Research Article

Positive Solutions for a Mixed-Order Three-Point Boundary Value Problem for p-Laplacian

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The author investigates the existence and multiplicity of positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator. The main tool is fixed point index theory and Leggett-Williams fixed point theorem.

1. Introduction

In this paper, we are interested in the existence of single and multiple positive solutions to nonlinear mixed-order three-point boundary value problem for p-Laplacian. Consider the following:

$$\left(\varphi_p \left(D_{0^+}^{\alpha} u(t) \right) \right)' + a(t) f(t, u(t)) = 0, \quad 0 < t < 1, \quad (1)$$

$$D_{0^+}^{\alpha} u(0) = u(0) = u''(0) = 0, \quad u'(1) = \gamma u'(\eta), \quad (2)$$

where $\eta, \gamma \in (0,1)$, $\alpha \in (2,3]$, $D_{0^+}^{\alpha}$ is the Caputo's derivative, $\varphi_p(s) = |s|^{p-2} s$, and $\varphi_q = (\varphi_p)^{-1}$ with $(1/p) + (1/q) = 1$. We assume the following conditions throughout.

(H1) $f \in C([0,1] \times [0, \infty), [0, \infty))$.

(H2) $a \in L^1(0,1)$ is nonnegative and $a(t) \not\equiv 0$ on any subinterval of $(0,1)$.

The equation with a p-Laplacian operator arises in the modeling of different physical and natural phenomena, non-Newtonian mechanics, nonlinear elasticity and glaciology, combustion theory, population biology, nonlinear flow laws, and so on. Liang et al. in [1] used the fixed point theorem of Avery and Henderson to show the existence of at least two positive solutions. Zhao et al. [2] studied the existence of at least three positive solutions by using Leggett-Williams fixed point theorem. Chai [3] obtain results for the existence of at least one nonnegative solution and two positive solutions by

using fixed point theorem on cone. Su et al. [4] studied the existence of one and two positive solutions by using the fixed point index theory. Su [5] studied the existence of one and two positive solution by using the method of defining operator by the reverse function of Green function and the fixed point index theory. Tang et al. [6] studied the existence of positive solutions of fractional differential equation with p-laplacian by using the coincidence degree theory.

Motivated by the above works, we obtain some sufficient conditions for the existence of at least one and three positive solutions for (1) and (2).

The organization of this paper is as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our results. In Section 3, we discuss the existence of at least one positive solution for (1) and (2). In Section 4, we discuss the existence of multiple positive solutions for (1) and (2). Finally, we give some examples to illustrate our results in Section 5.

2. Preliminaries

Definition 1. Let E be a real Banach space. A nonempty closed convex set $K \subset E$ is called cone if

(1) $x \in K$, $\lambda \geq 0$ then $\lambda x \in K$,

(2) $x \in K$, $-x \in K$ then $x = 0$.

Definition 2. An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Remark 3. By the positive solution of (1) and (2) we understand a function $u(t)$ which is positive on $[0, 1]$ and satisfies the differential equation (1) and the boundary conditions (2).

We will consider the Banach space $E = C[0, 1]$ equipped with standard norm:

$$\|u\| = \max_{0 \leq t \leq 1} |u(t)| \tag{3}$$

The proof of existence of solution is based upon an application of the following theorem.

Theorem 4 (see [7, 8]). *Let E be a Banach space and let K be a cone of E. For $r > 0$, define $K_r = \{u \in K : \|u\| \leq r\}$ and assume that $T : K_r \to K$ is a completely continuous operator such that $Tu \neq u$ for all $u \in \partial K_r$.*

(1) *If $\|Tu\| \leq \|u\|$ for all $u \in \partial K_r$, then $i(T, K_r, K) = 1$.*

(2) *If $\|Tu\| \geq \|u\|$ for all $u \in \partial K_r$, then $i(T, K_r, K) = 0$.*

Lemma 5 (see [9]). *Let $n \in \mathbb{N}$ with $n \geq 2$, $n - 1 < \alpha \leq n$. If $u \in C^{n-1}[a, b]$ and $D_{a^+}^\alpha u \in C(a, b)$, then*

$$\left(I_{a^+}^\alpha D_{a^+}^\alpha u\right)(t) = u(t) - \sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(t - a)^k \tag{4}$$

holds on (a, b).

