Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 891249, 8 pages
http://dx.doi.org/10.1155/2013/891249

Research Article

On the Geometry of the Unit Ball of a JB^*-Triple

Haifa M. Tahlawi, Akhlaq A. Siddiqui, and Fatmah B. Jamjoom

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Fatmah B. Jamjoom; fatmahj@yahoo.com

Received 14 February 2013; Revised 5 May 2013; Accepted 8 May 2013

Academic Editor: Ngai-Ching Wong

We explore a JB^*-triple analogue of the notion of quasi invertible elements, originally studied by Brown and Pedersen in the setting of C^*-algebras. This class of BP-quasi invertible elements properly includes all invertible elements and all extreme points of the unit ball and is properly included in von Neumann regular elements in a JB^*-triple; this indicates their structural richness. We initiate a study of the unit ball of a JB^*-triple investigating some structural properties of the BP-quasi invertible elements; here and in sequent papers, we show that various results on unitary convex decompositions and regular approximations can be extended to the setting of BP-quasi invertible elements. Some C^*-algebra and JB^*-algebra results, due to Kadison and Pedersen, Rørdam, Brown, Wright and Youngson, and Siddiqui, including the Russo-Dye theorem, are extended to JB^*-triples.

1. Introduction

Brown and Pedersen [1] introduced a notion of quasi invertible elements in a C^*-algebra. As is well explained in [1, 2], the Brown-Pedersen quasi (in short, BP-quasi) invertible elements bear many interesting properties similar to those of invertible elements. They have successfully demonstrated the significant role of BP-quasi invertible elements in studying geometry of the unit ball of a C^*-algebra; in particular, they obtained several results verifying that the relationships between the extreme convex decomposition theory and the distance $\alpha_q(x)$ from an element x to the set of BP-quasi invertible elements are analogous with the relationships in the earlier C^*-algebra theory of unitary convex decompositions and regular approximations.

It is widely believed that the underlying structure making several interesting results on C^*-algebras hold is not the presence of an associative product xy but the presence of the Jordan triple product $\{xyz\} = (1/2)(xyz + zyx)$. This provided one of the stimuli for the development of Jordan algebra or Jordan triple product generalizations of C^*-algebras; these include JB^*-algebras and JB^*-triples together with their subclasses which are Banach dual spaces (cf. [3]). In [4, 5], we introduced an exact analogue of the BP-quasi invertible elements for JB^*-triples. By using certain identities concerning the Bergman operators, we established that in case of nondegenrate Jordan triples $B(a,b) = 0 \Rightarrow B(b,a) = 0$ and $\{aba\} = a$ and that a is BP-quasi invertible with BP-quasi inverse $b \Leftrightarrow B(a,b) = 0$ [5]. Our aim in this paper is to study geometric consequences of these facts in the setting of complex JB^*-triples. After discussing some basics, we begin by recording a known result: $\{aba\} = a \Rightarrow a$ is von Neumann regular that admits a tripotent e_a, called its range tripotent. We prove that $B(a,b) = 0 \Leftrightarrow e_a$ is an extreme point of the unit ball and that the set of BP-quasi invertible elements of a JB^*-triple is open in the norm topology. Then we investigate the natural analogue of the Russo-Dye theorem [6, 7] on the representability of the elements with means of extreme points of the unit ball. Later on, we look at convex combinations of extreme points of the unit ball. In the course of our analysis, we obtain JB^*-triple analogues of some C^*-algebra results; this approach provides alternative proofs for some of the corresponding known results for C^*-algebras.

A Jordan triple system is a vector space $\mathcal{J}$ over a field of characteristic not 2, endowed with a triple product $\{xyz\}$ which is linear and symmetric in the outer variables x, z, and linear or antilinear in the inner variable y satisfying the Jordan triple identity: $\{xu\{yvz\}\} + \{\{xvy\}uz\} - \{yv\{xuz\}\} = \{x\{uyv\}z\}$ [3]. A JB^*-triple is a complex Banach space $\mathcal{J}$ together with a continuous, sesquilinear, operator-valued map $(x, y) \in \mathcal{J} \times \mathcal{J} \mapsto L(x, y)$ that defines a triple product $L(x, y)z := \{xy^*z\}$ in $\mathcal{J}$ making it a Jordan triple system such that each $L(x, x)$ is a positive hermitian operator on $\mathcal{J}$ and $\|\{xx^*x\}\| = \|x\|^3$ for all $x \in \mathcal{J}$. Any bounded symmetric domain in a complex Banach space is biholomorphically equivalent to the open unit ball of a JB^*-triple [8, 9]. An important example, from the viewpoint of the classical theory of operators and matrices, is the triple system $\mathcal{B}(\mathcal{H}, \mathcal{K})$ of all bounded linear operators between complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ under the product $\{xyz\} = (1/2)(xy^*z + zy^*x)$ and the operator norm, where x^* denotes the Hilbert adjoint of x (cf. [10]). In the finite-dimensional case, this can be viewed as adding an algebraic product to a linear matrix space $\mathcal{M}_{mn}(\mathbb{C})$ of all $m \times n$ matrices with complex number entries, namely, the triple product $\{xyz\} = (1/2)(xy^tz + zy^tx)$ where y^t denotes the conjugate transpose of the matrix y and the norm defined by $\|z\| = \sqrt{zz^t}$ (see [11, Example 4.7]). Any JB^*-algebra (cf. [3]) is a JB^*-triple under the triple product $\{xy^*z\} := (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$, where "$\circ$" denotes the underlying Jordan binary product.

A basic operator $P(x, y)$ on the JB^*-triple $\mathcal{J}$ is defined by $P(x, y)z = \{xz^*y\}$ for all $x, y, z \in \mathcal{J}$; we write $P(x, x)$ in short as $P(x)$. The Bergman operator $B(x, y)$ is defined on $\mathcal{J}$ by $B(x, y) = I - 2L(x, y) + P(x)P(y)$, where I is the identity operator [12, 2.11]. The operators $L(x, y)$ and $P(x, y)$ are the JB^*-triple analogues of the usual Jordan algebra operators $V_{x,y}z := \{xyz\}$ and $U_{x,y}z := \{xzy\}$, respectively; in fact, $L(x, y) = V_{x,y^*}$ and $P(x, y)z = U_{x,y}z^*$ for all $z \in \mathcal{J}$. The Bergman operator takes the form $B(x, y) = I - 2V_{x,y^*} + U_xU_{y^*}$, which translates to $B(x, y)z = (1 - xy^*)z(1 - y^*x)$ for the C^*-algebra case.

