^2 \alpha_1 \int_0^1 \cos^2 \pi x b_0(x) \, dx \\ & + \pi^2 \alpha_2 \int_0^1 a_0(x) \cos^2 \pi x \, dx = 0, \end{aligned} \quad (44)$$

from which we know that

$$\gamma_1(0) = \frac{3}{8} \pi \alpha_1^2 + \frac{\alpha \alpha_1^2 \pi}{24} - \frac{c_1 \pi}{2} + \frac{5}{24} \pi \alpha_2^2, \quad \alpha_1 \neq 0, \quad (45)$$

$\gamma_1(0)$ is an arbitrary constant, $\alpha_1 = 0$.

Next, we discuss the expression and the stability of the bifurcated solutions in different cases.

Case 1. $c = 0$, $\alpha_1 \neq 0$.

Substituting (41), (45) to (42), we have

$$a_1''(x) + \pi^2 a_1(x) + \left(\frac{\pi^2}{12} \alpha_1 - \frac{\pi^2}{12} \alpha \alpha_1^3 - \frac{\pi^2}{3} \alpha_1^3 \right) \cos 3\pi x = 0. \quad (46)$$

Because of the orthogonality condition

$$\int_0^1 a_1(x) \cos \pi x \, dx = 0, \quad (47)$$

and boundary condition

$$a_1'(0) = a_1'(1) = 0, \quad (48)$$

we get

$$a_1(x) = \left(\frac{\pi^2}{96} \alpha_1 - \frac{\pi^2}{96} \alpha \alpha_1^3 - \frac{\pi^2}{24} \alpha_1^3 \right) \cos 3\pi x. \quad (49)$$

Substituting (41) to (43), we have

$$\begin{aligned} & b_1''''(x) + \pi^2 b_1''(x) \\ & - \left(2\pi^3 \gamma_1(0) \alpha_2 + \frac{5}{6} \alpha \alpha_1^2 \alpha_2 \pi^4 \right) \cos \pi x \end{aligned} \quad (50)$$

$$+ \frac{3}{2} \alpha \alpha_1^2 \alpha_2 \pi^4 \cos 3\pi x = 0.$$

Taking the inner product of (50) with $\cos \pi x$, we obtain

$$\begin{aligned} & \int_0^1 b_1''''(x) \cos \pi x \, dx + \pi^2 \int_0^1 b_1''(x) \cos \pi x \, dx \\ & - \left(2\pi^3 \gamma_1(0) \alpha_2 + \frac{5}{6} \alpha \alpha_1^2 \alpha_2 \pi^4 \right) \int_0^1 \cos^2 \pi x \, dx \\ & + \frac{3}{2} \alpha \alpha_1^2 \alpha_2 \pi^4 \int_0^1 \cos 3\pi x \cos \pi x \, dx = 0, \end{aligned} \quad (51)$$

for $b_1(x)$ satisfies the orthogonality condition

$$\int_0^1 b_1(x) \cos \pi x dx = 0, \tag{52}$$

and the boundary condition

$$b_1'(0) = b_1'(1) = b_1'''(0) = b_1'''(1); \tag{53}$$

we know

$$\int_0^1 b_1''''(x) \cos \pi x dx + \pi^2 \int_0^1 b_1''(x) \cos \pi x dx = 0, \tag{54}$$

which forces

$$2\pi^3 \gamma_1(0) \alpha_2 + \frac{5}{6} \alpha \alpha_1^2 \alpha_2 \pi^4 = 0. \tag{55}$$

From (45) and (55), we get

$$\begin{aligned} \gamma_1(0) = -\frac{5}{12} \alpha \alpha_1^2 \pi = \frac{3}{8} \pi \alpha_1^2 + \frac{\alpha \alpha_1^2 \pi}{24} - \frac{c_1 \pi}{2} + \frac{5}{24} \pi \alpha_2^2, \\ \alpha_2 \neq 0, \end{aligned} \tag{56}$$

or

$$\begin{aligned} \gamma_1(0) = \frac{3}{8} \pi \alpha_1^2 + \frac{\alpha \alpha_1^2 \pi}{24} - \frac{c_1 \pi}{2} + \frac{5}{24} \pi \alpha_2^2, \\ \alpha_2 = 0. \end{aligned} \tag{57}$$