Lemma 6. *The three-point boundary value problem (1)-(2) has a unique solution*

$$u(t) = \int_0^1 G_1(t, s) \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau)) d\tau\right) ds$$

$$+ \frac{\gamma t}{1 - \gamma} \int_0^1 G_2(\eta, s) \tag{5}$$

$$\times \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau)) d\tau\right) ds,$$

where

$$G_1(t, s)$$

$$= \begin{cases} \dfrac{(\alpha - 1)t(1 - s)^{\alpha - 2} - (t - s)^{\alpha - 1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{(\alpha - 1)t(1 - s)^{\alpha - 2}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$G_2(\eta, s)$$

$$= \begin{cases} \dfrac{(\alpha - 1)(1 - s)^{\alpha - 2} - (\alpha - 1)(\eta - s)^{\alpha - 2}}{\Gamma(\alpha)}, & 0 \leq s \leq \eta \leq 1, \\ \dfrac{(\alpha - 1)(1 - s)^{\alpha - 2}}{\Gamma(\alpha)}, & 0 \leq \eta \leq s \leq 1. \end{cases} \tag{6}$$

Proof. Integrating both sides of (1) on $[0, 1]$, we have

$$\varphi_p\left(D^\alpha u(t)\right) - \varphi_p\left(D^\alpha u(0)\right) = -\int_0^t a(s) f(s, u) ds. \tag{7}$$

So

$$D^\alpha u(t) = -\varphi_q\left(\int_0^t a(s) f(s, u) ds\right). \tag{8}$$

From Lemma 5, we have

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \varphi_q\left(\int_0^s a(\tau) f(\tau, u) d\tau\right) ds$$

$$+ A + Bt + Ct^2,$$

$$u'(t) = -\frac{1}{\Gamma(\alpha - 1)} \int_0^t (t - s)^{\alpha - 2} \varphi_q\left(\int_0^s a(\tau) f(\tau, u) d\tau\right) ds$$

$$+ B + 2Ct,$$

$$u''(t) = -\frac{1}{\Gamma(\alpha - 2)} \int_0^t (t - s)^{\alpha - 3} \varphi_q$$

$$\times \left(\int_0^s a(\tau) f(\tau, u) d\tau\right) ds + 2C. \tag{9}$$

From (2), $A = 0$ and $C = 0$.

Now, consider the following:

$$u'(1) = -\frac{1}{\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha - 2} \varphi_q$$

$$\times \left(\int_0^s a(\tau) f(\tau, u) d\tau\right) ds + B,$$

$$u'(\eta) = -\frac{1}{\Gamma(\alpha - 1)} \int_0^\eta (\eta - s)^{\alpha - 2} \varphi_q$$

$$\times \left(\int_0^s a(\tau) f(\tau, u) d\tau\right) ds + B, \tag{10}$$

by the boundary value condition $u'(1) = \gamma u'(\eta)$, we have

$$B = \frac{1}{(1 - \gamma)\Gamma(\alpha - 1)} \int_0^1 (1 - s)^{\alpha - 2} \varphi_q$$

$$\times \left(\int_0^s a(\tau) f(\tau, u) d\tau\right) ds$$

$$- \frac{\gamma}{(1 - \gamma)\Gamma(\alpha - 1)} \int_0^\eta (\eta - s)^{\alpha - 2} \varphi_q$$

$$\times \left(\int_0^s a(\tau) f(\tau, u) d\tau\right) ds, \tag{11}$$

so

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} \varphi_q\left(\int_0^s a(\tau) f(\tau, u) d\tau\right) ds$$

$$+ \frac{t}{(1 - \gamma)\Gamma(\alpha - 1)}$$

$$\times \int_0^1 (1-s)^{\alpha-2} \varphi_q \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds$$

$$- \frac{\gamma t}{(1-\gamma) \Gamma(\alpha-1)}$$

$$\times \int_0^\eta (\eta-s)^{\alpha-2} \varphi_q \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds. \tag{12}$$

Splitting the second integral in two parts of the form

$$\frac{t}{\Gamma(\alpha-1)} + \frac{k}{(1-\gamma) \Gamma(\alpha-1)} = \frac{t}{(1-\gamma) \Gamma(\alpha-1)}, \tag{13}$$

we have $k = \gamma t$; thus,

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi_q$$

$$\times \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds$$

$$+ \frac{t}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \varphi_q$$

$$\times \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds$$

$$+ \frac{\gamma t}{(1-\gamma) \Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} \varphi_q$$

$$\times \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds$$

$$- \frac{\gamma t}{(1-\gamma) \Gamma(\alpha-1)} \int_0^\eta (\eta-s)^{\alpha-2} \varphi_q$$

$$\times \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds; \tag{14}$$

therefore,

$$u(t) = \int_0^t \left(\frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \varphi_q$$

$$\times \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds$$

$$+ \int_t^1 \frac{t(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \varphi_q \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds$$

$$+ \frac{\gamma t}{1-\gamma} \int_0^\eta \left(\frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} - \frac{(\eta-s)^{\alpha-2}}{\Gamma(\alpha-1)} \right) \varphi_q$$

$$\times \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds$$

$$+ \frac{\gamma t}{1-\gamma} \int_\eta^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha-1)} \varphi_q \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds. \tag{15}$$

This completes the proof. □

Lemma 7. *Let $\beta \in (0, 1)$ be fixed. The kernel, $G_1(t, s)$, satisfies the following properties.*

(1) $0 \leq G_1(t, s) \leq G_1(1, s)$ for all $s \in (0, 1)$,

(2) $\min_{\beta \leq t \leq 1} G_1(t, s) \geq \beta G_1(1, s)$ for all $s \in [0, 1]$.

