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Research Article

Bogdanov-Takens Bifurcation of a Delayed Ratio-Dependent Holling-Tanner Predator Prey System

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A delayed predator prey system with refuge and constant rate harvesting is discussed by applying the normal form theory of retarded functional differential equations introduced by Faria and Magalhães. The analysis results show that under some conditions the system has a Bogdanov-Takens singularity. A versal unfolding of the system at this singularity is obtained. Our main results illustrate that the delay has an important effect on the dynamical behaviors of the system.

1. Introduction

It is well known that the multiple bifurcations will occur when a predator prey system (ODE) with more interior positive equilibria, such as Bogdanov-Takens bifurcation, Hopf bifurcation, and backward bifurcation; see [1–5] for example. However, when the predator prey systems with delay and Bogdanov-Takens bifurcation are researched relative few (see [6–8] and the reference therein) using similar methods as [6–8], the authors of [9–11] consider the Bogdanov-Takens bifurcation of some delayed single inertial neuron or oscillator models.

Motivated by the works of [5, 6], we mainly consider the Bogdanov-Takens bifurcation of the following system:

$$\dot{x} = rx\left(1-\frac{x}{K}\right)-\frac{\alpha y(x-\overline{m})}{Ay+x-\overline{m}}-\overline{h},$$
$$\dot{y} = sy\left(1-\frac{by(t-\overline{\tau})}{x(t-\overline{\tau})-\overline{m}}\right), \tag{1}$$

where x and y stand for prey and predator population (or densities) at time t, respectively. The predator growth is of logistic type with growth rate r and carrying capacity K in the absence of predation; α and A stand for the predator capturing rate and half saturation constant, respectively; s is the intrinsic growth rate of predator; however, carrying capacity x/b (b is the conversion rate of prey into predators) is the function on the population size of prey. The parameters α, A, $\overline{m}$, $\overline{h}$, s, b, and $\overline{\tau}$ are all positive constants. $\overline{m}$ is a constant number of prey using refuges; $\overline{h}$ is the rate of prey harvesting.

For simplicity, we first rescale system (1). Let $X = x-\overline{m}$, $Y = y$; system (1) can be written as (still denoting X, Y as x, y)

$$\dot{x} = r(x+\overline{m})\left(1-\frac{x+\overline{m}}{K}\right)-\frac{\alpha xy}{Ay+x}-\overline{h},$$
$$\dot{y} = sy\left(1-\frac{by(t-\overline{\tau})}{x(t-\overline{\tau})}\right). \tag{2}$$

Next, let $\overline{t} = rt$, $X(\overline{t}) = x(t)/K$, and $Y(\overline{t}) = \alpha y(t)/rK$; then system (2) takes the form (still denoting X, Y, $\overline{t}$ as x, y, t)

$$\dot{x} = (x+m)(1-x-m)-\frac{xy}{ay+x}-h,$$
$$\dot{y} = \delta y\left(\beta-\frac{y(t-\tau)}{x(t-\tau)}\right), \tag{3}$$

where $m = \overline{m}/K$, $a = Ar/\alpha$, $\delta = s\overline{h}/\alpha$, $\beta = \alpha/br$, $h = \overline{h}/r$, and $\tau = r\overline{\tau}$.

When $\tau = 0$, we have known that for some parameter values system (3) exhibits Bogdanov-Takens bifurcation (see [5]). Summarizing the methods used by [6] and the formulae in [12], the sufficient conditions which depend on delay to guarantee that system (3) has a Bogdanov-Takens singularity will be given. Therefore, the delay has effect on the occurrence of Bogdanov-Takens bifurcation.

In the next section we will compute the normal form and give the versal unfolding of system (3) at the degenerate equilibrium.

2. Bogdanov-Takens Bifurcation

System (3) can also be written as

$$\dot{x} = \tau\left((x+m)(1-x-m) - \frac{xy}{ay+x} - h\right),$$
$$\dot{y} = \tau\delta y\left(\beta - \frac{y(t-1)}{x(t-1)}\right). \tag{4}$$

Let

$$0 < m < \frac{1}{2}\left(1 - \frac{\beta}{a\beta+1}\right),$$
$$h = \widetilde{h} = \frac{1}{4}\left(\frac{\beta}{a\beta+1} - 1\right)^2 + \frac{m\beta}{a\beta+1}. \tag{5}$$

Then $P_* = (x_*, y_*)$ is an interior positive equilibrium of systems (3) and (4), where $x_* = -(1/2)((\beta/(a\beta+1)) + 2m - 1)$, $y_* = \beta x_*$.

