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Research Article

On the Stability of Wave Equation

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We prove the generalized Hyers-Ulam stability of the wave equation, $\Delta u = (1/c^2)u_{tt}$, in a class of twice continuously differentiable functions under some conditions.

1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms.

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

The case of approximately additive functions was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. Indeed, he proved that each solution of the inequality $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, $f(x+y) = f(x) + f(y)$, is said to have the Hyers-Ulam stability.

Rassias [3] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon\left(\|x\|^p + \|y\|^p\right) \quad (1)$$

and proved the Hyers' theorem. That is, Rassias proved the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability) of the Cauchy additive functional equation. Since then, the stability of several functional equations has been extensively investigated [4–10].

The terminologies, the generalized Hyers-Ulam stability and the Hyers-Ulam stability, can also be applied to the case of other functional equations, of differential equations, and of various integral equations.

Given a real number $c > 0$, the partial differential equation

$$\Delta u(x,t) - \frac{1}{c^2}u_{tt}(x,t) = 0 \quad (2)$$

is called the wave equation, where $u_{tt}(x,t)$ and $\Delta u(x,t)$ denote the second time derivative and the Laplacian of $u(x,t)$, respectively.

For an integer $n \ge 2$, assume that U and T are open (connected) subsets of $\mathbb{R}^n$ and $\mathbb{R}$, respectively. Let $\varphi : U \times T \to [0,\infty)$ be a function. If, for each twice continuously differentiable function $u : U \times T \to \mathbb{R}$ satisfying

$$\left|\Delta u(x,t) - \frac{1}{c^2}u_{tt}(x,t)\right| \le \varphi(x,t) \quad (x \in U, t \in T), \quad (3)$$

there exist a solution $u_0 : U \times T \to \mathbb{R}$ of the wave equation (2) and a function $\Phi : U \times T \to [0,\infty)$ such that

$$\left|u(x,t) - u_0(x,t)\right| \le \Phi(x,t) \quad (x \in U, t \in T), \quad (4)$$

where $\Phi(x,t)$ is independent of $u(x,t)$ and $u_0(x,t)$, then we say that the wave equation (2) has the generalized Hyers-Ulam stability (or the Hyers-Ulam-Rassias stability).

In this paper, using ideas from [11, 12], we prove the generalized Hyers-Ulam stability of the wave equation (2).

2. Main Results

For a given integer $n \geq 2$, x_i denotes the ith coordinate of any point x in $\mathbb{R}^n$; that is $x = (x_1, \ldots, x_i, \ldots, x_n)$, and $|x|$ denotes the Euclidean distance between x and the origin; that is,

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{5}$$

Given a real number $c > 0$, assume that real numbers a and t_2 satisfy $a > c$ and $0 < t_2 < \infty$, and define

$$T := (0, t_2), \qquad U := \{x \in \mathbb{R}^n : |x| > at_2\}, \qquad R := (a, \infty). \tag{6}$$

We remark that $(x,t) \in U \times T$ if and only if $|x|/t \in R$. Using an idea from [11], we define a class W of all twice continuously differentiable functions $u : U \times T \to \mathbb{R}$ with the properties

(i) $u(x,t) = tv(|x|/t)$ for all $x \in U$ and $t \in T$ and for some $v : R \to \mathbb{R}$;

(ii) $\lim_{|x| \to at_2} \lim_{t \to t_2} u(x,t) = 0$.

If we define

$$(u_1 + u_2)(x,t) = u_1(x,t) + u_2(x,t), \qquad (\lambda u_1)(x,t) = \lambda u_1(x,t), \tag{7}$$

for all $u_1, u_2 \in W$ and $\lambda \in \mathbb{R}$, then W is a vector space over real numbers. That is, W is a large class such that it is a vector space.