Proof. (1) As $2 < \alpha \leq 3$ and $0 \leq s \leq t \leq 1$, we have

$$(\alpha-1) t(1-s)^{\alpha-2} > t(1-s)^{\alpha-2}$$

$$\geq (t-s)(t-s)^{\alpha-2} = (t-s)^{\alpha-1}; \tag{16}$$

thus, $G_1(t, s) > 0$. Note $\partial G_1(t, s)/\partial t \geq 0$ then, $G_1(t, s)$ is increasing as a function of t; therefore,

$$G_1(t, s) \leq G_1(1, s) \quad \forall s \in [0, 1]. \tag{17}$$

(2) For $\beta \leq t \leq 1$, we have

$$\min_{\beta \leq t \leq 1} G_1(t, s) = G_1(\beta, s), \tag{18}$$

where

$$G_1(\beta, s) = \begin{cases} \dfrac{(\alpha-1)\beta(1-s)^{\alpha-2} - (\beta-s)^{\alpha-1}}{\Gamma(\alpha)} & 0 \leq s \leq \beta, \\[3mm] \dfrac{(\alpha-1)\beta(1-s)^{\alpha-2}}{\Gamma(\alpha)} & \beta \leq s \leq 1. \end{cases} \tag{19}$$

(a) If $0 < s \leq \beta$,

$$\min_{\beta \leq t \leq 1} G_1(t, s) = \frac{\beta(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)} - \frac{(\beta-s)^{\alpha-1}}{\Gamma(\alpha)}; \tag{20}$$

on the other hand,

$$\beta G_1(1, s) = \frac{\beta(\alpha-1)(1-s)^{\alpha-2}}{\Gamma(\alpha)} - \frac{\beta(1-s)^{\alpha-1}}{\Gamma(\alpha)}. \tag{21}$$

Since $2 < \alpha \leq 3$ and

(i) $\alpha - 1 > 1, \beta \in (0, 1) \Rightarrow \beta^{\alpha-1} < \beta$,

(ii) $s \leq \beta \Rightarrow s/\beta \leq 1 \Rightarrow 1 - (s/\beta) \geq 0$,

(iii) $\beta < 1 \Rightarrow 1 < 1/\beta \Rightarrow -s(1/\beta) < -s \Rightarrow 1 - s(1/\beta) < 1 - s$;

thus, we have

$$\left(1 - \frac{s}{\beta} \right)^{\alpha-1} < (1-s)^{\alpha-1}. \tag{22}$$

From (20), we obtain

$$(\beta - s)^{\alpha-1} = \left(\beta \left(1 - \frac{s}{\beta} \right) \right)^{\alpha-1}$$

$$= \beta^{\alpha-1} \left(1 - \frac{s}{\beta} \right)^{\alpha-1}$$

$$\leq \beta \left(1 - \frac{s}{\beta} \right)^{\alpha-1} \tag{23}$$

$$< \beta(1-s)^{\alpha-1}.$$

It follows from (20), (21), and (23), that item 2 in the proof hold.

(b) If $\beta \leq s < 1$,

$$\min_{\beta \leq t \leq 1} G_1(t, s) = \frac{\beta(\alpha - 1)(1 - s)^{\alpha - 2}}{\Gamma(\alpha)},$$

$$\beta G_1(1, s) = \frac{\beta(\alpha - 1)(1 - s)^{\alpha - 2}}{\Gamma(\alpha)} - \frac{\beta(1 - s)^{\alpha - 1}}{\Gamma(\alpha)}. \tag{24}$$

It follows from (24) that item 2 in the proof holds. □

Lemma 8 (see [10]). *The unique solution $u(t)$ of (1), (2) is nonnegative and satisfies*

$$\min_{\beta \leq t \leq 1} u(t) \geq \beta \|u\|. \tag{25}$$

Define the cone K by

$$K = \left\{ u \in C[0, 1] : u(t) \geq 0, \min_{\beta \leq t \leq 1} u(t) \geq \beta \|u\| \right\} \tag{26}$$

and the operator $T : K \to E$ by

$$Tu(t) = \int_0^1 G_1(t, s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) d\tau \right) ds$$

$$+ \frac{\gamma t}{1 - \gamma} \int_0^1 G_2(\eta, s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) d\tau \right) ds. \tag{27}$$

Remark 9. By Lemma 6, the problem (1)-(2) has a positive solution $u(t)$ if and only if u is a fixed point of T.