Substituting (55) to (50), we get

$$b_1''''(x) + \pi^2 b_1''(x) + \frac{3}{2} \alpha \alpha_1^2 \alpha_2 \pi^4 \cos 3\pi x = 0, \tag{58}$$

which gives us

$$b_1(x) = \tilde{c}_1 - \frac{1}{48} \alpha \alpha_1^2 \alpha_2 \cos 3\pi x. \tag{59}$$

As has been discussed previously, we have the following.

For L near π , there are nontrivial steady-state solution branches of (11) bifurcated from the trivial solution:

$$\begin{aligned} A(\varepsilon) = \varepsilon \alpha_1 \cos \pi x + \varepsilon^2 \left(\frac{\alpha_1 \alpha_2}{6} \cos \pi x - \frac{\alpha_1 \alpha_2}{2} \right) \\ + \varepsilon^3 \left(\frac{\pi^2}{96} \alpha_1 - \frac{\pi^2}{96} \alpha \alpha_1^3 - \frac{\pi^2}{24} \alpha_1^3 \right) \cos 3\pi x \\ + o(\varepsilon^3), \\ h(\varepsilon) = \varepsilon \alpha_2 \cos \pi x + \varepsilon^2 \left(c_1 - \frac{\alpha \alpha_1^2}{6} \cos 2\pi x \right) \\ + \varepsilon^3 \left(\tilde{c}_1 - \frac{1}{48} \alpha \alpha_1^2 \alpha_2 \cos 3\pi x \right) + o(\varepsilon^3), \\ L = \pi + \varepsilon^2 \gamma_1(0) + o(\varepsilon^2). \end{aligned} \tag{60}$$

$\alpha_1 \neq 0, \alpha_2, \gamma_1(0)$ satisfy (56), (57). $c_1, \tilde{c}_1 \in R$. c_1, \tilde{c}_1 are arbitrary constants.

Next, we wish to consider the stability of the nontrivial solutions given in (60).

Considering the eigenvalue problem

$$F_{A,h}(A(\varepsilon), h(\varepsilon), L(\varepsilon)) \begin{pmatrix} \phi(\varepsilon) \\ \psi(\varepsilon) \end{pmatrix} = \lambda(\varepsilon) \begin{pmatrix} \phi(\varepsilon) \\ \psi(\varepsilon) \end{pmatrix}, \tag{61}$$

since $\lambda(0) = 0, \phi(0) = \cos \pi x, \psi(0) = 0$, we assume

$$\begin{pmatrix} \phi(\varepsilon) \\ \psi(\varepsilon) \end{pmatrix} = \begin{pmatrix} \cos \pi x \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} + \dots,$$

$$\lambda(\varepsilon) = \lambda_1 \varepsilon + \lambda_2 \varepsilon^2 + \dots,$$

$$F_{A,h}(A(\varepsilon), h(\varepsilon), L(\varepsilon))$$

$$= \begin{pmatrix} L^2 + \frac{\partial^2}{\partial x^2} - 3L^2 A^2(\varepsilon) + L^2 h(\varepsilon) & L^2 A(\varepsilon) \\ 2\alpha L^2 \left(2 \frac{\partial A(\varepsilon)}{\partial x} \frac{\partial}{\partial x} + \frac{\partial^2 A(\varepsilon)}{\partial x^2} + A(\varepsilon) \frac{\partial^2}{\partial x^2} \right) & -L^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^4}{\partial x^4} \end{pmatrix}.$$

