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Research Article

Existence and Decay Estimate of Global Solutions to Systems of Nonlinear Wave Equations with Damping and Source Terms

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The initial-boundary value problem for a class of nonlinear wave equations system in bounded domain is studied. The existence of global solutions for this problem is proved by constructing a stable set and obtain the asymptotic stability of global solutions through the use of a difference inequality.

1. Introduction

In this paper, we are concerned with the global solvability and decay stabilization for the following nonlinear wave equations system:

$$u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u_t|^{q-2}u_t - \Delta u_t = |v|^{r+2}|u|^r u, \quad (x,t) \in \Omega \times R^+, \tag{1}$$

$$v_{tt} - \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) + |v_t|^{q-2}v_t - \Delta v_t = |u|^{r+2}|v|^r v, \quad (x,t) \in \Omega \times R^+ \tag{2}$$

with the initial-boundary value conditions

$$u(x,0) = u_0(x) \in W_0^{1,p}(\Omega), \qquad u_t(x,0) = u_1(x) \in L^2(\Omega) \quad x \in \Omega, \tag{3}$$

$$v(x,0) = v_0(x) \in W_0^{1,p}(\Omega), \qquad v_t(x,0) = v_1(x) \in L^2(\Omega) \quad x \in \Omega, \tag{4}$$

$$u(x,t) = 0, \quad v(x,t) = 0, \quad (x,t) \in \partial\Omega \times R^+, \tag{5}$$

where Ω is a bounded open domain in R^n with a smooth boundary $\partial\Omega$, $p, q \geq 2$, $r > 0$ and $p < 2(r+2) \leq np/(n-p)$ for $n \geq p$ and $p < 2(r+2) < +\infty$ for $n < p$.

When $p = 2$, Medeiros and Miranda [1] proved the existence and uniqueness of global weak solutions. Cavalcanti et al. in [2–4] considered the asymptotic behavior for wave equation and an analogous hyperbolic-parabolic system with boundary damping and boundary source term. In paper [5, 6], the authors dealt with the existence, uniform decay rates, and blowup for solutions of systems of nonlinear wave equations with damping and source terms.

Rammaha and Wilstein [7] and Yang [8] are concerned with the initial boundary value problem for a class of quasilinear evolution equations with nonlinear damping and source terms. Under appropriate conditions, by a Galerkin approximation scheme combined with the potential well method, they proved the existence and asymptotic behavior of global weak solutions when $m < p$, where $m \geq 0$ and p are, respectively, the growth orders of the nonlinear strain terms and the source term.

Ono [9] considers the following initial-boundary value problem for nonlinear wave equations with nonlinear dissipative terms:

$$\begin{gathered} u_{tt} - \Delta u + \delta_1 u_t + \delta_2 |u_t|^\beta u_t - \delta_3 \Delta u_t = |u|^\alpha u, \quad (x,t) \in \Omega \times R^+, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ u(x,t) = 0, \quad x \in \partial\Omega, \ t \geq 0, \end{gathered} \tag{6}$$

where $\delta_i \geq 0$, $i = 1, 2, 3$, and $\alpha, \beta > 0$ are constants. The author mainly investigates on the blowup phenomenon to problem (6). On the other hand, in the case of $\delta_1 + \delta_2 + \delta_3 > 0$, he shows that the problem (6) admits a unique global solution, and its energy has some decay properties under some assumptions on u_0 and initial energy $E(0) \equiv E(u_0, u_1)$. In particular, when $\delta_2 > 0$ and $\delta_1 + \delta_3 > 0$ in (6), the energy $E(t) \equiv E(u(t), u_t(t))$ has some polynomial and exponential decay rates, respectively.

For the following strongly damped nonlinear wave equation

$$u_{tt} - \Delta u_t - \Delta u + f(u_t) + g(u) = h, \tag{7}$$

Dell'Oro and Pata [10] obtain the long-time behavior of the related solution semigroup, which is shown to possess the global attractor in the natural weak energy space. In addition, the existence of global and local solutions, decay estimates, and blowup for solutions of nonlinear wave equation with source and damping terms and exponential nonlinearities are studied in [11–14].

In this paper, we prove the global existence for the problem (1)–(5) by applying the potential well theory introduced by Sattinger [15] and Payne and Sattinger [16]. Meanwhile, we obtain the asymptotic stabilization of global solutions by using a difference inequality [17].

For simplicity of notations, hereafter we denote by $\|\cdot\|_p$ the norm of $L^p(\Omega)$; $\|\cdot\|$ denotes $L^2(\Omega)$ norm, and we write equivalent norm $\|\cdot\nabla\|_p$ instead of $W_0^{1,p}(\Omega)$ norm $\|\cdot\|_{W_0^{1,p}(\Omega)}$. Moreover, C denotes various positive constants depending on the known constants and may be different at each appearance.

2. Local Existence

In this section, we investigate the local existence and uniqueness of the solutions of the problem (1)–(5). For this purpose, we list up two useful lemmas which will be used later and give the definition of weak solutions.

Lemma 1. *Let $u \in W_0^{1,p}(\Omega)$, then $u \in L^s(\Omega)$; and the inequality $\|u\|_s \leq C\|u\|_{W_0^{1,p}(\Omega)}$ holds with a constant $C > 0$ depending on Ω, p, and s, provided that $2 \leq s < +\infty$, $2 \leq n \leq p$ and $2 \leq s \leq np/(n-p)$, $2 < p < n$.*

Lemma 2 (Young inequality). *Let $a, b \geq 0$ and $1/p + 1/q = 1$ for $1 < p, q < +\infty$; then one has the inequality*

$$ab \leq \delta a^p + C(\delta) b^q, \tag{8}$$

where $\delta > 0$ is an arbitrary constant, and $C(\delta)$ is a positive constant depending on δ.