Lemma 10. *T is completely continuous and $T(K) \subseteq K$.*

Proof. By Lemma 8, $T(K) \subseteq K$. In view of the assumption of nonnegativeness and continuity of functions $G_i(x, y)$ with $i = 1, 2$ and $a(t) f(t, u(t))$, we conclude that $T : K \to K$ is continuous.

Let $\Omega \subseteq K$ be bounded; that is, there exists $M > 0$ such that $\|u\| \leq M$ for all $u \in \Omega$.

Let

$$L = \max_{0 \leq t \leq 1, 0 \leq u \leq M} |f(t, u)|. \tag{28}$$

Then from $u \in \Omega$ and from Lemmas 6 and 7, we have

$$|Tu(t)| = \left| \int_0^1 G_1(t, s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u) d\tau \right) ds \right.$$

$$+ \frac{\gamma t}{1 - \gamma} \int_0^1 G_2(\eta, s)$$

$$\left. \times \varphi_q \left(\int_0^s a(\tau) f(\tau, u) d\tau \right) ds \right|$$

$$\leq \int_0^1 G_1(1, s) \varphi_q \left(\int_0^1 a(\tau) L d\tau \right) ds$$

$$+ \frac{\gamma}{1 - \gamma} \int_0^1 G_2(\eta, s)$$

$$\times \varphi_q \left(\int_0^1 a(\tau) L d\tau \right) ds$$

$$\leq L^{q-1} \varphi_q \left(\int_0^1 a(\tau) d\tau \right)$$

$$\times \left[\int_0^1 \frac{(\alpha - 1)}{\Gamma(\alpha)} (1 - s)^{\alpha - 2} ds + \frac{\gamma}{1 - \gamma} \right.$$

$$\left. \times \int_0^1 \frac{(\alpha - 1)}{\Gamma(\alpha)} (1 - s)^{\alpha - 2} ds \right]$$

$$= \frac{(\alpha - 1) L^{q-1}}{(1 - \gamma) \Gamma(\alpha)} \varphi_q \left(\int_0^1 a(\tau) d\tau \right) \int_0^1 (1 - s)^{\alpha - 2} ds$$

$$= \frac{L^{q-1}}{(1 - \gamma) \Gamma(\alpha)} \varphi_q \left(\int_0^1 a(\tau) d\tau \right) \doteq l. \tag{29}$$

Hence, $T(\Omega)$ is bounded.

On the other hand, let $u \in \Omega$, $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$; Then

$$|Tu(t_2) - Tu(t_1)|$$

$$\leq L^{q-1} \left[\int_0^1 |G_1(t_2, s) - G_1(t_2, s)| \varphi_q \right.$$

$$\left. \times \left(\int_0^s a(\tau) f(\tau, u) d\tau \right) ds \right]$$

$$+ \frac{L^{q-1} \gamma |t_2 - t_1|}{1 - \gamma}$$

$$\times \int_0^1 G_2(\eta, s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u) d\tau \right) ds. \tag{30}$$

The continuity of G_1 implies that the right-side of the above inequality tends to zero if $t_2 \to t_1$. Therefore, T is completely continuous by Arzela-Ascoli theorem.

We introduce the notation

$$f_a := \liminf_{u \to a} \min_{\beta \leq t \leq 1} \frac{f(t, u)}{u^{p-1}}, \qquad f^b := \limsup_{u \to b} \max_{0 \leq t \leq 1} \frac{f(t, u)}{u^{p-1}}, \tag{31}$$

where $a, b = 0^+, \infty$. Consider the following:

$$\Lambda_1 = \left[\beta \left(\int_\beta^1 G_1(1,s) \varphi_q \left(\int_0^s a(\tau) d\tau \right) ds + \frac{\gamma}{1-\gamma} \right. \right.$$
$$\left. \left. \times \int_\beta^1 G_2(\eta, s) \varphi_q \left(\int_0^s a(\tau) d\tau \right) ds \right) \right]^{-1},$$

$$\Lambda_2 = \left[\frac{1}{(1-\gamma)\Gamma(\alpha)} \varphi_q \left(\int_0^1 a(\tau) d\tau \right) \right]^{-1}.$$

$$(32)$$

□

3. Single Solutions

In what follows, the number $\beta \in (0, 1)$.

Theorem 11. *Suppose that conditions (H1) and (H2) hold. Assume that f also satisfy*

(A_1) $f(t, u) \geq (mr)^{p-1}$, $\beta r \leq u \leq r$, $\beta \leq t \leq 1$,

(A_2) $f(t, u) \leq (MR)^{p-1}$, $0 \leq u \leq R$, $0 \leq t \leq 1$,

where $m \in (\Lambda_1, \infty)$ and $M \in (0, \Lambda_2)$. Then (1)-(2) has at least one positive solution u such that $r \leq \|u\| \leq R$.