Theorem 1. *Let a function $\varphi : U \times T \to [0,\infty)$ be given such that there exists a positive real number s with*

$$s := \sup_{x \in U, t \in T} t\varphi(x,t). \tag{8}$$

If a $u \in W$ satisfies the inequality

$$\left| \Delta u(x,t) - \frac{1}{c^2} u_{tt}(x,t) \right| \leq \varphi(x,t), \tag{9}$$

for all $x \in U$ and $t \in T$, then there exists a solution $u_0 : U \times T \to \mathbb{R}$ of the wave equation (2) *which belongs to W and satisfies*

$$\left| u(x,t) - u_0(x,t) \right| \leq t \int_a^{|x|/t} \left(\frac{a^2}{z^2} \cdot \frac{z^2 - c^2}{a^2 - c^2} \right)^{(n-1)/2} \int_z^\infty \frac{c^2 s}{q^2 - c^2} \, dq \, dz, \tag{10}$$

for all $x \in U$ and $t \in T$.

Proof. Let $v : \mathbb{R} \to \mathbb{R}$ be a function which satisfies

$$u(x,t) = tv\left(\frac{|x|}{t}\right), \tag{11}$$

for all $x \in U$ and $t \in T$. For any $i \in \{1, 2, \ldots, n\}$, we differentiate $u(x,t)$ with respect to x_i to get

$$u_{x_i}(x,t) = \frac{x_i}{|x|} v'\left(\frac{|x|}{t}\right). \tag{12}$$

Similarly, we obtain the second partial derivative of $u(x,t)$ with respect to x_i as follows:

$$u_{x_i x_i}(x,t) = \left(\frac{1}{|x|} - \frac{x_i^2}{|x|^3} \right) v'\left(\frac{|x|}{t}\right) + \frac{1}{t} \frac{x_i^2}{|x|^2} v''\left(\frac{|x|}{t}\right). \tag{13}$$

Hence, we have

$$\Delta u(x,t) = \sum_{i=1}^n u_{x_i x_i}(x,t) = \frac{n-1}{t} \frac{t}{|x|} v'\left(\frac{|x|}{t}\right) + \frac{1}{t} v''\left(\frac{|x|}{t}\right). \tag{14}$$

By a similar way, we further get the second derivative of $u(x,t)$ with respect to t as follows:

$$u_{tt}(x,t) = \frac{1}{t} \frac{|x|^2}{t^2} v''\left(\frac{|x|}{t}\right). \tag{15}$$

Therefore, it follows from (14) and (15) that

$$\begin{aligned} \Delta u(x,t) - \frac{1}{c^2} u_{tt}(x,t) &= \frac{n-1}{t} \frac{t}{|x|} v'\left(\frac{|x|}{t}\right) \\ &\quad + \frac{1}{t} v''\left(\frac{|x|}{t}\right) - \frac{1}{c^2 t} \frac{|x|^2}{t^2} v''\left(\frac{|x|}{t}\right) \\ &= \frac{n-1}{t} \frac{1}{r} v'(r) + \left(\frac{1}{t} - \frac{1}{c^2 t} r^2 \right) v''(r) \\ &= \frac{1}{t} \left(1 - \frac{r^2}{c^2} \right) \left(v''(r) + \frac{n-1}{r} \frac{c^2}{c^2 - r^2} v'(r) \right), \end{aligned} \tag{16}$$

for any $x \in U$, $t \in T$, and $r := |x|/t \in R$, and it follows from (8) and (9) that

$$\left| v''(r) + \frac{n-1}{r} \frac{c^2}{c^2 - r^2} v'(r) \right| \leq \frac{c^2}{r^2 - c^2} t\varphi(x,t) \leq \frac{c^2 s}{r^2 - c^2} \tag{17}$$

or

$$\left| w'(r) + \frac{n-1}{r} \frac{c^2}{c^2 - r^2} w(r) \right| \leq \frac{c^2 s}{r^2 - c^2}, \tag{18}$$

for all $r \in R$, where we set $w(r) := v'(r)$.