In order to discuss the properties of system (4) in the neighborhood of the equilibrium $P_* = (x_*, y_*)$, let $\overline{x} = x - x_*$, $\overline{y} = y - y_*$; then P_* is translated to $(0,0)$, and system (4) becomes (still denoting $\overline{x}$, $\overline{y}$ as x, y)

$$\dot{x} = \frac{\tau\beta}{(a\beta+1)^2}x - \frac{\tau}{(a\beta+1)^2}y + g_1x^2 + g_2xy + g_3y^2 + \text{h.o.t},$$
$$\dot{y} = \tau\delta\beta^2 x(t-1) - \tau\delta\beta y(t-1) + g_4x^2(t-1) + g_5x(t-1)y(t-1) + g_6x(t-1)y + g_7y(t-1)y + \text{h.o.t}, \tag{6}$$

where h.o.t denotes the higher order terms and

$$g_1 = -\left(\frac{2a\delta\beta^2\tau}{\beta+(2m-1)(a\beta+1)} + \tau\right),$$
$$g_2 = \frac{4a\delta\beta\tau}{\beta+(2m-1)(a\beta+1)},$$
$$g_3 = \frac{-2a\delta\tau}{\beta+(2m-1)(a\beta+1)},$$
$$g_4 = \frac{2\tau\delta\beta^2(a\beta+1)}{\beta+(2m-1)(a\beta+1)},$$
$$g_5 = \frac{-2\tau\delta\beta(a\beta+1)}{\beta+(2m-1)(a\beta+1)},$$
$$g_6 = -\frac{2\tau\delta\beta(a\beta+1)}{\beta+(2m-1)(a\beta+1)},$$
$$g_7 = \frac{2\tau\delta(a\beta+1)}{\beta+(2m-1)(a\beta+1)}. \tag{7}$$

The characteristic equation of the linearized part of system (6) is

$$F(\lambda) = \lambda^2 + \left[\tau\delta\beta e^{-\lambda} - \frac{\tau\beta}{(a\beta+1)^2}\right]\lambda. \tag{8}$$

Clearly, if

$$\delta = \frac{1}{(a\beta+1)^2},$$
$$\tau \neq \frac{1}{\delta\beta}, \quad \text{that is, } \tau \neq \frac{(a\beta+1)^2}{\beta}; \tag{9}$$

then $\lambda = 0$ is double zero eigenvalue; if $\delta = 1/(a\beta+1)^2$ and $\tau = 1/\delta\beta$, that is, $\tau = (a\beta+1)^2/\beta$, then $\lambda = 0$ is triple zero eigenvalue. We will mainly discuss the first case in this paper.

According to the normal form theory developed by Faria and Magalhães [13], first, rewrite system (4) as $\dot{X}(t) = L(X_t)$, where $X(t) = (x_1(t), x_2(t))$, $L(\phi) = L\left(\begin{smallmatrix}\phi_1(-1)\\ \phi_2(0)\end{smallmatrix}\right)$, and $\phi = (\phi_1, \phi_2)$. Take A_0 as the infinitesimal generator of system. Let $\Lambda = \{0\}$, and denote by P the invariant space of A_0 associated with the eigenvalue $\lambda = 0$; using the formal adjoint theory of RFDE in [13], the phase space C_1 can be decomposed by Λ as $C_1 = P \oplus Q$. Define Φ and Ψ as the bases for P and P^*, the space associated with the eigenvalue $\lambda = 0$ of the adjoint equation, respectively, and to be normalized such that $(\Psi, \Phi) = I$, $\dot{\Phi} = \Phi J$, and $\dot{\Psi} = -J\Psi$, where Φ and Ψ are 2×2 matrices.

Next we will find the $\Phi(\theta)$ and $\Psi(s)$ based on the techniques developed by [14].