Proof. Without loss of generality, we suppose that $r < R$. For any $u \in K$, we have

$$u(t) \geq \beta \|u\|, \quad \beta \leq t \leq 1. \tag{33}$$

We define two open subsets of E,

$$\Omega_1 = \{u \in K : \|u\| < r\}, \qquad \Omega_2 = \{u \in K : \|u\| < R\}. \tag{34}$$

For $u \in \partial\Omega_1$, by (33), we have

$$r = \|u\| \geq u(t) \geq \beta \|u\| = \beta r, \quad t \in [\beta, 1]. \tag{35}$$

For $t \in [\beta, 1]$, by (A_1), (27), and Lemma 7, we have

$$Tu(t) = \int_0^1 G_1(t,s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) d\tau \right) ds$$
$$+ \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) \varphi_q$$
$$\times \left(\int_0^s a(\tau) f(\tau, u(\tau)) d\tau \right) ds$$

$$\geq \int_\beta^1 \beta G_1(1,s) \varphi_q \left(\int_0^s a(\tau)(mr)^{p-1} d\tau \right) ds$$
$$+ \frac{\gamma\beta}{1-\gamma} \int_\beta^1 G_2(\eta, s) \varphi_q \left(\int_0^s a(\tau)(mr)^{p-1} d\tau \right) ds$$

$$= rm\beta \left[\int_\beta^1 G_1(1,s) \varphi_q \left(\int_0^s a(\tau) d\tau \right) ds \right.$$
$$\left. + \frac{\gamma}{1-\gamma} \int_\beta^1 G_2(\eta, s) \varphi_q \left(\int_0^s a(\tau) d\tau \right) ds \right]$$

$$> r = \|u\|.$$

$$(36)$$

Therefore,

$$\|Tu\| > \|u\|, \quad \forall u \in \partial\Omega_1 \cap K. \tag{37}$$

Then by Theorem 4,

$$i(T, \Omega_1, K) = 0. \tag{38}$$

On the other hand, as $u \in \partial\Omega_2$, we have

$$0 \leq u(t) \leq \|u\| = R, \tag{39}$$

thus, by (A_2), (27), Lemma 7, and (29), we have

$$Tu(t) = \int_0^1 G_1(t,s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) d\tau \right) ds$$
$$+ \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) d\tau \right) ds$$

$$\leq MR \left[\int_0^1 G_1(t,s) \varphi_q \left(\int_0^s a(\tau) d\tau \right) ds \right.$$
$$\left. + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) \varphi_q \left(\int_0^s a(\tau) d\tau \right) ds \right]$$

$$\leq MR \left[\int_0^1 G_1(t,s) \varphi_q \left(\int_0^1 a(\tau) d\tau \right) ds \right.$$
$$\left. + \frac{\gamma t}{1-\gamma} \int_0^1 G_2(\eta, s) \varphi_q \left(\int_0^1 a(\tau) d\tau \right) ds \right]$$

$$\leq \frac{MR}{(1-\gamma)\Gamma(\alpha)} \varphi_q \left(\int_0^1 a(\tau) d\tau \right) ds$$

$$< R = \|u\|.$$

$$(40)$$

Therefore,

$$\|Tu\| < \|u\|, \quad \forall u \in \partial\Omega_2 \cap K. \tag{41}$$

Then by Theorem 4,

$$i(T, \Omega_1, K) = 1. \tag{42}$$

By (38) and (42):

$$i\left(T, \Omega_2 \setminus \overline{\Omega_1}, K\right) = 1. \tag{43}$$

Then, T has a fixed point $u \in \Omega_2 \setminus \overline{\Omega_1}$, u is positive solution of problem (1)-(2), and $r < \|u\| < R$. □

Corollary 12. *Suppose that conditions (H1) and (H2) hold. Assume that f also satisfy*

(A_3) $f_\infty = \lambda \in ((2\Lambda_1/\beta)^{p-1}, \infty)$,

(A_4) $f^0 = \psi \in [0, (\Lambda_2/2)^{p-1})$.

Then (1)-(2) has at least one positive solution u such that $r \le \|u\| \le R$.

Proof. By (A_4), for $\varepsilon = (\Lambda_2/2)^{p-1} - \psi$, there exists a suitably small positive number H_1, as $0 < u \le H_1$ and $0 \le t \le 1$, such that

$$f(t, u) \le (\psi + \varepsilon) u^{p-1} \le \left(\frac{\Lambda_2}{2}\right)^{p-1} u^{p-1}$$
$$\le \left(\frac{\Lambda_2}{2} H_1\right)^{p-1}. \tag{44}$$

Let $R = H_1$ and $M = (\Lambda_2/2) \in (0, \Lambda_2)$, then, by (44), condition (A_2) holds.