2. BP-Quasi Invertible Elements

We begin with the following characterization of BP-quasi invertible elements of a JB^*-triple (cf. [5, Theorem 6]).

Definition 1. In any JB^*-triple, an element x is BP-quasi invertible with BP-quasi inverse y if (and only if) the Bergman operator $B(x, y) = 0$.

In any JB^*-triple $\mathcal{J}$, $B(x, y) = 0 \Leftrightarrow B(y, x) = 0$ by [5, Theorem 2], and so the relation of being BP-quasi inverse of some element is symmetric in $\mathcal{J}$. Generally, BP-quasi inverse in not unique: if $x \in \mathcal{J}_q^{-1}$ with BP-quasi inverse y, then $P(y)x$ is also a BP-quasi inverse of x since $B(x, P(y)x) = I - 2L(x, P(y)x) + P(x)P(P(y)x) = I - 2L(P(x)y, y) + P(x)P(y)P(x)P(y) = I - 2L(x, y) + P(x)P(y)$ (cf. [12]) $= B(x, y) = 0$. We denote the set of BP-quasi invertible elements in $\mathcal{J}$ by $\mathcal{J}_q^{-1}$. $\mathcal{J}_q^{-1}$ includes the set of all invertible elements $\mathcal{J}^{-1}$ and the set of all extreme points $\mathcal{E}(\mathcal{J})_1$ of the closed unit ball $(\mathcal{J})_1$; and in case of a JB^*-algebra $\mathcal{J}$, the BP-quasi invertibility is invariant under the involution "$*$" (cf. [4]).

In the JB^*-triple $\mathcal{M}_{23}(\mathbb{C})$, all matrices of the form $\left[\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \end{smallmatrix}\right]$, where α, β are any nonzero complex numbers, are BP-quasi invertible with BP-quasi inverse of the form $\left[\begin{smallmatrix} 1/\alpha & 0 & 0 \\ 0 & 1/\beta & 0 \end{smallmatrix}\right]$.

An element x in a JB^*-triple $\mathcal{J}$ is called von Neumann regular if there exists $y \in \mathcal{J}$ with $x = P(x)y$. Such an element y is called generalized inverse of x [13]. If $\mathcal{J}$ is a JB^*-algebra, then $x = P(x)y = \{xy^*x\}$, and so y is a generalized inverse of x in $\mathcal{J}$ considered as a JB^*-triple if and only if y^* is a generalized inverse of x in the JB^*-algebra $\mathcal{J}$. From [5, Theorem 3], we know that x and y are von Neumann regular with generalized inverses of each other if $B(x, y) = 0$. Hence, any BP-quasi invertible element is necessarily von Neumann regular. For a von Neumann regular element in the C^*-algebra $\mathcal{M}_2(\mathbb{C})$ of 2×2 matrices that is not BP-quasi invertible, see [5, Example 9]. Thus, the class of BP-quasi invertible elements is a proper subclass of the von Neumann regular elements.

For any fixed element a in a JB^*-triple $\mathcal{J}$, the underlying linear space $\mathcal{J}$ becomes a complex Jordan algebra $\mathcal{J}_{[a]}$ with respect to the Jordan product $x \cdot_a y := \{xay\}$, called a-homotope of $\mathcal{J}$ (cf. [3]). An element u in a JB^*-triple $\mathcal{J}$ is called unitary if $L(u, u)z = z$ [3]; the set of all unitary elements is denoted by $\mathcal{U}(\mathcal{J})$. For any $u \in \mathcal{U}(\mathcal{J})$, the u-homotope of $\mathcal{J}$ is a JB^*-algebra with respect to the original norm and the involution "$*_u$" given by $x^{*_u} = P(u)x$ (cf. [3]); such a homotope is denoted by $\mathcal{J}^{[u]}$, called a unitary isotope of $\mathcal{J}$. The symbols $P_u(x, y)$, $L_u(x, y)$, $B_u(x, y)$ will denote the respective analogues of the operators $P(x, y)$, $L(x, y)$, and $B(x, y)$ on the isotope $\mathcal{J}^{[u]}$. Thus, $P_u(x, y)z = \{xz^{*_u}y\}_u$ and $L_u(x, y)z = \{xy^{*_u}z\}_u$ for all $x, y, z \in \mathcal{J}$; in particular, $P_u(x)z = P_u(x, x)z = \{xz^{*_u}x\}_u$. Here, $\{\cdots\}_u$ denotes the induced Jordan triple product in $\mathcal{J}^{[u]}$.

Theorem 2. *For any fixed unitary element u of a JB^*-triple $\mathcal{J}$ and for all $x, y, z \in \mathcal{J}$, one has the following:*

(i) *the Jordan triple product $\{xy^{*_u}z\}_u$ in $\mathcal{J}^{[u]}$ coincides with $\{xy^*z\}$;*

(ii) $P_u(x) = P(x)$;

(iii) $L_u(x, y) = L(x, y)$;

(iv) $B_u(x, y) = B(x, y)$;

(v) $(\mathcal{J}^{[u]})_q^{-1} = (\mathcal{J})_q^{-1}$.

Proof. (i) By using the basic Jordan triple identity, we get that

$$\begin{aligned} \{xy^{*_u}z\}_u &= \left\{x\left\{u\left(y^{*_u}\right)^*u\right\}^*z\right\} \\ &\quad + \left\{y^{*_u}u^*\{xu^*z\}\right\} - \left\{\{xu^*y^{*_u}\}u^*z\right\} \\ &\quad - \left\{y^{*_u}u^*\{xu^*z\}\right\} + \left\{zu^*\{xu^*y^{*_u}\}\right\} \\ &= \left\{x\left\{u\left(y^{*_u}\right)^*u\right\}^*z\right\} = \left\{x\left(\left(y^{*_u}\right)^{*_u}\right)^*z\right\} \\ &= \{xy^*z\}. \end{aligned} \tag{1}$$

(ii) By the part (i), $P_u(x)z = \{xz^{*_u}x\}_u = \{xz^*x\} = P(x)z$ for all $x \in \mathcal{J}$.

(iii) For any $a, b \in \mathcal{J}$, $P(a, b) = P(a + b) - P(a) - P(b)$ (cf. [12]). This together with the part (ii) gives $L_u(x, y)z = P_u(x, z)y = P_u(x + z)y - P_u(x)y - P_u(z)y = P(x + z)y - P(x)y - P(z)y = P(x, z)y = L(x, y)z$ for all $x, y, z \in \mathcal{J}$.