Substituting (62) to (61), we get

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \phi(\varepsilon) + L^2 \phi(\varepsilon) - 3L^2 A^2(\varepsilon) \phi(\varepsilon) \\ + L^2 h(\varepsilon) \phi(\varepsilon) + L^2 A(\varepsilon) \psi(\varepsilon) \\ - \lambda(\varepsilon) \phi(\varepsilon) = 0, \\ 2\alpha L^2 \left(2 \frac{\partial A(\varepsilon)}{\partial x} \frac{\partial \phi(\varepsilon)}{\partial x} + \frac{\partial^2 A(\varepsilon)}{\partial x^2} \phi(\varepsilon) + A(\varepsilon) \frac{\partial^2 \phi(\varepsilon)}{\partial x^2} \right) \\ - L^2 \frac{\partial^2 \psi(\varepsilon)}{\partial x^2} - \frac{\partial^4 \psi(\varepsilon)}{\partial x^4} - \lambda(\varepsilon) \psi(\varepsilon) = 0. \end{aligned} \tag{63}$$

On the branches given in (60), equating the coefficient of ε in (63) to 0, we obtain

$$\begin{aligned} \phi_1'' + \pi^2 \phi_1 + \alpha_2 \pi^2 \cos^2 \pi x - \lambda_1 \cos \pi x = 0, \\ 4\alpha \alpha_1 \pi^4 \cos 2\pi x + \pi^2 \psi_1'' + \psi_1'''' = 0. \end{aligned} \tag{64}$$

From (64) and the boundary conditions

$$\begin{aligned} \phi_1'(0) = \phi_1'(1) = \phi_1''''(0) = \phi_1''''(1), \\ \psi_1'(0) = \psi_1'(1) = \psi_1''''(0) = \psi_1''''(1), \end{aligned} \tag{65}$$

we get

$$\begin{aligned} \lambda_1 = 0, \\ \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} p_1 \cos \pi x + \frac{\alpha_2}{6} \cos 2\pi x - \frac{\alpha_2}{2} \\ q_1 + q_2 \cos \pi x - \frac{\alpha \alpha_1}{3} \cos 2\pi x \end{pmatrix} \cdot P_1, \end{aligned} \tag{66}$$

$$q_1, q_2 \in R.$$

Using the second-order term of ε in (63), we obtain

$$\begin{aligned} \phi_2'' + \pi^2 \phi_2 - \left[\left(-\frac{3}{2} \alpha_1^2 + \alpha_1 q_1 - \frac{\alpha}{6} \alpha_1^2 \right) \pi^2 - \lambda_2 \right] \cos \pi x \\ + \left(\frac{p_1 \alpha_2}{2} + \frac{q_2 \alpha_1}{2} \right) \pi^2 \cos 2\pi x \\ + \left(-\frac{3}{4} \alpha_1^2 + \frac{1}{12} \alpha_1^2 - \frac{\pi^2}{4} \alpha \alpha_1^2 \right) \cos 3\pi x \end{aligned} \tag{67}$$

$$\begin{aligned} + \frac{\pi^2}{2} p_1 \alpha_2 + \frac{\pi^2}{2} q_2 \alpha_1 = 0, \\ \psi_2'''' + \pi^2 \psi_2'' - \frac{5}{3} \alpha_1 \alpha_2 \alpha \pi^4 \cos \pi x + 4 \alpha \alpha_1 p_1 \pi^4 \cos 2\pi x \\ + 3 \alpha_1 \alpha_2 \alpha \pi^4 \cos 3\pi x = 0, \end{aligned}$$

and considering the boundary condition

$$\begin{aligned} \phi_2'(0) = \phi_2'(1) = \phi_2''(0) = \phi_2''(1), \\ \psi_2'(0) = \psi_2'(1) = \psi_2''(0) = \psi_2''(1), \end{aligned} \tag{68}$$

we deduce that (67) has solution if and only if $\alpha_2 = 0$; in this case, we have $\alpha_1 = \pm 1$ ($\alpha_1^2 + \alpha_2^2 = 1$). Taking the sign of ε into account in (15), we consider $\alpha_1 = 1$ only. Therefore, we have