Definition 3. A pair of functions (u, v) is said to be a weak solution of (1)–(5) on $[0, T]$ if $u, v \in C([0, T], W_0^{1,p}(\Omega))$, $u_t, v_t \in C([0, T], L^2(\Omega))$, $[u(0), v(0)] = [u_0, v_0] \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, $[u_t(0), v_t(0)] = [u_1, v_1] \in L^2(\Omega) \times L^2(\Omega)$, and $[u, v]$ satisfies

$$\begin{aligned}
&\langle u_t(t), \phi\rangle_{L^2(\Omega)} - \langle u_1, \phi\rangle_{L^2(\Omega)} \\
&\quad + \int_0^t \left\langle \left(|\nabla u|^{p-2}\nabla u\right), \nabla\phi\right\rangle_{L^2(\Omega)} d\tau \\
&\quad + \int_0^t \left\langle |u_t|^{q-2} u_t, \phi\right\rangle_{L^2(\Omega)} d\tau + \int_0^t \langle \nabla u_t, \nabla\phi\rangle_{L^2(\Omega)} \\
&= \int_0^t \left\langle |v|^{r+2}|u|^r u, \phi\right\rangle_{L^2(\Omega)} d\tau, \\
&\langle v_t(t), \psi\rangle_{L^2(\Omega)} - \langle v_1, \psi\rangle_{L^2(\Omega)} \\
&\quad + \int_0^t \left\langle \left(|\nabla v|^{p-2}\nabla v\right), \nabla\psi\right\rangle_{L^2(\Omega)} d\tau \\
&\quad + \int_0^t \left\langle |v_t|^{q-2} v_t, \psi\right\rangle_{L^2(\Omega)} d\tau + \int_0^t \langle \nabla v_t, \nabla\psi\rangle_{L^2(\Omega)} \\
&= \int_0^t \left\langle |u|^{r+2}|v|^r v, \psi\right\rangle_{L^2(\Omega)} d\tau,
\end{aligned} \tag{9}$$

for all test functions $\phi, \psi \in W_0^{1,p}(\Omega)$ and for almost all $t \in [0, T]$.

The local existence and uniqueness of solutions for problem (1)–(5) can be proved through the use of Galerkin method. The result reads as follows.

Theorem 4 (local solution). *Supposed that $[u_0, v_0] \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, $[u_1, v_1] \in L^2(\Omega) \times L^2(\Omega)$, and $p < 2(r+2) \leq np/(n-p)$ if $n \geq p$ and $p < 2(r+2) < +\infty$ for $n < p$, then there exists $T > 0$ such that the problem (1)–(5) has a unique local solution $[u(t), v(t)]$ satisfying*

$$\begin{aligned}
&[u, v] \in L^\infty\left([0, T);\ W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)\right); \\
&[u_t, v_t] \in L^\infty\left([0, T);\ L^2(\Omega) \times L^2(\Omega)\right),
\end{aligned} \tag{10}$$

$$\begin{aligned}
E(t) &+ \int_0^t \left(\|\nabla u_\tau(\tau)\|^2 + \|\nabla v_\tau(\tau)\|^2\right. \\
&\left. + \|u(\tau)\|_q^q + \|v(\tau)\|_q^q\right) d\tau = E(0),
\end{aligned} \tag{11}$$

where

$$\begin{aligned}
E(t) = &\frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{1}{p}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) \\
&- \frac{1}{r+2}\|uv\|_{r+2}^{r+2}.
\end{aligned} \tag{12}$$

Proof. Let $\{\omega_i\}_{i=1}^\infty$ be a basis for $W_0^{1,p}(\Omega)$. Supposed that V_k is the subspace of $W_0^{1,p}(\Omega)$ generated by $\{\omega_1, \omega_2, \ldots, \omega_k\}$, $k \in N$. We are going to look for the approximate solution

$$u_k(t) = \sum_{i=1}^k g_{ik}(t)\,\omega_i, \qquad v_k(t) = \sum_{i=1}^k h_{ik}(t)\,\omega_i \tag{13}$$

which satisfies the following Cauchy problem:

$$\int_\Omega \left(u_k'' - \operatorname{div}\left(|\nabla u_k|^{p-2}\nabla u_k\right) + |u_k'|^{q-2}u_k' - \Delta u_k'\right)\omega_i dx = \int_\Omega |v_k|^{r+2}|u_k|^r u_k \omega_i dx, \tag{14}$$

$$\int_\Omega \left(v_k'' - \operatorname{div}\left(|\nabla v_k|^{p-2}\nabla v_k\right) + |v_k'|^{q-2}v_k' - \Delta v_k'\right)\omega_i dx = \int_\Omega |u_k|^{r+2}|v_k|^r v_k \omega_i dx, \tag{15}$$

$$u_k(0) = u_{0k} = \sum_{i=1}^{k}(u_0,\omega_i)\,\omega_i \longrightarrow u_0, \quad \text{in } W_0^{1,p}(\Omega), \quad k \longrightarrow \infty, \tag{16}$$

$$v_k(0) = v_{0k} = \sum_{i=1}^{k}(v_0,\omega_i)\,\omega_i \longrightarrow v_0 \quad \text{in } W_0^{1,p}(\Omega), \quad k \longrightarrow \infty, \tag{17}$$

$$u_k'(0) = u_{1k} = \sum_{i=1}^{k}(u_1,\omega_i)\,\omega_i \longrightarrow u_1 \quad \text{in } L^2(\Omega), \quad k \longrightarrow \infty, \tag{18}$$

$$v_k'(0) = v_{1k} = \sum_{i=1}^{k}(v_1,\omega_i)\,\omega_i \longrightarrow v_1 \quad \text{in } L^2(\Omega), \quad k \longrightarrow \infty. \tag{19}$$