Lemma 1 (see Xu and Huang [14]). *The bases of P and its dual space P^* have the following representations:*

$$P = \text{span}\,\Phi, \Phi(\theta) = (\varphi_1(\theta), \varphi_2(\theta)), \quad -1 \le \theta \le 0,$$
$$P^* = \text{span}\,\Psi, \Psi(s) = \text{col}(\psi_1(s), \psi_2(s)), \quad 0 \le s \le 1, \tag{10}$$

where $\varphi_1(\theta) = \varphi_1^0 \in R^n \setminus \{0\}$, $\varphi_2(\theta) = \varphi_2^0 + \varphi_1^0\theta$, and $\varphi_2^0 \in R^n$ and $\psi_2(s) = \psi_2^0 \in R^{n*} \setminus \{0\}$, and $\psi_1(s) = \psi_1^0 - s\psi_2^0$, $\psi_1^0 \in R^{n*}$, which satisfy

(1) $(A+B)\varphi_1^0 = 0$,
(2) $(A+B)\varphi_2^0 = (B+I)\varphi_1^0$,
(3) $\psi_2^0(A+B) = 0$,
(4) $\psi_1^0(A+B) = \psi_2^0(B+I)$,

(5) $\psi_2^0\varphi_2^0 - (1/2)\psi_2^0 B\varphi_1^0 + \psi_2^0 B\varphi_2^0 = 1$,

(6) $\psi_1^0\varphi_2^0 - (1/2)\psi_1^0 B\varphi_1^0 + \psi_1^0 B\varphi_2^0 + (1/6)\psi_2^0 B\varphi_1^0 - (1/2)\psi_2^0 B\varphi_2^0 = 0$.

By (6), we have $A = \begin{pmatrix} \tau\delta\beta & -\tau\delta \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ \tau\delta\beta^2 & -\tau\delta\beta \end{pmatrix}$; using Lemma 1, we obtain

$$\Phi(\theta) = \begin{pmatrix} 1 & \dfrac{1}{\tau\delta\beta} + \theta \\ \beta & \beta\theta \end{pmatrix}, \quad -1 \le \theta \le 0,$$

$$\Psi(s) = \begin{pmatrix} m_1 + s\beta n & m_2 - sn \\ -\beta n & n \end{pmatrix}, \quad 0 \le s \le 1, \tag{11}$$

where $m_1 = \tau\delta\beta(\tau\delta\beta - 2)/2(\tau\delta\beta - 1)^2 = \tau\beta[\tau\beta - 2(a\beta + 1)^2]/2[\tau\beta - (a\beta + 1)^2]^2$, $m_2 = (1 + (\tau\delta\beta - 1)^2)/2\beta(\tau\delta\beta - 1)^2 = (1/2\beta)(1 + (a\beta + 1)^4/(\tau\beta - (a\beta + 1))^2)$, and $n = \tau\delta/(\tau\delta\beta - 1) = \tau/(\tau\beta - (a\beta + 1)^2)$.

The matrix J satisfying $\dot{\Phi} = \Phi J$ is given by $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. System (6) in the center manifold is equivalent to the system $\dot{z} = Jz + \Psi(0)F(\Phi z)$, where

$$F(\phi) = \begin{pmatrix} g_1\phi_1^2(0) + g_2\phi_1(0)\phi_2(0) + g_3\phi_2^2(0) + \text{h.o.t} \\ g_4\phi_1^2(-1) + g_5\phi_1(-1)\phi_2(-1) + g_6\phi_1(-1)\phi_2(0) + g_7\phi_2(-1)\phi_2(0) + \text{h.o.t} \end{pmatrix}. \tag{12}$$

It is easy to obtain

$$\begin{gathered} \Phi(\theta) z = \begin{pmatrix} z_1 + \left(\dfrac{1}{\tau\delta\beta} + \theta\right) z_2 \\ \beta(z_1 + x_2\theta) \end{pmatrix}, \\ \phi_1(0) = z_1 + \frac{1}{\tau\delta\beta} z_2, \\ \phi_1(-1) = z_1 + \left(\frac{1}{\tau\delta\beta} - 1\right) z_2, \\ \phi_2(0) = \beta z_1, \qquad \phi_2(-1) = \beta(z_1 - z_2). \end{gathered} \tag{13}$$

Hence (6) becomes

$$\begin{aligned} \dot{z}_1 &= z_2 - \tau m_1 z_1^2 - \frac{2m_1}{\delta\beta} z_1 z_2 + l_1 z_2^2 + \text{h.o.t}, \\ \dot{z}_2 &= n\tau\beta z_1^2 + \frac{2n}{\delta} z_1 z_2 + l_2 z_2^2 + \text{h.o.t}, \end{aligned} \tag{14}$$

where $l_1 = m_1 g_1/\tau^2\delta^2\beta^2 + 2m_2(\delta+\mu_1)(a\beta+1)(1-\tau\delta\beta)/\tau\delta^2(\beta + (a\beta+1)(2m-1))$, $l_2 = -(ng_1/\beta\tau^2\delta^2) + 2n(\delta+\mu_1)(a\beta+1)(1-\tau\delta\beta)/\tau\delta^2(\beta + (a\beta+1)(2m-1))$.