$$\begin{aligned} \lambda_2 = \left(-\frac{3}{2} - \frac{\alpha}{6} + q_1 \right) \pi^2, \\ \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} p_2 \cos \pi x + \frac{q_2}{6} \cos 2\pi x - \left(\frac{3}{32} + \frac{\alpha}{32} \right) \cos 3\pi x - \frac{q_2}{2} \\ \bar{q}_1 + \bar{q}_2 \cos \pi x - \frac{p_1 \alpha}{3} \cos 2\pi x \end{pmatrix} \cdot p_2, \\ \bar{q}_1, \bar{q}_2 \in \mathbb{R}. \end{aligned} \tag{69}$$

So in Case 1, on the branches (60), the eigenvalue of $F_{A,h}(A(\varepsilon), h(\varepsilon), L(\varepsilon))$ is

$$\lambda(\varepsilon) = \left(-\frac{3}{2} - \frac{\alpha}{6} + q_1 \right) \pi^2 \varepsilon^2 + o(\varepsilon^2). \tag{70}$$

The corresponding eigenfunction is

$$\begin{aligned} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \cos \pi x \\ 0 \end{pmatrix} + \begin{pmatrix} p_1 \cos \pi x \\ q_1 + q_2 \cos \pi x - \frac{\alpha}{3} \cos 2\pi x \end{pmatrix} \varepsilon \\ + \begin{pmatrix} p_2 \cos \pi x - \left(\frac{3}{32} + \frac{\alpha}{32} \right) \cos 3\pi x - \frac{q_2}{2} + \frac{q_2}{6} \cos 2\pi x \\ \bar{q}_1 + \bar{q}_2 \cos \pi x - \frac{p_1 \alpha}{3} \cos 2\pi x \end{pmatrix} \varepsilon^2 \\ + o(\varepsilon^2). \end{aligned} \tag{71}$$

From (70), we know that the eigenvalue of the linearized operator $F_{A,h}(A(\varepsilon), h(\varepsilon), L(\varepsilon))$ at the nontrivial solutions given in (60) is negative if $q_1 < (3/2 + \alpha/6)$, and positive

if $q_1 > (3/2 + \alpha/6)$. Finally, for small ε , if $q_1 < (3/2 + \alpha/6)$, then the corresponding solution branches are stable; if $q_1 > (3/2 + \alpha/6)$, then the corresponding solution branches are unstable; if $q_1 = (3/2 + \alpha/6)$, we need higher-order items of λ to determine the result.

Thus, we have proved the following theorems.

Theorem 1. For L near π , there are nontrivial steady-state solution branches of (4) bifurcated from the trivial solution:

$$\begin{aligned} A(\varepsilon) = \varepsilon \alpha_1 \cos \frac{\pi x}{L} + \varepsilon^2 \left(\frac{\alpha_1 \alpha_2}{6} \cos \frac{2\pi x}{L} - \frac{\alpha_1 \alpha_2}{2} \right) \\ + \varepsilon^3 \left(\frac{\pi^2}{96} \alpha_1 - \frac{\pi^2}{96} \alpha \alpha_1^3 - \frac{\pi^2}{24} \alpha_1^3 \right) \cos \frac{3\pi x}{L} + o(\varepsilon^3), \\ h(\varepsilon) = \varepsilon \alpha_2 \cos \frac{\pi x}{L} + \varepsilon^2 \left(c_1 - \frac{\alpha \alpha_1^2}{6} \cos \frac{2\pi x}{L} \right) \\ + \varepsilon^3 \left(\bar{c}_1 - \frac{1}{48} \alpha \alpha_1^2 \alpha_2 \cos 3\pi x \right) + o(\varepsilon^3), \\ L = \pi + \varepsilon^2 \gamma_1(0) + o(\varepsilon^2), \end{aligned} \tag{72}$$

where $\alpha_1 = 1, \alpha_2 = 0, \gamma_1(0) = 3/8 + \alpha/24 - c_1/2, c_1, \bar{c}_1 \in \mathbb{R}$.