Note that, we can solve the problem (14)–(19) by a Picard's iteration method in ordinary differential equations. Hence, there exists a solution in $[0,T_k)$ for some $T_k > 0$, and we can extend this solution to the whole interval $[0,T]$ for any given $T > 0$ by making use of the a priori estimates below.

Multiplying (14) by $g_{ik}'(t)$ and (15) by $h_{ik}'(t)$ and summing over i from 1 to k, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u_k'(t)\|^2 + \|\nabla u_k\|_p^p\right) + \|u_k'(t)\|_q^q + \|\nabla u_k'(t)\|^2 = \int_\Omega |v_k|^{r+2}|u_k|^r u_k u_k'\,dx, \tag{20}$$

$$\frac{1}{2}\frac{d}{dt}\left(\|v_k'(t)\|^2 + \|\nabla v_k\|_p^p\right) + \|v_k'(t)\|_q^q + \|\nabla v_k'(t)\|^2 = \int_\Omega |u_k|^{r+2}|v_k|^r v_k v_k'\,dx. \tag{21}$$

By summing (20) and (21) and integrating the resulting identity over $[0,t]$, we have

$$\begin{aligned}
&\frac{1}{2}\left(\|u_k'(t)\|^2 + \|v_k'(t)\|^2 + \|\nabla u_k\|_p^p + \|\nabla v_k\|_p^p\right) \\
&\quad + \int_0^t \left(\|\nabla u_k'(t)\|^2 + \|\nabla v_k'(t)\|^2 + \|u_k'(\tau)\|_q^q + \|v_k'(\tau)\|_q^q\right) d\tau \\
&\le C_0 + \int_0^t\int_\Omega \left(|v_k|^{r+2}|u_k|^r u_k u_k' + |u_k|^{r+2}|v_k|^r v_k v_k'\right) dx\,d\tau.
\end{aligned} \tag{22}$$

We estimate the right-hand terms of (22) as follows: we get from Hölder inequality and Lemmas 1 and 2 that

$$\begin{aligned}
&\left|\int_0^t\int_\Omega \left(|v_k|^{r+2}|u_k|^r u_k u_k' + |u_k|^{r+2}|v_k|^r v_k v_k'\right) dx\,d\tau\right| \\
&\le \int_0^t \left(\|u_k'(\tau)\|^2 + \|v_k'(\tau)\|^2\right) d\tau + \int_0^t\int_\Omega |u_k v_k|^{2(r+1)}\left(|u_k|^2 + |v_k|^2\right) dx\,d\tau \\
&\le \int_0^t \left(\|u_k'(\tau)\|^2 + \|v_k'(\tau)\|^2\right) d\tau + C\int_0^t \left(\|u_k\|_{2(r+2)}^{2(r+2)} + \|v_k\|_{2(r+2)}^{2(r+2)}\right) d\tau \\
&\le C\int_0^t \left(\|u_k'(\tau)\|^2 + \|v_k'(\tau)\|^2 + \|\nabla u_k\|_p^{2(r+2)} + \|\nabla v_k\|_p^{2(r+2)}\right) d\tau \\
&\le C\int_0^t \left(\|u_k'(\tau)\|^2 + \|v_k'(\tau)\|^2 + \|\nabla u_k\|_p^p + \|\nabla v_k\|_p^p\right)^{2(r+2)/p} d\tau.
\end{aligned} \tag{23}$$

It follows from (22) and (23) that

$$\begin{aligned}
&\|u_k'(t)\|^2 + \|v_k'(t)\|^2 + \|\nabla u_k\|_p^p + \|\nabla v_k\|_p^p \\
&\quad + 2\int_0^t \left(\|u_k'(\tau)\|_q^q \,\| + \|v_k'(\tau)\|_q^q + \|\nabla u_k'(t)\|^2 + \|\nabla v_k'(t)\|^2\right) d\tau \\
&\le 2C_0 + C\int_0^t \left(\|u_k'(\tau)\|^2 + \|v_k'(\tau)\|^2 + \|\nabla u_k\|_p^p + \|\nabla v_k\|_p^p\right)^{2(r+2)/p} d\tau,
\end{aligned} \tag{24}$$

which implies that

$$\begin{aligned}
&\|u_k'(t)\|^2 + \|v_k'(t)\|^2 + \|\nabla u_k\|_p^p + \|\nabla v_k\|_p^p \\
&\le 2C_0 + C\int_0^t \left(\|u_k'(\tau)\|^2 + \|v_k'(\tau)\|^2 + \|\nabla u_k\|_p^p + \|\nabla v_k\|_p^p\right)^{2(r+2)/p} d\tau.
\end{aligned} \tag{25}$$

We get from (25) and Gronwall type inequality that

$$\|u_k'(t)\|^2 + \|v_k'(t)\|^2 + \|\nabla u_k\|_p^p + \|\nabla v_k\|_p^p \le \left[2C_0 - \frac{2(r+2)-p}{p}Ct\right]^{-p/(2(r+2)-p)}. \tag{26}$$

Thus, we deduce from (26) that there exists a time $T > 0$ such that

$$\left\|u_k'(t)\right\|^2 + \left\|v_k'(t)\right\|^2 + \|\nabla u_k\|_p^p + \|\nabla v_k\|_p^p \le C_1, \quad \forall t \in [0, T], \tag{27}$$

where C_1 is a positive constant independent of k.