Set

$$g(r) := \frac{n-1}{r} \frac{c^2}{c^2 - r^2}, \qquad h(r) := 0, \qquad \phi(r) := \frac{c^2 s}{r^2 - c^2}, \tag{19}$$

for each $r \in R$. Then we have

$$\begin{gathered}
\int_a^r g(p)\,dp = \ln\left(\frac{r^2}{a^2}\cdot\frac{a^2-c^2}{r^2-c^2}\right)^{(n-1)/2}, \\
\int_a^\infty \phi(r)\exp\left\{\Re\left(\int_a^r g(p)\,dp\right)\right\}dr \\
= \int_a^\infty \left(\frac{r^2}{a^2}\cdot\frac{a^2-c^2}{r^2-c^2}\right)^{(n-1)/2}\frac{c^2 s}{r^2-c^2}dr \\
< \int_a^\infty \frac{c^2 s}{r^2-c^2}dr < \infty.
\end{gathered} \tag{20}$$

According to (18) and [13, Theorem 1], there exists a unique real number α such that

$$\begin{aligned}
&\left| w(r) - \alpha\left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-1)/2}\right| \\
&\quad\le \left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-1)/2} \\
&\quad\quad\times \int_r^\infty \frac{c^2 s}{q^2-c^2}\left(\frac{q^2}{a^2}\cdot\frac{a^2-c^2}{q^2-c^2}\right)^{(n-1)/2}dq \\
&\quad\le \left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-1)/2}\int_r^\infty \frac{c^2 s}{q^2-c^2}dq
\end{aligned} \tag{21}$$

or

$$\begin{aligned}
&\left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-1)/2}\left(\alpha - \int_r^\infty \frac{c^2 s}{q^2-c^2}dq\right) \\
&\quad\le v'(r) \\
&\quad\le \left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-1)/2}\left(\alpha + \int_r^\infty \frac{c^2 s}{q^2-c^2}dq\right),
\end{aligned} \tag{22}$$

for all $r \in R$.

Hence, it follows from the last inequalities that

$$\begin{aligned}
&\int_a^r \left(\frac{a^2}{z^2}\cdot\frac{z^2-c^2}{a^2-c^2}\right)^{(n-1)/2}\left(\alpha - \int_z^\infty \frac{c^2 s}{q^2-c^2}dq\right)dz \\
&\quad\le v(r) - \lim_{z\to a^+} v(z) \\
&\quad\le \int_a^r \left(\frac{a^2}{z^2}\cdot\frac{z^2-c^2}{a^2-c^2}\right)^{(n-1)/2}\left(\alpha + \int_z^\infty \frac{c^2 s}{q^2-c^2}dq\right)dz,
\end{aligned} \tag{23}$$

for any $r \in R$.

Due to (ii), it holds that $\lim_{z\to a^+} v(z) = 0$. Replacing r with $|x|/t$ in the last inequalities, we get

$$\begin{aligned}
&\left| u(x,t) - \alpha t\int_a^{|x|/t}\left(\frac{a^2}{z^2}\cdot\frac{z^2-c^2}{a^2-c^2}\right)^{(n-1)/2}dz\right| \\
&\quad\le t\int_a^{|x|/t}\left(\frac{a^2}{z^2}\cdot\frac{z^2-c^2}{a^2-c^2}\right)^{(n-1)/2}\int_z^\infty \frac{c^2 s}{q^2-c^2}dq\,dz,
\end{aligned} \tag{24}$$

for all $x \in U$ and $t \in T$.