Remark 2. Since $\gamma_1(0) = 3/8 + \alpha/24 - c_1/2$, when $c_1 < 3/4 + \alpha/12, L = \pi$ is a supercritical bifurcation point; when $c_1 > (3/4 + \alpha/12), L = \pi$ is a supercritical bifurcation point.

Theorem 3. The eigenvalue of the linearized operator $F_{A,h}(A(\varepsilon), h(\varepsilon), L(\varepsilon))$ at the nontrivial solutions given in Theorem 1 is negative if $q_1 < (3/2 + \alpha/6)$, positive if $q_1 > (3/2 + \alpha/6)$. Finally, for small ε , if $q_1 < (3/2 + \alpha/6)$, then the corresponding solution branches are stable; if $q_1 > (3/2 + \alpha/6)$, then the corresponding solution branches are unstable.

Case 2. $c = 0, \alpha_1 = 0$.

If $c = 0, \alpha_1 = 0$, then $\alpha_2 = \pm 1$; taking the sign of ε into account in (15), we consider $\alpha_2 = 1$ only. In this case, from (40), (41), (45), (49), (56), and (59), we have

$$\begin{aligned} a_0(x) = 0, \quad b_0(x) = c_1, \quad \gamma(0) = 0, \\ a_1(x) = 0, \quad b_1(x) = \bar{c}_1, \quad \gamma_1(0) = 0. \end{aligned} \tag{73}$$

Similarly discussed as in Case 1 previously, we get

$$\gamma_2(0) = \gamma_3(0) = \gamma_4(0) = \dots = 0. \tag{74}$$

Remark 4. Since $\gamma(0) = \gamma_1(0) = \gamma_2(0) = \gamma_3(0) = \gamma_4(0) = \dots = 0$, there are no solution branches of (4) bifurcated from the trivial solution in Case 2.

Case 3. $c \neq 0, \alpha_1 \neq 0$.

If $c \neq 0, \alpha_1 \neq 0$, then $\alpha_2 = 0, \alpha_1 = \pm 1$; taking the sign of ε into account in (15), we consider $\alpha_2 = 1$ only. In this case,

from (27), (31), and (37), we have

$$\begin{aligned} a_0(x) &= -2cc_1 \cos \pi x, \\ b_0(x) &= c_1 - \frac{\alpha}{6} \cos 2\pi x, \\ \gamma(0) &= -\frac{1}{2}c\pi. \end{aligned} \tag{75}$$

Letting the coefficient of ε^3 of (16) vanish, we get

$$\begin{aligned} a_1''(x) + \pi^2 a_1(x) + \left(c_1 \pi^2 + 2\pi\gamma_1 - \frac{3}{4}c^2\pi^2 \right. \\ \left. - \frac{3}{4}\pi^2 - \frac{\alpha}{12}\pi^2 \right) \cos \pi x \\ + \left(\frac{\alpha}{12}\pi^2 - \frac{\pi^2}{4} \right) \cos 3\pi x = 0. \end{aligned} \tag{76}$$

Taking the inner product of (76) with $\cos \pi x$, we obtain

$$\begin{aligned} \int_0^1 a_1''(x) \cos \pi x \, dx + \pi^2 \int_0^1 a_1(x) \cos \pi x \, dx \\ + \int_0^1 \left(c_1 \pi^2 + 2\pi\gamma_1 - \frac{3}{4}c^2\pi^2 \right. \\ \left. - \frac{3}{4}\pi^2 - \frac{\alpha}{12}\pi^2 \right) \cos^2 \pi x \, dx \\ + \left(\frac{\alpha}{12}\pi^2 - \frac{\pi^2}{4} \right) \int_0^1 \cos 3\pi x \cos \pi x \, dx, \end{aligned} \tag{77}$$

from which we get

$$\begin{aligned} \gamma_1(0) &= \frac{3}{8}c^2\pi + \frac{3}{8}\pi - \frac{c_1\pi}{2} + \frac{\alpha}{24}\pi, \\ a_1(x) &= k_2 \cos \pi x + \left(\frac{\alpha}{96} - \frac{1}{32} \right) \cos 3\pi x. \end{aligned} \tag{78}$$