We have from (24) and (26) that

$$2\int_0^t \left(\left\|u_k'(\tau)\right\|_q^q + \left\|v_k'(\tau)\right\|_q^q + \left\|\nabla u_k'(\tau)\right\|^2 + \left\|\nabla v_k'(\tau)\right\|^2\right) d\tau \le C_2, \quad \forall t \in [0, T]. \tag{28}$$

It follows from (27) and (28) that

$$\begin{gathered} \left\|u_k'(t)\right\|^2 \le C_1, \qquad \left\|v_k'(t)\right\|^2 \le C_1, \\ \|\nabla u_k\|_p^p \le C_1, \qquad \|\nabla v_k\|_p^p \le C_1. \\ u_k'(t) \text{ and } v_k'(t) \text{ are bounded in } L^2\left([0,T];\ L^q(\Omega)\right) \\ \text{and } L^2\left([0,T];\ H_0^1(\Omega)\right). \end{gathered} \tag{29}$$

Using the same process as the proof of Theorem 2.1 in paper [18], we derive that $[u(t), v(t)]$ is a local solution of the problem (1)–(5). By (20) and (21), we conclude that (11) is valid. □

3. Global Existence

In order to state our main results, we first introduce the following functionals:

$$J([u,v]) = \frac{1}{p}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) - \frac{1}{r+2}\|uv\|_{r+2}^{r+2}, \tag{30}$$

$$K([u,v]) = \left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) - 2\|uv\|_{r+2}^{r+2} \tag{31}$$

for $[u,v] \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$.

We put that

$$d = \inf\left\{\sup_{\lambda \ge 0} J(\lambda[u,v]) : [u,v] \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) / \{[0,0]\}\right\}. \tag{32}$$

Then, we are able to define the stable set as follows for problem (1)–(5):

$$W = \left\{[u,v] \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \mid K([u,v]) > 0, \ J([u,v]) < d\right\} \cup \{[0,0]\}. \tag{33}$$

We denote the total energy related to (1) and (2) by (12), and

$$E(0) = \frac{1}{2}\left(\|u_1\|^2 + \|v_1\|^2\right) + \frac{1}{p}\left(\|\nabla u_0\|_p^p + \|\nabla v_0\|_p^p\right) - \frac{1}{r+2}\|u_0 v_0\|_{r+2}^{r+2} \tag{34}$$

is the total energy of the initial data.

Lemma 5. *Let $[u,v]$ be a solution to problem* (1)–(5)*; then, $E(t)$ is a nonincreasing function for $t > 0$ and*

$$\frac{d}{dt}E(t) = -\left(\|u_t\|_q^q + \|v_t\|_q^q + \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2\right). \tag{35}$$

We have from (11) that $E(t)$ is the primitive of an integrable function. Therefore, $E(t)$ is absolutely continuous, and equality (35) is satisfied.

Lemma 6. *Supposed that $[u,v] \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$, and $p < 2(r+2) \le np/(n-p)$ if $n \ge p$; $p < 2(r+2) < +\infty$ if $n < p$, then $d > 0$.*

Proof. Since

$$J(\lambda[u,v]) = \frac{\lambda^p}{p}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) - \frac{\lambda^{2(r+2)}}{r+2}\|uv\|_{r+2}^{r+2}, \tag{36}$$

so we get

$$\frac{d}{d\lambda}J(\lambda[u,v]) = \lambda^{p-1}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) - 2\lambda^{2r+3}\|uv\|_{r+2}^{r+2}. \tag{37}$$

In case $uv \ne 0$, let $(d/d\lambda)J(\lambda[u,v]) = 0$, which implies that

$$\lambda_1 = \left(\frac{\|\nabla u\|_p^p + \|\nabla v\|_p^p}{2\|uv\|_{r+2}^{r+2}}\right)^{1/(2r-p+4)}. \tag{38}$$

As $\lambda = \lambda_1$, an elementary calculation shows that $(d^2/d\lambda^2)J(\lambda[u,v])|_{\lambda=\lambda_1} < 0$. Therefore, we have that

$$\begin{aligned} \sup_{\lambda \ge 0} J(\lambda[u,v]) &= J(\lambda_1[u,v]) \\ &= \frac{2(r+2)-p}{2p(r+2)}\left(\frac{\|\nabla u\|_p^p + \|\nabla v\|_p^p}{2^{p/(2r+4)}\|uv\|_{r+2}^{p/2}}\right)^{(2r+4)/(2r-p+4)}. \end{aligned} \tag{39}$$

It follows from Hölder inequality and Lemma 1 that

$$\begin{aligned} \|uv\|_{r+2}^{p/2} &\le \|u\|_{2(r+2)}^{p/2}\|v\|_{2(r+2)}^{p/2} \\ &\le \frac{1}{2}\left(\|u\|_{2(r+2)}^p + \|v\|_{2(r+2)}^p\right) \\ &\le C\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right). \end{aligned} \tag{40}$$