By (A_3), for $\varepsilon = \lambda - (2\Lambda_1/\beta)^{p-1}$, there exists a sufficiently large $r \ne R$ such that

$$\frac{f(t, u)}{u^{p-1}} \ge \lambda - \varepsilon = \left(\frac{2\Lambda_1}{\beta}\right)^{p-1},$$
$$\beta r \le u < \infty, \quad \beta \le t \le 1. \tag{45}$$

Thus, when $\beta r \le u \le r$, one has

$$f(t, u) \ge \left(\frac{2\Lambda_1}{\beta}\right)^{p-1} u^{p-1} \ge (2\Lambda_1 r)^{p-1}. \tag{46}$$

Let $m = 2\Lambda_1 \in (\Lambda_1, \infty)$. Then, by (46), condition (A_1) holds.
Hence, from Theorem 11 the desired result hold. □

Corollary 13. *Suppose that conditions (H1) and (H2) hold. Assume that f also satisfy*

(A_5) $f_0 = \lambda \in ((2\Lambda_1/\beta)^{p-1}, \infty)$,

(A_6) $f^\infty = \psi \in [0, (\Lambda_2/2)^{p-1})$.

Then (1)-(2) has at least one positive solution u such that $r \le \|u\| \le R$.

Proof. By (A_5), for $\varepsilon = \lambda - (2\Lambda_1/\beta)^{p-1}$, there exists a sufficiently small $r > 0$, such that

$$\frac{f(t, u)}{u^{p-1}} \ge \lambda - \varepsilon = \left(\frac{2\Lambda_1}{\beta}\right)^{p-1}, \quad 0 < u < r, \ \beta \le t \le 1. \tag{47}$$

Thus, when $\beta r \le u \le r$, we have

$$f(t, u) \ge \left(\frac{2\Lambda_1}{\beta}\right)^{p-1} u^{p-1} \ge (2\Lambda_1 r)^{p-1}. \tag{48}$$

Let $m = 2\Lambda_1 \in (\Lambda_1, \infty)$. Then by (48), condition (A_1) holds.
By (A_6), for $\varepsilon = (\Lambda_2/2)^{p-1} - \psi$, there exist a sufficiently large $H_2 \ne r$ such that

$$\frac{f(t, u)}{u^{p-1}} \le \psi + \varepsilon = \left(\frac{\Lambda_2}{2}\right)^{p-1}, \quad H_2 \le u < \infty, \ 0 \le t \le 1. \tag{49}$$

We consider the following two cases.
(a) Suppose that $f(t, u)$ is unbounded, then we know from (H1) that there is a $R \ne r(> H_2)$ such that

$$f(t, u) \le f(t, R), \quad 0 \le u \le R, \ 0 \le t \le 1. \tag{50}$$

Since $R > H_2$, then from (49) and (50) we have

$$f(t, u) \le f(t, R) \le \left(\frac{\Lambda_2}{2} R\right)^{p-1},$$
$$H_2 \le u \le R, \quad 0 \le t \le 1. \tag{51}$$

Letting $M = (\Lambda_2/2) \in (0, \Lambda_2)$, we have

$$f(t, u) \le (MR)^{p-1}, \quad 0 \le u \le R, \ 0 \le t \le 1. \tag{52}$$

Thus, (A_2) holds.
(b) Suppose that $f(t, u)$ is bounded, say

$$f(t, u) \le L^{p-1}, \quad 0 \le u < \infty, \ 0 \le t \le 1. \tag{53}$$

In this case, taking sufficiently large $R \ge (2/\Lambda_2)L$, then letting $M = \Lambda_2/2 \in (0, \Lambda_2)$, we have

$$f(t, u) \le L^{p-1} \le \left(\frac{\Lambda_2}{2} R\right)^{p-1} \le (MR)^{p-1},$$
$$0 \le u \le R, \quad 0 \le t \le 1. \tag{54}$$

Thus, (A_2) holds.
Hence, from Theorem 11 the desired result holds. □

4. Multiple Solutions

To show the existence of multiple solutions we will use the Leggett-Williams fixed point theorem [11]. To this end define the following subsets of a cone K as

$$K_c = \{u \in K : \|u\| < c\},$$
$$K(\psi, b, d) = \{u \in K : b \le \psi(u), \ \|u\| \le d\}. \tag{55}$$

Definition 14. A map $\alpha : K \to [0, +\infty)$ is said to be a nonnegative continuous concave functional on a cone K of a real Banach space E if α is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y) \tag{56}$$

for all $x, y \in K$ and $t \in [0, 1]$.

Theorem 15 (see [11]). *Suppose $T : \overline{K_c} \to \overline{K_c}$ is completely continuous and suppose that there exists a concave positive functional ψ on K such that $\psi(u) \leq \|u\|$ for $u \in \overline{K_c}$. Suppose that there exist constants $0 < a < b < d \leq c$ such that*

(B1) $\{u \in K(\psi, b, d) : \psi(u) > b\} \neq \emptyset$ and $\psi(Tu) > b$ if $u \in K(\psi, b, d)$,

(B2) $\|Tu\| < a$ if $u \in \overline{K_a}$,

(B3) $\psi(Tu) > b$ for $u \in K(\psi, b, c)$ with $\|Tu\| > d$.