Letting the coefficient of ε^3 of (17) vanish, we get

$$b_1''''(x) + \pi^2 b_1''(x) - \left(8c_1 + \frac{8}{3} \right) \pi^4 \alpha c \cos 2\pi x = 0, \tag{79}$$

from which we get

$$b_1(x) = \left(\frac{2}{3}c_1\alpha c + \frac{2}{9}\alpha c \right) \cos 2\pi x + c_2. \tag{80}$$

Substituting (78), (80) to the orthogonality condition

$$\int_0^1 \cos \pi x a_1(x) + c b_1(x) \, dx = 0, \tag{81}$$

we get

$$k_2 = -2cc_2. \tag{82}$$

Thus, we have that for L near π , there are nontrivial steady-state solution branches of (11) bifurcated from the trivial solution:

$$\begin{aligned} A(\varepsilon) &= \varepsilon \cos \pi x + \varepsilon^2 (-2cc_1) \cos \pi x \\ &\quad + \varepsilon^3 \left[-2cc_2 \cos \pi x + \left(\frac{\alpha}{96} - \frac{1}{32} \right) \cos 3\pi x \right] + o(\varepsilon^3), \\ h(\varepsilon) &= \varepsilon^2 \left(c_1 - \frac{\alpha}{6} \cos 2\pi x \right) \\ &\quad + \varepsilon^3 \left(c_2 + \frac{2}{3}c_1\alpha c + \frac{2}{9}\alpha c \right) \cos 2\pi x + o(\varepsilon^3), \\ L &= \pi - \frac{1}{2}c\pi\varepsilon + \left(\frac{3}{8}c^2\pi + \frac{3}{8}\pi \right. \\ &\quad \left. - \frac{c_1\pi}{2} + \frac{\alpha}{24}\pi \right) \varepsilon^2 + o(\varepsilon^2). \end{aligned} \tag{83}$$

Next, we wish to consider the stability of the nontrivial solutions given in (83).

Considering the eigenvalue problem

$$F_{A,h}(A(\varepsilon), h(\varepsilon), L(\varepsilon)) \begin{pmatrix} \phi(\varepsilon) \\ \psi(\varepsilon) \end{pmatrix} = \lambda(\varepsilon) \begin{pmatrix} \phi(\varepsilon) \\ \psi(\varepsilon) \end{pmatrix}, \tag{84}$$

since $\lambda(0) = 0$, $\phi(0) = \cos \pi x$, $\psi(0) = c$, we assume that

$$\begin{aligned} \begin{pmatrix} \phi(\varepsilon) \\ \psi(\varepsilon) \end{pmatrix} &= \begin{pmatrix} \cos \pi x \\ c \end{pmatrix} + \varepsilon \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \phi_2 \\ \psi_2 \end{pmatrix} + \dots, \\ \lambda(\varepsilon) &= \lambda_1\varepsilon + \lambda_2\varepsilon^2 + \dots, \end{aligned}$$

$$F_{A,h}(A(\varepsilon), h(\varepsilon), L(\varepsilon))$$

$$= \begin{pmatrix} L^2 + \frac{\partial^2}{\partial x^2} - 3L^2 A^2(\varepsilon) + L^2 h(\varepsilon) & L^2 A(\varepsilon) \\ 2\alpha L^2 \left(2 \frac{\partial A(\varepsilon)}{\partial x} \frac{\partial}{\partial x} + \frac{\partial^2 A(\varepsilon)}{\partial x^2} + A(\varepsilon) \frac{\partial^2}{\partial x^2} \right) & -L^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^4}{\partial x^4} \end{pmatrix}. \tag{85}$$