We get from (39) and (40) that

$$\sup_{\lambda \ge 0} J(\lambda[u,v]) \ge \frac{2(r+2)-p}{2p(r+2)}\left(2^{p/(2r+4)}C\right)^{-(2r+4)/(2r-p+4)} > 0. \tag{41}$$

In case $uv = 0$ and $u = 0$ or $v = 0$, then

$$J(\lambda[u,v]) = \frac{\lambda^p}{p}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right). \tag{42}$$

Therefore, we have

$$J(\lambda[u,v]) = +\infty. \tag{43}$$

We conclude from (41) and (43) that

$$d \geq \frac{2(r+2)-p}{2p(r+2)}\left(2^{p/(2r+4)}C\right)^{-(2r+4)/(2r-p+4)} > 0. \tag{44}$$

Thus, we complete the proof of Lemma 6. □

Lemma 7. *Supposed that $p < 2(r+2) \leq np/(n-p)$ for $n \geq p$ and $p < 2(r+2) < +\infty$ for $n < p$, if $[u_0, v_0] \in W$, $[u_1, v_1] \in L^2(\Omega) \times L^2(\Omega)$ and $E(0) < d$, then $[u, v] \in W$ for $\forall t \in [0, T)$.*

Proof. Assume that there exists a number $t^* \in (0, T)$ such that $[u(t), v(t)] \in W$ on $[0, t^*)$ and $u(t^*) \notin W$. Then, in virtue of the continuity of $u(t)$, we see $u(t^*) \in \partial W$, where ∂W denotes the boundary of domain W. From the definition of W and the continuity of $J([u(t), v(t)])$ and $K([u(t), v(t)])$ in t, we have either

$$J([u(t^*), v(t^*)]) = d \tag{45}$$

or

$$K([u(t^*), v(t^*)]) = 0. \tag{46}$$

It follows from (12) and (30) that

$$J([u(t^*), v(t^*)]) \leq E(t^*) \leq E(0) < d. \tag{47}$$

So, case (45) is impossible.

Assume that (46) holds; then, we get that

$$\begin{aligned} &\frac{d}{d\lambda}J(\lambda[u(t^*), v(t^*)]) \\ &\quad = \lambda^{p-1}\left(1-\lambda^{2r-p+4}\right)\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right). \end{aligned} \tag{48}$$

We obtain from $(d/d\lambda)J(\lambda[u(t^*), v(t^*)]) = 0$ that $\lambda = 1$. Since

$$\begin{aligned} &\left.\frac{d^2}{d\lambda^2}J(\lambda[u(t^*), v(t^*)])\right|_{\lambda=1} \\ &\quad = -\left[(2(r+2)-p) + (2r+3)\right] < 0. \end{aligned} \tag{49}$$

Consequently, we get from (47) that

$$\sup_{\lambda\geq 0} J(\lambda[u(t^*), v(t^*)]) = J([u(t^*), v(t^*)]) < d, \tag{50}$$

which contradicts the definition of d. Hence, case (46) is impossible as well. Thus we conclude that $[u(t), v(t)] \in W$ on $[0, T)$. □

Theorem 8 (global solution). *Supposed that $p < 2(r+2) \leq np/(n-p)$ as $n \geq p$ and $p < 2(r+2) < +\infty$ as $n < p$, and $[u(t), v(t)]$ is a local solution of problem (1)–(5) on $[0, T)$. If $[u_0, v_0] \in W$, $[u_1, v_1] \in L^2(\Omega) \times L^2(\Omega)$ and $E(0) < d$, then $[u(t), v(t)]$ is a global solution of problem (1)–(5).*

Proof. It suffices to show that $\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|_p^p + \|\nabla v\|_p^p$ is bounded uniformly with respect to t. Under the hypotheses in Theorem 8, we get from Lemma 7 that $[u, v] \in W$ on $[0, T)$. So the following formula holds on $[0, T)$:

$$\begin{aligned} J([u,v]) &= \frac{1}{p}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) - \frac{1}{r+2}\|uv\|_{r+2}^{r+2} \\ &\geq \frac{2(r+2)-p}{2p(r+2)}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right). \end{aligned} \tag{51}$$

We have from (51) that

$$\begin{aligned} &\frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{2(r+2)-p}{2p(r+2)}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) \\ &\quad \leq \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + J([u(t), v(t)]) \\ &\quad = E(t) \leq E(0) < d. \end{aligned} \tag{52}$$

Hence, we get

$$\begin{aligned} &\left(\|u_t\|^2 + \|v_t\|^2\right) + \left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) \\ &\quad \leq \max\left(2, \frac{2p(r+2)}{2(r+2)-p}\right)d < +\infty. \end{aligned} \tag{53}$$

The above inequality and the continuation principle lead to the global existence of the solution $[u, v]$ for problem (1)–(5). □

4. Asymptotic Behavior of Global Solutions

The following lemma plays an important role in studying the decay estimate of global solutions for the problem (1)–(5).

Lemma 9 (see [9]). *Suppose that $\varphi(t)$ is a nonincreasing nonnegative function on $[0, +\infty)$ and satisfies*

$$\varphi(t)^{r+1} \leq k(\varphi(t) - \varphi(t+1)), \quad \forall t \geq 0. \tag{54}$$

Then, $\varphi(t)$ has the decay property

$$\varphi(t) \leq \left[\frac{r}{k}(t-1) + M^{-r}\right]^{-1/r}, \quad \forall t \geq 1, \tag{55}$$

where $k, r > 0$ are constants and $M = \max_{t\in[0,1]}\varphi(t)$.