(iv) Follows from parts (ii) and (iii) since $B(x,y) = I - 2L(x,y) + P(x)P(y)$.

(v) $x \in \mathcal{J}_q^{-1} \Leftrightarrow B(x,y) = 0 \Leftrightarrow B_u(x,y) = 0$ (by (iv)) $\Leftrightarrow x \in (\mathcal{J}^{[u]})_q^{-1}$. □

We close this section with the following observation on images of BP-quasi invertible elements under triple homomorphisms.

Theorem 3. *For any closed ideal V in JB^*-triple $\mathcal{J}$, the triple homomorphism $\phi : \mathcal{J} \to \mathcal{J}/V$ given by $\phi(x) = x + V$ preserves the BP-quasi invertibility.*

Proof. V being closed ideal of $\mathcal{J}$ is a JB^*-triple, and so is the quotient $\mathcal{J}/V$. The quotient JB^*-triple $\mathcal{J}/V$ admits the canonical surjective triple homomorphism $\phi : \mathcal{J} \to \mathcal{J}/V$ defined by $\phi(x) = x + V$ such that $\phi(\{xyz\}) = \{\phi(x)\phi(y^*)\phi(z)\}$ for all $x, y, z \in \mathcal{J}$ [8, Proposition 5.5].

Now, for any fixed $x \in \mathcal{J}_q^{-1}$ with inverse $y \in \mathcal{J}$ and for all $z \in \mathcal{J}$, $B(\phi(x),\phi(y))\phi(z) = \phi(z) - 2\{\phi(x)(\phi(y))^*\phi(z)\} + \{\phi(x)\{\phi(y)(\phi(z))^*\phi(y)\}^*\phi(x)\} = \phi(z) - 2\phi(\{xy^*z\}) + \phi(\{x\{yz^*y\}^*x\}) = \phi(B(x,y)(z)) = \phi(0) = 0$. Hence, $\phi(x) \in (\mathcal{J}/V)_q^{-1}$. □

3. Positivity of BP-Quasi Invertibles

In this section, we prove that any BP-quasi invertible element in a JB^*-triple $\mathcal{J}$ is positive invertible in the Peirce 1-space $\mathcal{J}_1(v) := \{x \in \mathcal{J} \mid L(v,v)x = x\}$ of the operator $L(v,v)$ for certain extreme point v of the closed unit ball. Besides other results, we obtain another characterization of the BP-quasi invertible elements. The following result is known in pieces; we include it with a unified proof.

Theorem 4. *For each von Neumann regular element x of a JB^*-triple $\mathcal{J}$, there exists a unique tripotent $e_x \in \mathcal{J}$ such that x is positive invertible in $\mathcal{J}_1(e_x)$.*

Proof. Since x is von Neumann regular, there exists a unique tripotent $e_x \in \mathcal{J}$, called the range tripotent of x (cf. [10]) such that x is invertible in the Peirce space $\mathcal{J}_1(e_x)$ (cf. [10, Lemma 3.2], [14, page 540], and [15, Theorem 1]). Let $\mathcal{J}(x)$ denotes the norm closure of $P(x)(\mathcal{J}) = \{x\mathcal{J}x\}$, which is the smallest norm closed inner ideal of $\mathcal{J}$ containing x (cf. [16, pages 19-20]). From [10, Lemma 3.2], we see that $P(x)(\mathcal{J})$ is norm closed in $\mathcal{J}$. Hence, $\mathcal{J}(x) = P(x)(\mathcal{J}) = \mathcal{J}_1(e_x)$ since x is von Neumann regular in $\mathcal{J}$ (cf. [10, page 572]). Thus, x is positive in $\mathcal{J}_1(e_x)$ by [16, Proposition 2.1]. □

Remark 5. The range tripotent e_x of x is defined as the least tripotent for which x is a positive element in the JB^*-algebra $\mathcal{J}_1(e_x)$. If u and v are two tripotents in $\mathcal{J}$ such that x is positive invertible in both the Peirce spaces $\mathcal{J}_1(u)$ and $\mathcal{J}_1(v)$, then $u = v$; for details, see [17, Lemma 3.3] and [16, Proposition 2.1].

Theorem 6. *Let $\mathcal{J}$ be a JB^*-triple and $x \in \mathcal{J}_q^{-1}$. Then the range tripotent e_x is a unique extreme point of $(\mathcal{J})_1$ such that x is positive invertible in $\mathcal{J}_1(e_x)$. Moreover, there exists a unique $y \in \mathcal{J}_q^{-1}$ satisfying $P(x)y = x$, $P(y)x = y$, $P(x)P(y) = P(y)P(x)$, and $L(x,y) = L(e_x,e_x) = L(y,x)$.*

Proof. Since x being BP-quasi invertible is a von Neumann regular element, Theorem 4 gives the existence of a unique tripotent e_x in $\mathcal{J}$ such that x is positive invertible in $\mathcal{J}_1(e_x)$. By [10, Lemma 3.2], there exists a unique generalized inverse $y \in \mathcal{J}$ satisfying all the conditions of the theorem; in fact, this generalized inverse is called the Moore-Penrose inverse, usually denoted by $x^\dagger$. Since $\{xx^\dagger x\} = x$, $\{zx^\dagger x\} = \{zx^\dagger\{xx^\dagger x\}\} = 2\{\{zx^\dagger x\}x^\dagger x\} - \{x\{x^\dagger zx^\dagger\}x\}$ for all $z \in \mathcal{J}$. So that $P(x)P(x^\dagger) = 2L(x,x^\dagger)^2 - L(x,x^\dagger)$. Hence, $B(x,x^\dagger) = 1 - 3L(x,x^\dagger) + 2L(x,x^\dagger)^2 = (2L(x,x^\dagger) - 1)(L(x,x^\dagger) - 1)$. In view of [18, Lemma 2.1] and [10, page 573], we have $L(x,x^\dagger) = L(e_x,e_x) = L(x^\dagger,x)$ since e_x is the unique von Neumann regular element in $(\mathcal{J})_1(e_x)$. Therefore, $B(x,x^\dagger) = B(e_x,e_x) = \Pi_0$, where Π_μ stands for the Peirce projection associated with the tripotent e_x onto the subtriple $\mathcal{J}_\mu(e_x) = \{z : L(e_x,e_x)z = \mu z\}$ (cf. [12]). In [19, page 192], authors show that e_x is unitary in $P(x)\mathcal{J}$, hence $P(x)\mathcal{J} = \mathcal{J}_1(e_x)$. Therefore, $e_x \in \mathcal{E}(\mathcal{J})_1$ by [20, Lemma 4]. Thus, $B(x,x^\dagger) = B(e_x,e_x) = 0$. We conclude that $y \in \mathcal{J}_q^{-1}$. □