If we define a function $u_0 : U \times T \to \mathbb{R}$ by

$$u_0(x,t) := \alpha t\int_a^{|x|/t}\left(\frac{a^2}{z^2}\cdot\frac{z^2-c^2}{a^2-c^2}\right)^{(n-1)/2}dz, \tag{25}$$

then we have

$$\begin{aligned}
\frac{\partial}{\partial x_i}u_0(x,t) &= \frac{\alpha x_i}{|x|}\left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-1)/2}, \\
\frac{\partial^2}{\partial x_i^2}u_0(x,t) &= \alpha\left(\frac{1}{|x|}-\frac{x_i^2}{|x|^3}\right)\left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-1)/2} \\
&\quad + \frac{(n-1)\alpha a^2 c^2 x_i^2}{(a^2-c^2)tr^3|x|^2}\left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-3)/2}, \\
\Delta u_0(x,t) &= \frac{(n-1)\alpha a^2}{(a^2-c^2)|x|}\left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-3)/2}, \\
\frac{\partial}{\partial t}u_0(x,t) &= \alpha\int_a^{|x|/t}\left(\frac{a^2}{z^2}\cdot\frac{z^2-c^2}{a^2-c^2}\right)^{(n-1)/2}dz \\
&\quad - \frac{\alpha|x|}{t}\left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-1)/2}, \\
\frac{\partial^2}{\partial t^2}u_0(x,t) &= \frac{(n-1)\alpha a^2 c^2}{(a^2-c^2)|x|}\left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-3)/2},
\end{aligned} \tag{26}$$

for all $x \in U$ and $t \in T$, which implies that $u_0(x,t)$ is a solution of the wave equation (2).

It is now to show that $u_0 \in W$. Let $F : R \to \mathbb{R}$ be a function with the property

$$F(r) := \int\left(\frac{a^2}{r^2}\cdot\frac{r^2-c^2}{a^2-c^2}\right)^{(n-1)/2}dr. \tag{27}$$

Then we have

$$u_0(x,t) = \alpha t\left(F\left(\frac{|x|}{t}\right) - F(a)\right), \tag{28}$$

which implies that $u_0(x,t)$ can be expressed as $tv(|x|/t)$, where $v(r) = \alpha F(r) - \alpha F(a)$. Moreover, we get

$$\begin{aligned}
&\lim_{|x|\to at_2}\lim_{t\to t_2} u_0(x,t) \\
&\quad = \lim_{|x|/t\to a^+}\alpha t\int_a^{|x|/t}\left(\frac{a^2}{z^2}\cdot\frac{z^2-c^2}{a^2-c^2}\right)^{(n-1)/2}dz = 0,
\end{aligned} \tag{29}$$

which verifies that $u_0 \in W$. Finally, by (24), the inequality (10) holds true. □

Assume now that b and t_1 are given real numbers satisfying $0 < b < c$ and $0 < t_1 < \infty$. We then set

$$T' := (t_1, \infty), \qquad U' := \{x \in \mathbb{R}^n : 0 < |x| < bt_1\}, \qquad R' := (0, b) \tag{30}$$

and define a class W' of all twice continuously differentiable functions $u : U' \times T' \to \mathbb{R}$ with the properties

(iii) $u(x,t) = tv(|x|/t)$ for all $x \in U'$ and $t \in T'$ and for some $v : R' \to \mathbb{R}$;

(iv) $\lim_{|x| \to bt_1} \lim_{t \to t_1} u(x,t) = 0$.

It might be remarked that $(x,t) \in U' \times T'$ if and only if $|x|/t \in R'$. If we define

$$\left(u_1 + u_2\right)(x,t) = u_1(x,t) + u_2(x,t), \qquad \left(\lambda u_1\right)(x,t) = \lambda u_1(x,t), \tag{31}$$

for all $u_1, u_2 \in W'$ and $\lambda \in \mathbb{R}$, then W' is a vector space over real numbers.

Theorem 2. *Let a function $\varphi : U' \times T' \to [0, \infty)$ be given such that there exists a positive real number s' with*

$$s' := \sup_{x \in U', t \in T'} t\varphi(x,t). \tag{32}$$

If a $u \in W'$ satisfies the inequality

$$\left|\Delta u(x,t) - \frac{1}{c^2} u_{tt}(x,t)\right| \le \varphi(x,t), \tag{33}$$

for all $x \in U'$ and $t \in T'$, then there exists a solution $u_0 : U' \times T' \to \mathbb{R}$ of the wave equation (2) *which belongs to W' and satisfies*

$$\left|u(x,t) - u_0(x,t)\right| \le t \int_{|x|/t}^{b} \left(\frac{b^2}{z^2} \cdot \frac{c^2 - z^2}{c^2 - b^2}\right)^{(n-1)/2} \int_0^z \frac{c^2 s'}{c^2 - q^2} dq\, dz \tag{34}$$

for all $x \in U'$ and $t \in T'$.