Then, T has at least three fixed points u_1, u_2, and u_3 such that $\|u_1\| < a$, $b < \psi(u_2)$ and $\|u_3\| > a$ with $\psi(u_3) < b$.

Theorem 16. *Suppose that there exist a, b, c with $0 < a < \beta b < b \leq c$ such that*

(C1) $f(t, u) < (a\Lambda_2)^{p-1}$, $(t, u) \in [0, 1] \times [0, a]$,

(C2) $f(t, u) > (b\Lambda_1)^{p-1}$, $(t, u) \in [\beta, 1] \times [\beta b, b]$,

(C3) $f(t, u) < (c\Lambda_2)^{p-1}$, $(t, u) \in [0, 1] \times [0, c]$.

Then (1)-(2) has at least three positive solutions.

Proof. By Lemma 10, $T : K \to K$ is completely continuous. Let

$$\psi(u) = \min_{\beta \leq t \leq 1} u(t). \tag{57}$$

It is obvious that ψ is a nonnegative continuous concave functional on K with $\psi(u) \leq \|u\|$, for $u \in \overline{K_c}$. Now we will show that the conditions of Theorem 15 are satisfied. For $u \in \overline{K_c}$, then $\|u\| \leq c$. For $t \in [0, 1]$ by (27), (29), and (C3), one has

$$
\begin{aligned}
Tu(t) &= \int_0^1 G_1(t, s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds + \frac{\gamma t}{1 - \gamma} \\
&\quad \times \int_0^1 G_2(\eta, s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u) \, d\tau \right) ds \\
&< \int_0^1 G_1(t, s) \varphi_q \left(\int_0^1 a(\tau) (c\Lambda_2)^{p-1} d\tau \right) ds + \frac{\gamma}{1 - \gamma} \\
&\quad \times \int_0^1 G_2(\eta, s) \varphi_q \left(\int_0^1 a(\tau) (c\Lambda_2)^{p-1} d\tau \right) ds \\
&< \frac{c\Lambda_2}{(1 - \gamma) \Gamma(\alpha)} \varphi_q \left(\int_0^1 a(\tau) \, d\tau \right) = c.
\end{aligned}
\tag{58}
$$

This implies $T : \overline{K_c} \to K_c$. By the same method, if $u \in \overline{K_a}$, then we can get $\|Tu\| < a$ and therefore (B2) is satisfied. Next, we assert that $\{u \in K(\psi, \beta b, b) : \psi(u) > \beta b\} \neq \phi$ and $\psi(Tu) > \beta b$ for all $u \in K(\psi, \beta b, b)$. In fact, the constant function

$$\frac{\beta b + b}{2} \in \{u \in K(\psi, \beta b, b) : \psi(u) > \beta b\}. \tag{59}$$

On the other hand, for $u \in K(\psi, \beta b, b)$, we have

$$\beta b \leq \psi(u) = \min_{\beta \leq t \leq 1} u(t) \leq u(t) \leq \|u\| \leq b, \quad \forall t \in [\beta, 1]. \tag{60}$$

Thus, in view of (27), Lemma 7, and (C2), one has

$$
\begin{aligned}
\psi(Tu) \\
= \min_{\beta \leq t \leq 1} &\left[\int_0^1 G_1(t, s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) \, d\tau \right) ds + \frac{\gamma t}{1 - \gamma} \right. \\
&\left. \times \int_0^1 G_2(\eta, s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) \, d\tau \right) ds \right] \\
\geq \int_\beta^1 &\beta G_1(1, s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) \, d\tau \right) ds \\
+ \frac{\beta \gamma}{1 - \gamma} &\int_\beta^1 G_2(\eta, s) \varphi_q \left(\int_0^s a(\tau) f(\tau, u(\tau)) \, d\tau \right) ds \\
> b\Lambda_1 \beta &\left[\int_\beta^1 G_1(1, s) \varphi_q \left(\int_0^s a(\tau) \, d\tau \right) ds \right. \\
&\left. + \frac{\gamma}{1 - \gamma} \int_\beta^1 G_2(\eta, s) \varphi_q \left(\int_0^s a(\tau) \, d\tau \right) ds \right] = b \\
> \beta b &
\end{aligned}
\tag{61}
$$

as required and therefore (B1) is satisfied.

Finally, we assert that if $u \in K(\psi, \beta b, c)$ with $\|Tu\| > b$ then $\psi(u) > \beta b$. To see this, suppose that $u \in K(\psi, \beta b, c)$ and $\|Tu\| > b$, then it follows from Lemma 10 that

$$\psi(Tu) = \min_{\beta \leq t \leq 1} (Tu)(t) \geq \beta \|Tu\| > \beta b. \tag{62}$$

Thus, (B3) is satisfied.