Remark 7. The unique BP-quasi inverse of any BP-quasi invertible matrix x in the JB^*-triple $\mathcal{M}_{23}(\mathbb{C})$, satisfying the conditions of Theorem 6, is precisely the Moore-Penrose inverse $x^\dagger$ (cf. [10, Lemma 3.2]), which can be calculated according to the rank of the matrix (cf. [21]). If $\operatorname{rank}(x) = 2$ then xx^t is invertible in $\mathcal{M}_2(\mathbb{C})$, and the Moore-Penrose inverse of x is given by $x^\dagger = x^t(xx^t)^{-1}$. For example, let $x = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Then x is BP-quasi invertible and its unique BP-quasi inverse, via Theorem 6, is given by $x^\dagger = (1/3)\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Since $x(x^\dagger)^t = (x^\dagger)^t x = e$ (the unit in $\mathcal{M}_2(\mathbb{C})$), $x(x^\dagger)^t y = (x^\dagger)^t xy = y$ for all $y \in M_{23}$ and so $L(x,x^\dagger) = I$. Thus, $B(x,x^\dagger) = 0$.

Next result establishes an important topological property of the BP-quasi invertible elements in a JB^*-triple.

Theorem 8. *For any JB^*-triple $\mathcal{J}$, the set $\mathcal{J}_q^{-1}$ is open in the norm topology.*

Proof. Assume $B(a,c) = 0$ where $c = a^\dagger + z$ for some nonzero $z \in \mathcal{J}$, and $a^\dagger$ denotes the Moore-Penrose type inverse of a (cf. [4, 10]). Consequently, $P(a)c = \{ac^*a\} = a$, and so $a = \{a(a^\dagger + z)^*a\} = \{a(a^\dagger)^*a\} + \{az^*a\} = a + \{az^*a\}$. Thus, $P(a)z = \{az^*a\} = 0$, which means $P(a^\dagger)P(a)z = 0$. Recall that $\Pi_1 = P^2(v) = P(a^\dagger)P(a)$ (see [10] and [19, page 192]). Also, since $\{a(\Pi_1 h)^*a\} = 0$ implies that $\Pi_1 h = 0$, the nonzero element $z \notin \operatorname{range}(\Pi_1)$. Moreover, as a projection, Π_0 is the identity operator on $\mathcal{J}_0(v)$, so $\Pi_0 B(a,c)\Pi_0 = I\Pi_0 = \Pi_0 = 0$ since $B(a,c) = 0$. Thus, $\operatorname{range}(\Pi_0) = 0$ and $z \in \operatorname{range}(\Pi_{1/2}) = \mathcal{J}_{1/2}(v)$. Conversely, if a is BP-quasi invertible with the unique BP-quasi inverse $a^\dagger$ and $z \in \operatorname{range}(\Pi_{1/2}) = \mathcal{J}_{1/2}(v)$, then $B(a^\dagger, a+z) = 1 - 3L(a^\dagger, a+z) + 2L(a^\dagger, a+z)^2 = 1 - 3L(a^\dagger,a) + 2L(a^\dagger,a)^2 - 3L(a^\dagger,z) + 2(L(a^\dagger,a)L(a^\dagger,z) + L(a^\dagger,z)L(a^\dagger,a) + L(a^\dagger,z)^2) = B(a^\dagger,a) + 2(L(a^\dagger,a)L(a^\dagger,z) + L(a^\dagger,z)L(a^\dagger,a) + L(a^\dagger,z)^2)$.

Since $B(a^\dagger, a) = 0$ and $\{\mathcal{J}_1\mathcal{J}_0\mathcal{J}\} = 0$ [12, Theorem 5.4 (9)], $L(a^\dagger, z) = 0$, and we get the value 0 for the right hand side. Hence $a+z$ is BP-quasi invertible. The elements $x \in \mathcal{J}_1(v) :=$ range(Π_1) with spectrum $\mathrm{Sp}(L(x,x))|_{\mathcal{J}_1} > 0$ form an open subset of $\mathcal{J}_1$. It follows that $\{x+z : \mathrm{Sp}(L(x,x))|_{\mathcal{J}_1(v)} > 0, z \in \mathcal{J}_{1/2}(v)\}$ is an open subset of $\mathcal{J}$. Thus, $\mathcal{J}_q^{-1}$ is an open set. □

Now, we give the following improvement of [22, Theorem 4.12].

Theorem 9. *Let x be an invertible element in a unital JB^*-algebra $\mathcal{J}$. Then there exists unique $u \in \mathcal{U}(\mathcal{J})$ such that x is positive and invertible in $\mathcal{J}^{[u]}$.*

Proof. Since $x \in \mathcal{J}^{-1}$, there is a unique element $y \in \mathcal{J}$ with $x \circ y = e$ and $x^2 \circ y = x$ where "$\circ$" denotes the Jordan binary product and e is the unit in $\mathcal{J}$. Since $\mathcal{J}^{-1} \subseteq \mathcal{J}_q^{-1}$ and that any JB^*-algebra is a JB^*-triple under the triple product $\{ab^*c\} := a\circ(b^*\circ c) - b^*\circ(a\circ c) + c\circ(a\circ b^*)$, so, by Theorem 6, there exists a unique $u \in \mathcal{E}(\mathcal{J})_1$ (viz, the range tripotent e_x) such that x is positive and invertible in $\mathcal{J}_1(u)$ with the inverse of the same y in $\mathcal{J}_1(u)$ (cf. [15, Theorem 6]). Moreover, $x^\dagger = x^{-1}$, and so $V_{u,u^*} = L(u,u) = L(x, x^\dagger) = I$. Hence, the Peirce decomposition of $\mathcal{J}$ reduces to $\mathcal{J} = \mathcal{J}_1(u)$. Of course, the product and the involution defined on both the JB^*-algebra $\mathcal{J}_1(u)$ and the unitary isotope $\mathcal{J}^{[u]}$ of $\mathcal{J}$ are the same. Further, $e = e\circ_u u = u\circ_u e = \{uu^*e\} = u\circ u^*$ and $u = u\circ_u u = \{uu^*u\} = 2u\circ(u\circ u^*) - u^2\circ u^*$, so that $u^2\circ u^* = u$. Hence, $u \in \mathcal{J}^{-1}$ with inverse u^*; that is, $u \in \mathcal{U}(\mathcal{J})$. We conclude that $\mathcal{J}_1(u)$ and $\mathcal{J}^{[u]}$ coincide as JB^*-algebras. □

The following result shows that positive invertibility of an element x in the Peirce 1-space for some extreme point of the unit ball is sufficient for the BP-quasi invertibility of x.