Proof. If $v : \mathbb{R} \to \mathbb{R}$ is given by (11), then we can simply follow the lines in the first part of the proof of Theorem 1 to obtain

$$\left|w'(r) + \frac{n-1}{r} \frac{c^2}{c^2 - r^2} w(r)\right| \le \frac{c^2 s'}{c^2 - r^2}, \tag{35}$$

for all $r \in R'$, where $w(r) := v'(r)$.

Set

$$g(r) := \frac{n-1}{r} \frac{c^2}{c^2 - r^2}, \qquad h(r) := 0, \qquad \phi(r) := \frac{c^2 s'}{c^2 - r^2}, \tag{36}$$

for any $r \in R'$. Then we get

$$\begin{aligned} &\int_r^b g(p)\, dp = \ln\left(\frac{b^2}{r^2} \cdot \frac{c^2 - r^2}{c^2 - b^2}\right)^{(n-1)/2}, \\ &\int_0^b \phi(r) \exp\left\{\Re\left(\int_b^r g(p)\, dp\right)\right\} dr \\ &\quad = \int_0^b \left(\frac{r^2}{b^2} \cdot \frac{c^2 - b^2}{c^2 - r^2}\right)^{(n-1)/2} \frac{c^2 s'}{c^2 - r^2} dr \\ &\quad < \int_0^b \frac{c^2 s'}{c^2 - r^2} dr < \infty. \end{aligned} \tag{37}$$

According to (35) and [13, Corollary 2], there exists a unique real number α such that

$$\begin{aligned} &\left|w(r) - \alpha\left(\frac{b^2}{r^2} \cdot \frac{c^2 - r^2}{c^2 - b^2}\right)^{(n-1)/2}\right| \\ &\quad \le \left(\frac{b^2}{r^2} \cdot \frac{c^2 - r^2}{c^2 - b^2}\right)^{(n-1)/2} \int_0^r \frac{c^2 s'}{c^2 - q^2} dq \end{aligned} \tag{38}$$

or

$$\begin{aligned} &\left(\frac{b^2}{r^2} \cdot \frac{c^2 - r^2}{c^2 - b^2}\right)^{(n-1)/2} \left(\alpha - \int_0^r \frac{c^2 s'}{c^2 - q^2} dq\right) \\ &\quad \le v'(r) \\ &\quad \le \left(\frac{b^2}{r^2} \cdot \frac{c^2 - r^2}{c^2 - b^2}\right)^{(n-1)/2} \left(\alpha + \int_0^r \frac{c^2 s'}{c^2 - q^2} dq\right), \end{aligned} \tag{39}$$

for all $r \in R'$.

From the last inequalities, it follows that

$$\begin{aligned} &\int_r^b \left(\frac{b^2}{z^2} \cdot \frac{c^2 - z^2}{c^2 - b^2}\right)^{(n-1)/2} \left(\alpha - \int_0^z \frac{c^2 s'}{c^2 - q^2} dq\right) dz \\ &\quad \le \lim_{z \to b^-} v(z) - v(r) \\ &\quad \le \int_r^b \left(\frac{b^2}{z^2} \cdot \frac{c^2 - z^2}{c^2 - b^2}\right)^{(n-1)/2} \left(\alpha + \int_0^z \frac{c^2 s'}{c^2 - q^2} dq\right) dz, \end{aligned} \tag{40}$$

for each $r \in R'$.