Lemma 10. *Under the assumptions of Theorem 8, if initial value $[u_0, v_0] \in W$ and $[u_1, v_1] \in L^2(\Omega)\times L^2(\Omega)$ are sufficiently small such that*

$$C^{2(r+2)}\left(\frac{2p(r+2)}{2p(r+2)-p}E(0)\right)^{(2(r+2)-p)/p} < 1, \tag{56}$$

then

$$\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) \leq \frac{1}{\theta}K([u,v]), \tag{57}$$

where $\theta = 1 - C^{2(r+2)}((2p(r+2)/(2p(r+2)-p))E(0))^{(2(r+2)-p)/p} > 0$ is a positive constant and C is the optimal Sobolev's constant from $W_0^{1,p}(\Omega)$ to $L^{2(r+2)}(\Omega)$.

Proof. We have from Lemma 1 and (52) that

$$
\begin{aligned}
2\|uv\|_{r+2}^{r+2} &\le 2\|u\|_{2(r+2)}^{r+2}\|v\|_{2(r+2)}^{r+2} \\
&\le \|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)} \\
&\le C^{2(r+2)}\left(\|\nabla u\|_p^{2(r+2)} + \|\nabla v\|_p^{2(r+2)}\right) \\
&\le C^{2(r+2)}\left(\|\nabla u\|_p^{2(r+2)-p}\|\nabla u\|_p^p\right. \\
&\qquad \left. + \|\nabla v\|_p^{2(r+2)-p}\|\nabla v\|_p^p\right) \\
&\le C^{2(r+2)}\left(\frac{2p(r+2)}{2p(r+2)-p}E(0)\right)^{(2(r+2)-p)/p} \\
&\qquad \times \left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right).
\end{aligned} \tag{58}
$$

Therefore, we get from (58) and (31) that

$$
\begin{aligned}
&\left[1 - C^{2(r+2)}\left(\frac{2p(r+2)}{2p(r+2)-p}E(0)\right)^{(2(r+2)-p)/p}\right] \\
&\quad \times \left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) \le K([u,v]).
\end{aligned} \tag{59}
$$

Let

$$
\theta = 1 - C^{2(r+2)}\left(\frac{2p(r+2)}{2p(r+2)-p}E(0)\right)^{(2(r+2)-p)/p} > 0; \tag{60}
$$

then, we have from (59) that

$$
\|\nabla u\|_p^p + \|\nabla v\|_p^p \le \frac{1}{\theta}K([u,v]). \tag{61}
$$

□

Theorem 11. *Under the assumptions of Theorem 8, if $p < q < r+2$ and (56) hold, then the global solution $[u,v]$ in W of the problem (1)–(5) has the following decay property:*

$$
E(t) \le \left[\frac{p-2}{pC}(t-1) + M^{(p+q-pq)/p}\right]^{p/(p+q-pq)}, \quad \forall t > 1, \tag{62}
$$

where $M = \max_{t\in[0,1]}E(t) > 0$ is some constant depending only on $[u_0, v_0]$ and $[u_1, v_1]$.

Proof. Multiplying (1) by u_t and (2) by v_t and integrating over $\Omega \times [t, t+1]$, and summing up together, we get

$$
\begin{aligned}
&\int_t^{t+1}\left(\|u_t(s)\|_q^q + \|v_t(s)\|_q^q + \|\nabla u_t(s)\|_2^2\right. \\
&\qquad \left. + \|\nabla v_t(s)\|_2^2\right) ds = E(t) - E(t+1).
\end{aligned} \tag{63}
$$

Thus, there exists $t_1 \in [t, t+1/4]$, $t_2 \in [t+3/4, t+1]$ such that

$$
\begin{aligned}
&4\left(\|u_t(t_i)\|_q^q + \|v_t(t_i)\|_q^q + \|\nabla u_t(t_i)\|_2^2 + \|\nabla v_t(t_i)\|_2^2\right) \\
&\quad = E(t) - E(t+1), \quad i = 1, 2.
\end{aligned} \tag{64}
$$

On the other hand, we multiply (1) by u and (2) by v and integrate over $\Omega \times [t_1, t_2]$. Adding them together, we obtain

$$
\begin{aligned}
\int_{t_1}^{t_2} K([u,v])\,ds &= \int_{t_1}^{t_2}\|u_t\|^2 ds + \int_{t_1}^{t_2}\|v_t\|^2 ds \\
&\quad + (u_t(t_1), u(t_1)) - (u_t(t_2), u(t_2)) \\
&\quad + (v_t(t_1), v(t_2)) - (v_t(t_2)\, v(t_2)) \\
&\quad - \left(\int_{t_1}^{t_2}\int_\Omega |u_t|^{q-2}u_t u\,dx\,ds\right. \\
&\qquad \left. + \int_{t_1}^{t_2}\int_\Omega |v_t|^{q-2}v_t v\,dx\,ds\right) \\
&\quad - \int_{t_1}^{t_2}\int_\Omega \nabla u_t \nabla u\,dx\,ds - \int_{t_1}^{t_2}\int_\Omega \nabla v_t \nabla v\,dx\,ds.
\end{aligned} \tag{65}
$$