Theorem 10. *Let $\mathcal{J}$ be a JB^*-triple and $x \in \mathcal{J} \setminus \mathcal{J}_q^{-1}$. Then x is not positive invertible in the Peirce 1-space $\mathcal{J}_1(v)$, for all $v \in \mathcal{E}(\mathcal{J})_1$.*

Proof. Suppose x is positive invertible in $\mathcal{J}_1(v)$ for some $v \in \mathcal{E}(\mathcal{J})_1$. Then there exists a unique inverse $y \in \mathcal{J}_1(v)$ satisfying $x\cdot_v y = y\cdot_v x = v$ and $x^2\cdot_v y = x$. Since $(\mathcal{J}(v))^{-1} \subseteq (\mathcal{J}(v))_q^{-1}$, x is von Neumann regular with generalized inverse y satisfying $P_v(x)y = x$ and $P_v(y)x = y$, where $P_v(a)b = \{ab^{*_v}a\}_v$. Being positive, x is self-adjoint in $\mathcal{J}_1(v)$; that is, $x = x^{*_v}$. However, the invertibility is invariant under the involution (see above). Therefore, $x \in (\mathcal{J}_1(v))^{-1}$ with inverse y such that $y = x^{-1_v} = (x^{*_v})^{-1_v} = (x^{-1_v})^{*_v} = y^{*_v} = P_v y$ by Theorem 2. This together with the von Neumann regularity of x in $\mathcal{J}_1(v)$ gives $x = P_v(x)y = P(x)P(v)y = P(x)y$. Thus, the element x is von Neumann regular in $\mathcal{J}$.

Now, by Theorem 4, there exists a (unique) tripotent $e_x \in \mathcal{J}$ such that x is positive invertible in $\mathcal{J}_1(e_x)$. Hence, by Remark 5 and our supposition on x, we get the coincidence $\mathcal{J}_1(e_x) = \mathcal{J}_1(v)$.

Next, by Theorem 6, y is the unique generalized inverse of x satisfying $P(x)y = x$, $P(y)x = y$, and $L(x,y) = L(v,v) = L(y,x)$. Hence, $B(x,y) = B(v,v) = 0$. This means $x \in \mathcal{J}_q^{-1}$, a contradiction to the hypothesis. □

The above result together with Theorem 6 gives another characterization of the BP-quasi invertibility, as follows.

Theorem 11. *An element x in a JB^*-triple $\mathcal{J}$ is BP-quasi invertible if and only if x is positive and invertible in the Peirce 1-space $\mathcal{J}_1(v)$ for some extreme point $v \in \mathcal{E}(\mathcal{J})_1$.*

4. An Analogue of the Russo-Dye Theorem

We investigate the natural analogue of the Russo-Dye theorem on the representability of the elements with arithmetic means of extreme points of the unit ball in a JB^*-triple. We extend [7, Theorem 2.2] for JB^*-triples as follows.

Theorem 12. *Let $\mathcal{J}$ be a JB^*-triple with $v \in \mathcal{E}(\mathcal{J})_1$ and $s \in (\mathcal{J}_1(v))_1^\circ$ (the open unit ball of $\mathcal{J}_1(v)$). Then for any positive integer n, $v + (n-1)s = \sum_{i=1}^n v_i$ with $v_i \in \mathcal{E}(\mathcal{J})_1$ for each $i = 1,\ldots,n$.*

Proof. Since $v \in \mathcal{E}(\mathcal{J})_1$, $\mathcal{J}_1(v)$ is a JB^*-algebra with unit v. Since $\|-s\| < 1$ in $\mathcal{J}_1(v)$, $v+s$ and so $(1/2)(v+s)$ are invertible elements in $\mathcal{J}_1(v)$ (cf. [22, Lemma 2.1(iii)]). Clearly, $\|(1/2)(v+s)\| < 1$. Hence, by [7, Lemma 2.1], there exist unitaries $v_1, v_2 \in \mathcal{U}(\mathcal{J}_1(v))$ with $v + s = v_1 + v_2$. Moreover, $v_1, v_2 \in \mathcal{E}(\mathcal{J})_1$ by [20, Lemma 4]. The required result now follows by induction on n. □

Definition 13. For any element x in a JB^*-triple $\mathcal{J}$, the number $e_m(x)$ is defined by $e_m(x) := \min\{n : x = (1/n)\sum_{j=1}^n v_j,\ v_j \in \mathcal{E}(\mathcal{J})_1\}$. If x has no such decomposition, then $e_m(x) := \infty$.

Next result describes a basic connection between $e_m(x)$ and the distance to the set $\mathcal{E}(\mathcal{J})_1$.

Theorem 14. *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$ and $x \in \mathcal{J}$ with $nx - v \in \mathcal{J}_1(v)$ satisfying $\|nx - v\| < n-1$ for some $n \ge 2$. Then $e_m(x) \le n$. On the other hand, if $e_m(x) \le n$, then $\mathrm{dist}(nx, \mathcal{E}(\mathcal{J})_1) \le n-1$.*

Proof. Clearly, $(n-1)^{-1}(nx - v) \in (\mathcal{J}_1(v))_1^\circ$. Replacing s by $(n-1)^{-1}(nx-v)$ in Theorem 12, we get $x = (1/n)(v + (n-1)s) = (1/n)\sum_{i=1}^n v_i$ with $v_i \in \mathcal{E}(\mathcal{J})_1$ for $i = 1,\ldots,n$. Thus, $e_m(x) \le n$. On the other hand, suppose $e_m(x) \le n$. Then $x = r^{-1}\sum_{i=1}^r v_i$ for some $1 \le r \le n$ with v_is in $\mathcal{E}(\mathcal{J})_1$. Then $\|x\| \le r^{-1}\sum_{i=1}^r \|v_i\| = 1$. Further, $\|rx - v_1\| = \|\sum_{i=2}^r v_i\| \le r-1$. Hence, $\|nx - v_1\| = \|(n-r)x + rx - v_1\| \le \|(n-r)x\| + \|rx - v_1\| \le n - r + r - 1 = n-1$ since $\|x\| \le 1$. Thus, $\mathrm{dist}(nx, \mathcal{E}(\mathcal{J})_1) = \inf_{w\in\mathcal{E}(\mathcal{J})_1}\|nx - w\| \le \|nx - v_1\| \le n-1$. □

For any $v \in \mathcal{E}(\mathcal{J})_1$, the set $\mathcal{E}(\mathcal{J})_1$ includes the set of all unitaries in $\mathcal{J}_1(v)$, as seen above from [20, Lemma 4]; the following example shows that the inclusion generally is proper.