On account of (iv), we have $\lim_{z \to b^-} v(z) = 0$. Replacing r with $|x|/t$ in the last inequalities, we obtain

$$\begin{aligned} &\left|u(x,t) - \alpha t \int_b^{|x|/t} \left(\frac{b^2}{z^2} \cdot \frac{c^2 - z^2}{c^2 - b^2}\right)^{(n-1)/2} dz\right| \\ &\quad \le t \int_{|x|/t}^b \left(\frac{b^2}{z^2} \cdot \frac{c^2 - z^2}{c^2 - b^2}\right)^{(n-1)/2} \int_0^z \frac{c^2 s'}{c^2 - q^2} dq\, dz, \end{aligned} \tag{41}$$

for all $x \in U'$ and $t \in T'$.

Let us define a function $u_0 : U' \times T' \to \mathbb{R}$ by

$$u_0(x,t) := \alpha t \int_b^{|x|/t} \left(\frac{b^2}{z^2} \cdot \frac{c^2 - z^2}{c^2 - b^2} \right)^{(n-1)/2} dz. \tag{42}$$

Then, a similar argument to the last part of the proof of Theorem 1 shows that $u_0(x,t)$ is a solution of the wave equation (2) and it belongs to W'. Finally, the validity of (34) immediately follows from (41). □

3. Remarks

Remark 1. The inequality (10) in Theorem 1 can be rewritten as

$$\begin{aligned}
&\left| u(x,t) - u_0(x,t) \right| \\
&\quad \le t \int_a^{|x|/t} \left(\frac{a^2}{z^2} \cdot \frac{z^2 - c^2}{a^2 - c^2} \right)^{(n-1)/2} \int_z^{\infty} \frac{c^2 s}{q^2 - c^2} dq\, dz \\
&\quad \le t \int_a^{|x|/t} \left(\frac{a^2}{z^2} \cdot \frac{z^2 - c^2}{a^2 - c^2} \right)^{(n-1)/2} \frac{cs}{2} \left(\ln \frac{a+c}{a-c} \right) dz \\
&\quad \le \frac{cst}{2} \left(\frac{a^2}{a^2 - c^2} \right)^{(n-1)/2} \left(\ln \frac{a+c}{a-c} \right) \\
&\qquad \times \int_a^{|x|/t} \left(1 - \frac{c^2}{z^2} \right)^{(n-1)/2} dz,
\end{aligned} \tag{43}$$

for all $x \in U$ and $t \in T$. If we further substitute $\sin\theta$ for c/z in the previous inequality, then we obtain

$$\begin{aligned}
&\left| u(x,t) - u_0(x,t) \right| \\
&\quad \le \frac{cst}{2} \left(\frac{a^2}{a^2 - c^2} \right)^{(n-1)/2} \left(\ln \frac{a+c}{a-c} \right) \\
&\qquad \times \int_{\sin^{-1}(c/a)}^{\sin^{-1}(ct/|x|)} \cos^{n-1}\theta \left(-c \frac{\cos\theta}{\sin^2\theta} \right) d\theta \\
&\quad = -\frac{c^2 st}{2} \left(\frac{a^2}{a^2 - c^2} \right)^{(n-1)/2} \left(\ln \frac{a+c}{a-c} \right) \\
&\qquad \times \left[-\frac{\cos^{n-1}\theta}{\sin\theta} \right]_{\sin^{-1}(c/a)}^{\sin^{-1}(ct/|x|)} + \frac{(n-1)c^2 st}{2} \\
&\qquad \times \left(\frac{a^2}{a^2 - c^2} \right)^{(n-1)/2} \left(\ln \frac{a+c}{a-c} \right) \\
&\qquad \times \int_{\sin^{-1}(c/a)}^{\sin^{-1}(ct/|x|)} \cos^{n-2}\theta d\theta \\
&\quad = -\frac{c^2 st}{2} \left(\frac{a^2}{a^2 - c^2} \right)^{(n-1)/2} \left(\ln \frac{a+c}{a-c} \right) \\
&\qquad \times \left[\frac{a}{c} \left(1 - \left(\frac{c}{a} \right)^2 \right)^{(n-1)/2} - \frac{|x|}{ct} \left(1 - \left(\frac{ct}{|x|} \right)^2 \right)^{(n-1)/2} \right] \\
&\qquad + \frac{(n-1)c^2 st}{2} \left(\frac{a^2}{a^2 - c^2} \right)^{(n-1)/2} \\
&\qquad \times \left(\ln \frac{a+c}{a-c} \right) \int_{\sin^{-1}(c/a)}^{\sin^{-1}(ct/|x|)} \cos^{n-2}\theta\, d\theta,
\end{aligned} \tag{44}$$