Substituting (85) to (84), we get

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \phi(\varepsilon) + L^2 \phi(\varepsilon) - 3L^2 A^2(\varepsilon) \phi(\varepsilon) + L^2 h(\varepsilon) \phi(\varepsilon) \\ + L^2 A(\varepsilon) \psi(\varepsilon) - \lambda(\varepsilon) \phi(\varepsilon) = 0, \\ 2\alpha L^2 \left(2 \frac{\partial A(\varepsilon)}{\partial x} \frac{\partial \phi(\varepsilon)}{\partial x} + \frac{\partial^2 A(\varepsilon)}{\partial x^2} \phi(\varepsilon) + A(\varepsilon) \frac{\partial^2 \phi(\varepsilon)}{\partial x^2} \right) \\ - L^2 \frac{\partial^2 \psi(\varepsilon)}{\partial x^2} - \frac{\partial^4 \psi(\varepsilon)}{\partial x^4} - \lambda(\varepsilon) \psi(\varepsilon) = 0. \end{aligned} \tag{86}$$

On the branches given in (83), equating the coefficient of ε in (86) to 0, we obtain

$$\begin{aligned} \phi_1'' + \pi^2 \phi_1 - c\pi^2 \cos \pi x - \lambda_1 \cos \pi x = 0, \\ 4\alpha\pi^4 \cos 2\pi x + \pi^2 \psi_1'' + \psi_1'''' + \lambda_1 c = 0. \end{aligned} \tag{87}$$

From (87) and the boundary conditions

$$\begin{aligned}\phi_1'(0) &= \phi_1'(1) = \phi_1'''(0) = \phi_1'''(1), \\ \psi_1'(0) &= \psi_1'(1) = \psi_1'''(0) = \psi_1'''(1),\end{aligned}\quad (88)$$

we get

$$\begin{aligned}c &= 0, \\ \lambda_1 &= -c\pi^2 = 0, \\ \begin{pmatrix} \phi_1 \\ \psi_1 \end{pmatrix} &= \begin{pmatrix} \widetilde{\widetilde{P}}_1 \cos \pi x \\ \widetilde{\widetilde{Q}}_1 + \widetilde{\widetilde{Q}}_2 \cos \pi x - \frac{\alpha}{3} \cos 2\pi x \end{pmatrix} \cdot \widetilde{\widetilde{P}}_1, \\ \widetilde{\widetilde{Q}}_1, \widetilde{\widetilde{Q}}_2 &\in \mathbb{R}.\end{aligned}\quad (89)$$

We find that $c = 0$ is a contradiction to $c \neq 0$, so there are no solution branches of (4) bifurcated from the trivial solution in Case 3.

Case 4. $c \neq 0$, $\alpha_1 = 0$.

If $c \neq 0$, $\alpha_1 = 0$, then $\alpha_2 = \pm 1$; taking the sign of ε into account in (15), we consider $\alpha_2 = 1$ only. In this case, from (27), (31), and (37), we have

$$a_0(x) = -2cc_1 \cos \pi x, \quad b_0(x) = c_1, \quad \gamma(0) = 0. \quad (90)$$

Similarly discussed as in Case 1 previously, we get

$$\gamma_1(0) = \gamma_2(0) = \gamma_3(0) = \gamma_4(0) = \dots = 0. \quad (91)$$

Remark 5. Since $\gamma(0) = \gamma_1(0) = \gamma_2(0) = \gamma_3(0) = \gamma_4(0) = \dots = 0$, there were no solution branches of (4) bifurcated from the trivial solution in Case 4.

Remark 6. In this paper, trivial stationary solution and bifurcations at it are considered for the KS-GL equations with Neumann's boundary conditions on a finite domain $(0, L)$. The results we get in this paper are new and original. It would be very interesting and much more complicated to try and extend our study to stationary patterns and the corresponding bifurcations. Another interesting but challenging direction would be the case in several space dimensions.

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