Therefore, by the conclusion of Theorem 15, the operator T has at least three fixed points. This implies that (1)-(2) has at least three solutions. $\square$

5. Examples

Example 1. Consider the boundary value problem with p-Laplacian:

$$\left(\varphi_p \left(D_{0^+}^{5/2} u(t) \right) \right)' + \frac{1}{4} t^{-1/2} \frac{a u^{1/2} e^{2u}}{b + e^u + e^{2u}} = 0,$$
$$0 < t < 1, \tag{63}$$

$$D_{0^+}^{5/2} u(0) = u(0) = u''(0) = 0, \quad u'(1) = \frac{8}{9} u'\left(\frac{1}{2} \right),$$

where $\alpha = 5/2$, $\eta = 1/2$, $\gamma = 8/9$, $\beta = 1/2$, $p = 3/2$, $q = 3$, $f(t, u) = a u^{1/2} e^{2u} / (b + e^u + e^{2u})$, $a(t) = (1/4) t^{-1/2}$, and $\varphi_3(\int_0^1 a(t) dt) = 1/4$, $\varphi_3(\int_0^s a(t) dt) = (1/4)s$.

Then

$$f^0 = \psi = \frac{a}{b + 2}, \qquad f_\infty = \lambda = a. \tag{64}$$

Next,

$$\Lambda_1 = \left[\frac{1}{2} \left(\int_{1/2}^1 G_1(1,s) \frac{1}{4} s \, ds + 8 \int_{1/2}^1 G_2\left(\frac{1}{2},s\right) \frac{1}{4} s \, ds \right) \right]^{-1}$$
$$= \left[\frac{9}{35} \sqrt{\frac{2}{\pi}} \right]^{-1} \approx 4.873999426.$$

(65)

Then, for $a = 5$ we have $f_\infty \in ((2\Lambda_1/\beta)^{p-1}, \infty)$, so condition (A_3) holds.

Now, Consider

$$\Lambda_2 = \frac{\sqrt{\pi}}{3} \approx 0.5908179503.$$

(66)

Then, for $a = 5$ and $b = 8$ we have $f^0 \in [0, (\Lambda_2/2)^{p-1})$, so condition (A_4) holds. Therefore, by Corollary 12, (63) has at least one positive solution.

Example 2. Consider the boundary value problem with p-Laplacian:

$$\left(\varphi_p \left(D_{0^+}^{5/2} u(t) \right) \right)' + \frac{1}{4} t^{-1/2} f(t,u) = 0, \quad 0 < t < 1,$$
$$D_{0^+}^{5/2} u(0) = u(0) = u''(0) = 0, \quad u'(1) = \frac{8}{9} u'\left(\frac{1}{2}\right),$$

(67)

where $\alpha = 5/2$, $\eta = 1/2$, $\gamma = 8/9$, $\beta = 1/2$, $p = 3/2$, $q = 3$, $a(t) = (1/4)t^{-1/2}$, $\varphi_3(\int_0^1 a(t)dt) = 1/4$, and $\varphi_3(\int_0^s a(t)dt) = (1/4)s$.

Let

$$f(t,u) = \begin{cases} \dfrac{t}{40} + 14u^2, & u \leq 1 \\ 13 + \dfrac{t}{40} + u^{1/4}, & u > 1. \end{cases}$$

(68)

By Example 1, we have $\Lambda_1 \approx 4.873999426$ and $\Lambda_2 \approx 0.5908179503$.

Choosing $a = 1/14$, $b = 18$, and $c = 1296$, then

$$f(t,u) = \frac{t}{40} + 14u^2 \approx 0.096\ldots < (a\Lambda_2)^{1/2} \approx 0.20\ldots,$$
$$(t,u) \in [0,1] \times \left[0, \frac{1}{14}\right],$$
$$f(t,u) = \frac{t}{40} + 13 + u^{1/4} \approx 14.7\ldots > (9\Lambda_1)^{1/2} \approx 9.3\ldots,$$
$$(t,u) \in \left[\frac{1}{2},1\right] \times [9,18],$$
$$f(t,u) = \frac{t}{40} + 13 + u^{1/4} \approx 19\ldots < (1296\Lambda_2)^{1/2} \approx 27\ldots,$$
$$(t,u) \in [0,1] \times [0,1296].$$

(69)

Then the conditions $(C1)$–$(C3)$ are satisfied. Therefore, it follows from Theorem 16 that (67) has at least three positive solutions u_1, u_2, and u_3 such that

$$\|u_1\| < \frac{1}{14}, \qquad 18 < \psi(u_2), \qquad \|u_3\| > \frac{1}{14}$$

(70)

with $\psi(u_3) < 18$.

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