Example 15. Let $\mathcal{J}$ be the JB^*-triple $\mathcal{M}_{12}(\mathbb{C})$, and let v denotes the matrix $[1 \quad 0]$. Then $\|v\| = 1$ and $vv^tv = v$. So, v is a norm one tripotent matrix. Moreover, for any matrix $z = [z_1 \quad z_2]$, we have $L_{v,v}z = [z_1 \quad (1/2)z_2]$. Also note that $P_v z = [\overline{z_1} \quad 0]$, where $\overline{z_i}$ denotes the complex conjugate of z_i. So that, $P_vP_vz = [z_1 \quad 0]$. Hence, $B(v,v)z = [z_1 \quad z_2] - 2[z_1 \quad (1/2)z_2] + [z_1 \quad 0] = [0 \quad 0]$. Thus, $v \in \mathcal{E}(\mathcal{J})_1$.

Next, consider the Peirce spaces of $\mathcal{J}$, relative to the extreme point matrix $v = [1 \quad 0]$, given by $\mathcal{J}_1(v) = \{z : z = [z_1 \quad 0], z_1 \in \mathbb{C}\}$, $\mathcal{J}_{1/2}(v) = \{z : z = [0 \quad z_2], z_2 \in \mathbb{C}\}$ and $\mathcal{J}_0(v) = \{[0 \quad 0]\}$. Clearly, both the balls $(\mathcal{J}_1(v))_1$ and $(\mathcal{J}_{1/2}(v))_1$ are nontrivial. It is easily seen that the matrix $[0 \quad 1] \in \mathcal{E}(\mathcal{J})_1 \setminus \mathcal{M}_1(v)$.

From [6, 7], we know different proofs of strict analogue of the Russo-Dye theorem for general (unital) JB^*-algebras. Unfortunately, there exists no exact analogue of the Russo-Dye theorem for nonunital JB^*-triples. The following result is an appropriate analogue for an arbitrary JB^*-triple, where $\mathrm{co}\,\mathcal{E}(\mathcal{J})_1$ and $\overline{\mathrm{co}}\mathcal{E}(\mathcal{J})_1$ denote the convex hull of $\mathcal{E}(\mathcal{J})_1$ and its norm closure, respectively.

Theorem 16 (A Russo-Dye Theorem type for JB^*-triples). *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$.*

(i) *For any $x \in \mathcal{J}_1(v)$ with $\|x\| < 1 - 2n^{-1}$ for some $n \geq 3$, there exists $v_i \in \mathcal{E}(\mathcal{J})_1$ for all $i = 1, 2, 3, \ldots, n$ such that $x = (1/n)\sum_{i=1}^{n} v_i$.*

(ii) $(\mathcal{J}_1(v))_1^\circ \subseteq \mathrm{co}\mathcal{E}(\mathcal{J})_1$.

(iii) $(\mathcal{J}_1(v))_1 \subseteq \overline{\mathrm{co}}\mathcal{E}(\mathcal{J})_1$.

Proof. (i) Since $\|x\| < 1 - 2n^{-1}$, we have $\|nx\| < n - 2$, so that $\|(n-1)^{-1}(nx - v)\| < 1$. Also, note that $(n-1)^{-1}(nx - v) \in \mathcal{J}_1(v)$; recall that x is in the JB^*-algebra $\mathcal{J}_1(v)$. Hence, by taking v the same as in Theorem 12 while $s = (n-1)^{-1}(nx - v)$, we get $nx = \sum_{i=1}^{n} v_i$ for some extremes v_is in $\mathcal{J}$.

(ii) Suppose $x \in (\mathcal{J}_1(v))_1^\circ$. Then $\|x\| < 1 - 2n^{-1}$ for some positive integer $n \geq 3$. Therefore, $x \in \mathrm{co}\mathcal{E}(\mathcal{J})_1$ by the part (i).

(iii) Using the part (ii), we get $(\mathcal{J}_1(v))_1 = \overline{(\mathcal{J}_1(v))_1^\circ} \subseteq \overline{\mathrm{co}}\mathcal{E}(\mathcal{J})_1$. □

Remark 17. If a JB^*-triple $\mathcal{J}$ has a BP-quasi invertible element then $\mathcal{E}(\mathcal{J})_1$ is a nonempty set by Theorem 6. Hence, the above theorem holds in this case.

Corollary 18. *For any extreme point v of the closed unit ball in a JB^*-triple $\mathcal{J}$, $(\mathcal{J}_1(v))_1 \subset \overline{\mathrm{co}}\mathcal{E}(\mathcal{J})_1$. If v is a unitary element in $\mathcal{J}$, then $(\mathcal{J})_1 = \overline{\mathrm{co}}\mathcal{E}(\mathcal{J})_1 = \overline{\mathrm{co}}\mathcal{U}(\mathcal{J})$. Moreover, $\bigcup_{w \in \mathcal{E}(\mathcal{J})_1}(\mathcal{J}_1(w))_1 = \overline{\mathrm{co}}\mathcal{U}(\mathcal{J}) = \overline{\mathrm{co}}\mathcal{E}(\mathcal{J})_1 = (\mathcal{J})_1$.*

Proof. We know from [20, Lemma 4] that $\mathcal{U}(\mathcal{J}_1(v)) \subseteq \mathcal{E}(\mathcal{J})_1$. So, by the Russo-Dye Theorem for JB^*-algebras [7, Theorem 2.3], $(\mathcal{J}_1(v))_1 = \overline{(\mathcal{J}_1(v))_1^\circ} = \overline{\mathrm{co}}\mathcal{U}(\mathcal{J}_1(v)) \subseteq \overline{\mathrm{co}}\mathcal{E}(\mathcal{J})_1$. By Example 15, there exists a JB^*-triple $\mathcal{J}$ with $u, v \in \mathcal{E}(\mathcal{J})_1$ such that $u \notin \mathcal{J}_1(v)$. Hence, we have $(\mathcal{J}_1(v))_1 \subset \overline{\mathrm{co}}\mathcal{E}(\mathcal{J})_1$.