After a series of transformations we obtain

$$\begin{aligned} \dot{z}_1 &= z_2 + \text{h.o.t}, \\ \dot{z}_2 &= d_1 z_1^2 + d_2 z_1 z_2 + \text{h.o.t}, \end{aligned} \tag{15}$$

where $d_1 = \beta n\tau = \beta\tau^2/(\tau\beta - (a\beta + 1)^2)$, $d_2 = -\tau m_1 + n/\delta = -(\tau^2\beta^2 + 2(a\beta + 1)^2[(a\beta + 1)^2 - 2\tau\beta])/(\tau\beta - (a\beta + 1)^2)^2$; if $\tau \neq (2 \pm \sqrt{2})/\delta\beta$, then $d_2 \neq 0$.

Hence, we have the following theorem.

Theorem 2. *Let* (5), (9), *and* $\tau \neq (2 \pm \sqrt{2})/\delta\beta$ *hold. Then the equilibrium* P_* *of system* (4) *is a Bogdanov-Takens singularity.*

In the following, we will do versal unfolding of system (4) at P_*:

$$\begin{aligned} \dot{x} &= \tau\left((x+m)(1-x-m) - \frac{xy}{ay+x} - h\right), \\ \dot{y} &= \tau(\delta+\mu_1) y \left(\beta + \mu_2 - \frac{y(t-1)}{x(t-1)}\right). \end{aligned} \tag{16}$$

When $\mu_1 = \mu_2 = 0$, system (16) has a Bogdanov-Takens singularity P_* and a two-dimensional center manifold exists.

The Taylor expansion of system (16) at P_* takes the form

$$\begin{aligned} \dot{x} &= \tau\delta\beta x - \tau\delta y + g_1 x^2 + g_2 xy + g_3 y^2 + \text{h.o.t}, \\ \dot{y} &= w_0\mu_2 + \tau(\delta+\mu_1)\beta^2 x(t-1) - \tau(\delta+\mu_1)\beta y(t-1) \\ &\quad + \tau\delta\mu_2 y + w_4 x^2(t-1) + w_5 x(t-1) y(t-1) \\ &\quad + w_6 x(t-1) y + w_7 y(t-1) y + \text{h.o.t}, \end{aligned} \tag{17}$$

where

$$\begin{aligned} w_0 &= -\frac{\tau\delta\beta[\beta + (2m-1)(a\beta+1)]}{2(a\beta+1)}, \\ w_4 &= \frac{2\tau(\delta+\mu_1)\beta^2(a\beta+1)}{\beta + (2m-1)(a\beta+1)}, \\ w_5 &= -\frac{2\tau(\delta+\mu_1)\beta(a\beta+1)}{\beta + (2m-1)(a\beta+1)}, \\ w_6 &= -\frac{2\tau(\delta+\mu_1)(a\beta+1)}{\beta + (2m-1)(a\beta+1)}, \\ w_7 &= \frac{2\tau(\delta+\mu_1)(a\beta+1)}{\beta + (2m-1)(a\beta+1)}. \end{aligned} \tag{18}$$

We decompose the enlarged phase space BC of system as $BC = P \oplus \operatorname{Ker}\pi$. Then y in system can be decomposed as $y = \Phi z + u$ with $z \in R^2$ and $u \in Q'$. Hence, system is decomposed as

$$\dot{z} = B_1 + B_2 z + \Psi(0) G(\Phi z + u),$$

$$\dot{u} = A_{Q'} u + (I - \pi) X_0 \times [B_0 + B_2(\Phi_0 z + u(0)) + G(\Phi z + u)], \quad (19)$$

where

$$X_0(\theta) = \begin{cases} I, & \theta = 0 \\ 0, & -1 \le \theta < 0, \end{cases}$$