From (63), Sobolev inequality, and Hölder inequality, we have

$$
\begin{aligned}
\int_{t_1}^{t_2}\|u_t\|^2 ds &\le C\int_{t_1}^{t_2}\|\nabla u_t\|^2 ds \le C(E(t) - E(t+1)), \\
\int_{t_1}^{t_2}\|v_t\|^2 ds &\le C\int_{t_1}^{t_2}\|\nabla v_t\|^2 ds \le C(E(t) - E(t+1)).
\end{aligned} \tag{66}
$$

We get from (52), (64), and Lemmas 1 and 2 that

$$
\begin{aligned}
|u_t(t_i), u(t_i)| &\le \|u_t(t_i)\| \cdot \|u(t_i)\| \le C\|\nabla u_t(t_i)\| \cdot \|\nabla u(t_i)\|_p \\
&\le C(E(t) - E(t+1))^{1/2}\sup_{t\le s\le t+1}E(s)^{1/p} \\
&\le C(\varepsilon)(E(t) - E(t+1))^{p/2(p-1)} \\
&\quad + \varepsilon\sup_{t\le s\le t+1}E(s), \quad i = 1, 2, \\
|(v_t(t_i), v(t_i))| &\le \|v_t(t_i)\| \cdot \|v(t_i)\| \le C\|\nabla v_t(t_i)\| \cdot \|\nabla v(t_i)\|_p \\
&\le C(E(t) - E(t+1))^{1/2}\sup_{t\le s\le t+1}E(s)^{1/p} \\
&\le C(\varepsilon)(E(t) - E(t+1))^{p/2(p-1)} \\
&\quad + \varepsilon\sup_{t\le s\le t+1}E(s), \quad i = 1, 2.
\end{aligned} \tag{67}
$$

From Hölder inequality and Lemma 2,we get

$$
\begin{aligned}
\left|\int_{t_1}^{t_2}\int_\Omega |u_t|^{q-2}u_t u\,dx\,ds\right| &\le \int_{t_1}^{t_2}\|u_t\|_q^{q-1}\|u\|_q ds \\
&\le \left(\int_{t_1}^{t_2}\|u_t\|_q^q ds\right)^{(q-1)/q}\left(\int_{t_1}^{t_2}\|u\|_q^q ds\right)^{1/q} \\
&\le C(\varepsilon)\int_{t_1}^{t_2}\|u_t\|_q^q ds + \varepsilon\int_{t_1}^{t_2}\|u\|_q^q ds,
\end{aligned} \tag{68}
$$

$$\left|\int_{t_1}^{t_2}\int_\Omega |v_t|^{q-2}v_t v\,dx\,ds\right| \le \int_{t_1}^{t_2}\|v_t\|_q^{q-1}\|v\|_q ds$$
$$\le \left(\int_{t_1}^{t_2}\|v_t\|_q^q ds\right)^{(q-1)/q}\left(\int_{t_1}^{t_2}\|v\|_q^q ds\right)^{1/q}$$
$$\le C(\varepsilon)\int_{t_1}^{t_2}\|v_t\|_q^q ds + \varepsilon\int_{t_1}^{t_2}\|v\|_q^q ds. \tag{69}$$

Since $p < q < r+2$ and the property of the function $f(x) = \alpha^x/x$, $\alpha \ge 0$, $x > 0$, we obtain

$$\frac{\|u\|_q^q}{q} \le C\frac{\|u\|_p^p}{p} + C\frac{\|u\|_{r+2}^{r+2}}{r+2}, \qquad \frac{\|v\|_q^q}{q} \le C\frac{\|v\|_p^p}{p} + C\frac{\|v\|_{r+2}^{r+2}}{r+2}. \tag{70}$$

We conclude from (69), (70), $[u,v] \in W$, and Lemma 1 that

$$\|u\|_q^q + \|v\|_q^q \le C\left(\|u\|_p^p + \|u\|_{r+2}^{r+2} + \|v\|_p^p + \|v\|_{r+2}^{r+2}\right)$$
$$\le C\left(\|u\|_p^p + \|\nabla u\|_p^p + \|v\|_p^p + \|\nabla v\|_p^p\right) \tag{71}$$
$$\le C\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) \le CE(t).$$

It follows from (63), (68), (69), and (71) that

$$\left|-\left(\int_{t_1}^{t_2}\int_\Omega |u_t|^{q-2}u_t u\,dx\,ds + \int_{t_1}^{t_2}\int_\Omega |v_t|^{q-2}v_t v\,dx\,ds\right)\right|$$
$$\le C(\varepsilon)(E(t) - E(t+1)) + \varepsilon C\int_{t_1}^{t_2} E(s)\,ds, \tag{72}$$

and we obtain from (63), Sobolev inequality, Hölder inequality, and Lemma 2 that

$$\left|-\int_{t_1}^{t_2}\int_\Omega \nabla u_t \nabla u ds\right| \le \int_{t_1}^{t_2}\|\nabla u_t\|\cdot\|\nabla u\|\,ds$$
$$\le \left(\int_{t_1}^{t_2}\|\nabla u_t\|^2 ds\right)^{1/2}\left(\int_{t_1}^{t_2}\|\nabla u\|^2 ds\right)^{1/2}$$
$$\le C(E(t) - E(t+1))^{1/2}\left(\int_{t_1}^{t_2}\|\nabla u\|_p^2 ds\right)^{1/2}$$
$$\le C(E(t) - E(t+1))^{1/2}\left(\int_{t_1}^{t_2}\|\nabla u\|_p^p ds\right)^{1/p}$$
$$\le C(E(t) - E(t+1))^{p/2(p-1)} + \varepsilon\int_{t_1}^{t_2}\|\nabla u\|_p^p ds. \tag{73}$$