If $v \in \mathcal{U}(\mathcal{J})$, then $L_{v,v} = I$ (the identity operator on $\mathcal{J}$) (cf. [3]). So, $\mathcal{J} = \mathcal{J}_1(v)$ which is a unital JB^*-algebra, and hence $\mathcal{J}$ is a JB^*-algebra. Then, $(\mathcal{J})_1 = (\mathcal{J}_1(v))_1 \subseteq \overline{\mathrm{co}}\mathcal{E}(\mathcal{J})_1 \subseteq (\mathcal{J})_1 = \overline{\mathrm{co}}\mathcal{U}(\mathcal{J})$ by [7, Theorem 2.3]. Hence, $(\mathcal{J})_1 = \overline{\mathrm{co}}\mathcal{E}(\mathcal{J})_1 = \overline{\mathrm{co}}\mathcal{U}(\mathcal{J})$.

As mentioned above, every unitary element v in a JB^*-triple $\mathcal{J}$ satisfies $L_{v,v} = I$, and so $P(v)P(v) = I$ (cf. [10, page 582]). Hence, v is an extreme point of $(\mathcal{J})_1$ by [23, Lemma 3.2 and Proposition 3.5]. Thus, for any $v \in \mathcal{U}(\mathcal{J})$, $(\mathcal{J})_1 = (\mathcal{J}_1(v))_1 \subseteq \bigcup_{w \in \mathcal{E}(\mathcal{J})_1}(\mathcal{J}_1(w))_1 \subseteq (\mathcal{J})_1 = \overline{\mathrm{co}}\mathcal{U}(\mathcal{J})$. Therefore, $\bigcup_{w \in \mathcal{E}(\mathcal{J})_1}(\mathcal{J}_1(w))_1 = \overline{\mathrm{co}}\mathcal{U}(\mathcal{J}) = \overline{\mathrm{co}}\mathcal{E}(\mathcal{J})_1 = (\mathcal{J})_1$. □

Corollary 19. *Each element of the Peirce 1-space $\mathcal{J}_1(v)$ in a JB^*-triple $\mathcal{J}$ with $v \in \mathcal{E}(\mathcal{J})_1$ is a positive multiple of the sum of three extreme points of $(\mathcal{J})_1$.*

Proof. Let $x \in \mathcal{J}_1(v)$ and $\epsilon > 0$. Let $y = (3\|x\| + \epsilon)^{-1}x$. Then $\|y\| < 1/3$. Hence, by Theorem 16(i), there exist three extreme points u_1, u_2, u_3 in $\mathcal{J}$ such that $y = (1/3)(u_1 + u_2 + u_3)$. Thus, $x = (\|x\| + \epsilon/3)(u_1 + u_2 + u_3)$. □

5. Convex Combinations of Extreme Points

We continue investigating convex combinations, not necessarily means, of extreme points of the closed unit ball. The following result gives an analogue of [7, Lemma 2.1] for BP-quasi invertible elements in a JB^*-triple.

Theorem 20. *In any JB^*-triple $\mathcal{J}$, $\mathcal{J}_q^{-1} \cap (\mathcal{J})_1 \subseteq (1/2)(\mathcal{E}(\mathcal{J})_1 + \mathcal{E}(\mathcal{J})_1)$.*

Proof. Let $x \in \mathcal{J}_q^{-1} \cap (\mathcal{J})_1$. Then, there is a unique $v \in \mathcal{E}(\mathcal{J})_1$ such that x is positive and invertible in $\mathcal{J}_1(v)$ by Theorem 6. In particular, x is self-adjoint in the JB^*-algebra $\mathcal{J}_1(v)$. This together with $\|x\| \leq 1$ gives the existence of unitaries $v_1, v_2 \in \mathcal{J}_1(v)$ satisfying $x = (1/2)(v_1 + v_2)$ by [24, Theorem 2.11]. The result now follows from [20, Lemma 4]. □

The unit ball of a JB^*-triple $\mathcal{J}$ often has extreme points in abundance, as in the case when $\mathcal{J}$ is a Hilbert space H; however, the set $\mathcal{E}(\mathcal{J})_1$ may be empty as in the JB^*-triple $\mathcal{J} = C_\circ(H)$ of compact operators on H [25, page 151].

Theorem 21. *Let $\mathcal{J}$ be a JB^*-triple and $v \in \mathcal{E}(\mathcal{J})_1$. Let $x \in (\mathcal{J}_1(v))_1$ with $\mathrm{dist}(x, \mathcal{U}(\mathcal{J}_1(v)) < 2\alpha$ and $\alpha < 1/2$. Then $x \in \alpha\mathcal{E}(\mathcal{J})_1 + (1-\alpha)\mathcal{E}(\mathcal{J})_1$.*

Proof. Since $\mathcal{J}_1(v)$ is a JB^*-algebra with unit v, $x \in \alpha\mathcal{U}(\mathcal{J}_1(v)) + (1-\alpha)\mathcal{U}(\mathcal{J}_1(v))$ by [26, Corollary 3.4]. Hence, $x \in \alpha\mathcal{E}(\mathcal{J})_1 + (1-\alpha)\mathcal{E}(\mathcal{J})_1$ by [20, Lemma 4]. □

Let $\mathcal{J}$ be a JB^*-triple and $x \in (\mathcal{J})_1$. We define $V(x) := \{\beta \geq 1 : x \in \mathrm{co}_\beta\mathcal{E}(\mathcal{J})_1\}$, where $\mathrm{co}_\beta\mathcal{E}(\mathcal{J})_1 := \{\beta^{-1}\sum_{i=1}^{n-1} v_i + \beta^{-1}(1 + \beta - n)v_n : v_j \in \mathcal{E}(\mathcal{J})_1, j = 1, \ldots, n\}$ in which the positive integer n satisfies the condition $n - 1 < \beta \leq n$. We observe the following immediate connection of these constructs with the number $e_m(x)$.

Lemma 22. *Let $\mathcal{J}$ be a JB^*-triple. Let $x \in \mathcal{J}$ and $n \in \mathbb{N}$ (the set of positive integers). Then*

(i) $x \in \mathrm{co}_n\mathcal{E}(\mathcal{J})_1 \Leftrightarrow e_m(x) \leq n$.