$$B_0 = \begin{pmatrix} 0 \\ w_0 \mu_2 \end{pmatrix}, \qquad B_1 = \Psi(0) B_0 = \begin{pmatrix} w_0 m_2 \mu_2 \\ w_0 n \mu_2 \end{pmatrix},$$

$$B_2 = \Psi(0) \begin{pmatrix} \tau\delta\beta\phi_1(0) - \tau\delta\phi_2(0) \\ \tau(\delta + \mu_1)\beta^2\phi_1(-1) - \tau(\delta + \mu_1)\beta\phi_2(-1) + \tau\delta\mu_2\phi_2(0) \end{pmatrix}, \quad (20)$$

$$G(\phi) = \begin{pmatrix} g_1\phi_1^2(0) + g_2\phi_1(0)\phi_2(0) + g_3\phi_2^2(0) + \text{h.o.t} \\ w_4\phi_1^2(-1) + w_5\phi_1(-1)\phi_2(-1) + w_6\phi_1(-1)\phi_2(0) + w_7\phi_2(-1)\phi_2(0) + \text{h.o.t} \end{pmatrix}.$$

To compute the normal form of system at P_*, consider $\dot{z} = B_1 + B_2 z + \Psi(0)G(\Phi z)$; together with (13) we obtain

$$\begin{aligned} \dot{z}_1 &= w_0 m_2 \mu_2 + z_2 + m_2\tau\delta\beta\mu_2 z_1 + \frac{m_2\beta\mu_1}{\delta} z_2 \\ &\quad - \tau m_1 z_1^2 - \frac{2m_1}{\delta\beta} z_1 z_2 + l_1 z_2^2 + \text{h.o.t}, \\ \dot{z}_2 &= w_0 n \mu_2 + n\tau\delta\beta\mu_2 z_1 + \frac{n\beta\mu_1}{\delta} z_2 \\ &\quad + n\tau\beta z_1^2 + \frac{2n}{\delta} z_1 z_2 + l_2 z_2^2 + \text{h.o.t}. \end{aligned} \quad (21)$$

Following the normal form formula in Kuznetsov [12], system (21) can be reduced to

$$\begin{aligned} \dot{z}_1 &= z_2 + \text{h.o.t}, \\ \dot{z}_2 &= \gamma_1 + \gamma_2 z_1 + d_1 z_1^2 + d_2 z_1 z_2 + \text{h.o.t}, \end{aligned} \quad (22)$$

where

$$\begin{aligned} \gamma_1 &= -\frac{\tau\delta\beta[\beta + (2m-1)(a\beta+1)]}{2(a\beta+1)} n\mu_2 \\ &= \frac{\tau^2\beta[\beta + (2m-1)(a\beta+1)]}{2(a\beta+1)^3[(a\beta+1)^2 - \tau\beta]}\mu_2, \\ \gamma_2 &= \frac{\beta^2 n^2 \tau}{n - \tau m_1 \delta}\mu_1 \\ &\quad + \Bigg(n\tau\delta\beta - \frac{2w_0 n}{\delta\beta} - \frac{w_0 n(l_2\delta\beta - 2 + 2m_2\beta)}{\delta\beta} \\ &\quad + \big(n\tau\big(m_2\tau\delta^2\beta^2 + 2l_1 w_0 n\delta\beta + 2w_0 m_2 \\ &\quad - 2w_0 m_2^2\beta - 2w_0 m_2 l_2\delta\beta\big)\big) \\ &\quad \times(-\tau\delta + m_2\tau\delta\beta + n)^{-1}\Bigg)\mu_2. \end{aligned} \quad (23)$$

Hence system (4) exist the following bifurcation curves in a small neighborhood of the origin in the (μ_1, μ_2) plane.

Theorem 3. *Let* (5), (9), *and* $\tau \ne (2 \pm \sqrt{2})/\delta\beta$ *hold. Then system* (4) *admits the following bifurcations:*

(i) *a saddle-node bifurcation curve* $SN = \{(\mu_1, \mu_2) : \gamma_1 = (1/4d_1)\gamma_2^2\}$;

(ii) *a Hopf bifurcation curve* $H = \{(\mu_1, \mu_2) : \mu_2 = 0, \gamma_2 < 0\}$;

(iii) *a homoclinic bifurcation curve* $HL = \{(\mu_1, \mu_2) : \gamma_1 = -(6/25d_1)\gamma_2^2, \gamma_2 < 0\}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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