for any $x \in U$ and $t \in T$.

For the case of $n = 3$, the inequality (10) can be rewritten as

$$\begin{aligned}
&\left| u(x,t) - u_0(x,t) \right| \\
&\quad \le \frac{c^2 st}{2} \frac{a^2}{a^2 - c^2} \left(\ln \frac{a+c}{a-c} \right) \\
&\qquad \times \left(\frac{ct}{|x|} + \frac{|x|}{ct} - \frac{c}{a} - \frac{a}{c} \right),
\end{aligned} \tag{45}$$

for all $x \in U$ and $t \in T$.

Remark 2. As in Remark 1, the inequality (34) in Theorem 2 can be rewritten as

$$\begin{aligned}
&\left| u(x,t) - u_0(x,t) \right| \\
&\quad \le \frac{cs't}{2} \left(\frac{b^2}{c^2 - b^2} \right)^{(n-1)/2} \\
&\qquad \times \left(\ln \frac{c+b}{c-b} \right) \int_{|x|/t}^{b} \left(\frac{c^2}{z^2} - 1 \right)^{(n-1)/2} dz,
\end{aligned} \tag{46}$$

for all $x \in U'$ and $t \in T'$. If we substitute $c\cos\theta$ for z in the previous inequality, then we get

$$\begin{aligned}
&\left| u(x,t) - u_0(x,t) \right| \\
&\quad \le -\frac{cs't}{2} \left(\frac{b^2}{c^2 - b^2} \right)^{(n-1)/2} \left(\ln \frac{c+b}{c-b} \right) \\
&\qquad \times \left[\frac{1}{t} \frac{\left(c^2 t^2 - |x|^2 \right)^{(n-1)/2}}{|x|^{n-2}} - \frac{\left(c^2 - b^2 \right)^{(n-1)/2}}{b^{n-2}} \right] \\
&\qquad - \frac{(n-1)c^2 s't}{2} \left(\frac{b^2}{c^2 - b^2} \right)^{(n-1)/2} \\
&\qquad \times \left(\ln \frac{c+b}{c-b} \right) \int_{\cos^{-1}(|x|/ct)}^{\cos^{-1}(b/c)} \frac{\sin^{n-2}\theta}{\cos^{n-1}\theta} d\theta,
\end{aligned} \tag{47}$$

for any $x \in U'$ and $t \in T'$.

For the case of $n = 3$, the inequality (34) can be rewritten as

$$\left|u(x,t) - u_0(x,t)\right| \le \frac{c^2 s' t}{2} \frac{b^2}{c^2 - b^2} \left(\ln \frac{c+b}{c-b}\right) \left(\frac{ct}{|x|} + \frac{|x|}{ct} - \frac{b}{c} - \frac{c}{b}\right), \tag{48}$$

for all $x \in U'$ and $t \in T'$.

Remark 3. It is an open problem whether the wave equation (2) has the generalized Hyers-Ulam stability for the case of either $T = (0, t_2)$ and $U = \{x \in \mathbb{R}^n : 0 < |x| < at_2\}$ or $T = (t_1, \infty)$ and $U = \{x \in \mathbb{R}^n : |x| > bt_1\}$ or $T = (0, \infty)$ and $U = \{x \in \mathbb{R}^n : |x| > 0\}$.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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