Similarly, we have the following formula:

$$\left|-\int_{t_1}^{t_2}\int_\Omega \nabla v_t \nabla v ds\right| \le \int_{t_1}^{t_2}\|\nabla v_t\|\cdot\|\nabla v\|\,ds$$
$$\le \left(\int_{t_1}^{t_2}\|\nabla v_t\|^2 ds\right)^{1/2}\left(\int_{t_1}^{t_2}\|\nabla v\|^2 ds\right)^{1/2}$$
$$\le C(E(t) - E(t+1))^{1/2}\left(\int_{t_1}^{t_2}\|\nabla v\|_p^2 ds\right)^{1/2}$$
$$\le C(E(t) - E(t+1))^{1/2}\left(\int_{t_1}^{t_2}\|\nabla v\|_p^p ds\right)^{1/p}$$
$$\le C(E(t) - E(t+1))^{p/2(p-1)} + \varepsilon\int_{t_1}^{t_2}\|\nabla v\|_p^p ds. \tag{74}$$

We get from (57), (73), and (74) that

$$\left|\int_{t_1}^{t_2}\int_\Omega (\nabla u_t\nabla u + \nabla v_t \nabla v)\,ds\right|$$
$$\le C(E(t) - E(t+1))^{p/2(p-1)} + \varepsilon\int_{t_1}^{t_2}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right)ds$$
$$\le C(E(t) - E(t+1))^{p/2(p-1)} + \frac{\varepsilon}{\theta}\int_{t_1}^{t_2} K([u,v])\,ds. \tag{75}$$

Choosing small enough ε, we have from (65), (66), (67), (72), and (75) that

$$\int_{t_1}^{t_2} K([u,v])\,ds \le C\left[(E(t) - E(t+1)) + (E(t) - E(t+1))^{p/2(p-1)}\right] \tag{76}$$
$$+ \varepsilon \sup_{t\le s\le t+1} E(s) + \varepsilon\int_{t_1}^{t_2} E(s)\,ds.$$

It follows from (30) and (31) that

$$J([u,v]) = \frac{2(r+2)-p}{2p(r+2)}\left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) + \frac{1}{2(r+2)}K([u,v]). \tag{77}$$

On the other hand, from (12) and using (57) and (77), we deduce that

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + J([u,v])$$
$$= \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{2(r+2)-p}{2p(r+2)} \times \left(\|\nabla u\|_p^p + \|\nabla v\|_p^p\right) + \frac{1}{2(r+2)}K([u,v])$$
$$\le \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) + \left(\frac{2(r+2)-p}{2\theta p(r+2)} + \frac{1}{2(r+2)}\right) \times K([u,v]). \tag{78}$$

By integrating (78) over $[t_1, t_2]$, we obtain

$$\int_{t_1}^{t_2} E(s)\,ds \le \frac{1}{2}\int_{t_1}^{t_2} \left(\|u_t\|^2 + \|v_t\|^2\right) ds + \left(\frac{2(r+2)-p}{2\theta p(r+2)} + \frac{1}{2(r+2)}\right)\int_{t_1}^{t_2} K([u,v])\,ds. \tag{79}$$

For small enough ε, we have from (76) and (79) that

$$\int_{t_1}^{t_2} E(s)\,ds \le C\left[(E(t)-E(t+1)) + (E(t)-E(t+1))^{p/(2(p-1))}\right] + \varepsilon \sup_{t\le s\le t+1} E(s). \tag{80}$$

Thus, there exists $t^* \in [t_1, t_2]$, such that

$$E(t^*) \le C\left[(E(t)-E(t+1)) + (E(t)-E(t+1))^{p/2(p-1)}\right] + \varepsilon \sup_{t\le s\le t+1} E(s). \tag{81}$$

Multiplying (1) by u_t and (2) by v_t and integrating over $\Omega \times [t^*, t_2]$, and summing up, we get

$$E(t_2) = E(t^*) - \int_{t^*}^{t_2} \left(\|u_t\|_q^q + \|v_t\|_q^q + \|\nabla u_t\|^2 + \|\nabla v_t\|^2\right) ds. \tag{82}$$

Therefore, we obtain from (63), (81), and (82) that

$$\sup_{t\le s\le t+1} E(s) \le C\left[(E(t)-E(t+1)) + (E(t)-E(t+1))^{p/2(p-1)}\right] + \varepsilon \sup_{t\le s\le t+1} E(s). \tag{83}$$

Choosing small enough ε, we have from (83) that

$$\sup_{t\le s\le t+1} E(s) \le C\left[(E(t)-E(t+1)) + (E(t)-E(t+1))^{p/2(p-1)}\right]. \tag{84}$$

Since $p > 2$ and $E(t) < E(0)$, we get

$$\sup_{t\le s\le t+1} E(s) \le C(E(t)-E(t+1))^{p/2(p-1)}. \tag{85}$$

Consequently,

$$\sup_{t\le s\le t+1} E(s)^{(2(p-1))/p} \le C(E(t)-E(t+1)). \tag{86}$$

Thus, applying Lemma 9 to (86), we get

$$E(t) \le \left[\frac{p-2}{pC}(t-1) + M^{(p-2)/p}\right]^{p/(2-p)}, \quad \forall t > 1, \tag{87}$$

where $M = \max_{t\in[0,1]} E(t) > 0$ is some constant depending only on $[u_0, v_0]$ and $[u_1, v_1]$. □

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