(ii) $n \in V(x) \Leftrightarrow e_m(x) \leq n$.

(iii) $e_m(x) = \min(V(x) \cap \mathbb{N})$.

An extension of [27, Proposition 3.1] for unital JB^*-algebras appeared in [28, Theorem 2.2]. We use Theorem 6, to deduce an analogue of the same result for JB^*-triples, as follows.

Theorem 23. *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$ and $x \in (\mathcal{J})_1 \cap \mathcal{J}_1(v)$.*

(a) *Let $\|\gamma x - v_0\| \leq \gamma - 1$ for some $\gamma \geq 1$ and some $v_0 \in \mathcal{U}(\mathcal{J}_1(v))$. Let $(\delta_2, \ldots, \delta_m) \in \mathbb{R}^{m-1}$ such that $0 \leq \delta_j < \gamma - 1$ for all $j = 2, \ldots, m-1$ and $\gamma^{-1} + \sum_{j=2}^{m-1} \delta_j = 1$. Then there exist $v_1, \ldots, v_m \in \mathcal{E}(\mathcal{J})_1$ satisfying $x = \gamma^{-1}v_1 + \sum_{j=2}^{m} \delta_j v_j$. Moreover, $]\gamma, \infty[\subseteq V(x)$.*

(b) *On the other hand, if $]\gamma, \infty[\subseteq V(x)$ holds, then for each $r > \gamma$, there is $v_1 \in \mathcal{E}(\mathcal{J})_1$ such that $\|rx - v_1\| \leq r - 1$.*

Proof. Since $\mathcal{J}_1(v)$ is a unital JB^*-algebra, the part (a) follows from [20, Theorem 2.2] and [20, Lemma 4].

(b) Suppose $]\gamma, \infty[\subseteq V(x)$. If $r > \gamma$ then $x \in \mathrm{co}_r\mathcal{E}(\mathcal{J})_1$ with $x = r^{-1}(v_1 + \cdots + v_{n-1} + (1 + r - n)v_n)$ for some $v_1, \ldots, v_n \in \mathcal{E}(\mathcal{J})_1$ and positive integer n satisfying $n - 1 < r \leq n$. Hence, $\|rx - v_1\| = \|\sum_{j=2}^{n-1} v_j + (1 + r - n)v_n\| \leq (n - 2) + (1 + r - n) = r - 1$. □

Corollary 24. *For any JB^*-triple $\mathcal{J}$, $\mathrm{co}_\gamma\mathcal{E}(\mathcal{J})_1 \subseteq \mathrm{co}_\delta\mathcal{E}(\mathcal{J})_1$ if $1 \leq \gamma \leq \delta$. In particular, for each $x \in (\mathcal{J})_1 \cap \mathcal{J}_1(v)$, $V(x)$ is either empty or equal to $[\gamma, \infty)$ or (γ, ∞) for some $\gamma \geq 1$.*

6. Distance to the Extreme Points

In this section, we prove some results on distances from an element of a JB^*-triple $\mathcal{J}$ to the set $\mathcal{E}(\mathcal{J})_1$ and to the set $\mathcal{J}_q^{-1}$. We define the function $\alpha_q : \mathcal{J} \to \mathbb{R}$ by $\alpha_q(x) := \inf\{\|x - u\| : u \in \mathcal{J}_q^{-1}\}$.

Lemma 25. *Let $\mathcal{J}$ be a JB^*-triple with nonempty $\mathcal{E}(\mathcal{J})_1$ and $x \in \mathcal{J}$. Then*

(i) $\alpha_q(rx) = |r|\alpha_q(x)$ *for all* $r \in \mathbb{C}$;

(ii) $\alpha_q(x) \leq \|x\|$;

(iii) $|\alpha_q(x) - \alpha_q(y)| \leq \|x - y\|$ *for all* $y \in \mathcal{J}$;

(iv) *if $\mathcal{J}$ is a JB^*-algebra, then* $\alpha_q(x) = \alpha_q(x^*)$.

Proof. (i) For any complex number $r \neq 0$, $x \in \mathcal{J}_q^{-1}$ if and only if $rx \in \mathcal{J}_q^{-1}$. So, the part (i) is clear for all nonzero complex numbers. If $r = 0$, then $\alpha_q(rx) = \alpha_q(0) = 0$, one may consider the BP-quasi invertible element $(1/n)v$ with $v \in \mathcal{E}(\mathcal{J})_1$ and positive integer $n \to 0$.

(ii) Since for any $v \in \mathcal{E}(\mathcal{J})_1$ and any positive integer n, $(1/n)v \in \mathcal{J}_q^{-1}$; hence, $\alpha_q(x) = \mathrm{dist}(x, \mathcal{J}_q^{-1}) \leq \|x - (1/n)v\| \leq \|x\| + 1/n$. It follows that $\alpha_q(x) \leq \|x\|$.

(iii) Since the set $\mathcal{E}(\mathcal{J})_1$ is nonempty, and since any extreme point is BP-quasi invertible, $\alpha(x) < \infty$. Now, by definition of $\alpha_q(x)$, for each $n \in \mathbb{N}$, there exists an element $x_n \in \mathcal{J}_q^{-1}$ such that $\|x - x_n\| \leq \alpha_q(x) + 1/n$. Hence, $\alpha_q(y) \leq \|y - x_n\| \leq \|y - x\| + \|x - x_n\| \leq \|x - y\| + \alpha_q(x) + 1/n$ for all $n \in \mathbb{N}$, so that $\alpha_q(y) \leq \|x - y\| + \alpha_q(x)$. Interchanging x and y, we get $\alpha_q(x) \leq \|x - y\| + \alpha_q(y)$. Thus, $|\alpha_q(x) - \alpha_q(y)| \leq \|x - y\|$ for all $y \in \mathcal{J}$.

(iv) Now, let the $\mathcal{J}$ be a JB^*-algebra. If $x \in \mathcal{J}_q^{-1}$, then $x^* \in \mathcal{J}_q^{-1}$ by [4, Theorem 3.2], and hence $\alpha_q(x) = 0 = \alpha_q(x^*)$. Next, if x is not BP-quasi invertible, then $\alpha_q(x) = \mathrm{dist}(x, \mathcal{J}_q^{-1}) = \inf\{\|x - u\| : u \in \mathcal{J}_q^{-1}\} = \inf\{\|x^* - u\| : u \in \mathcal{J}_q^{-1}\} = \alpha_q(x^*)$ since the involution "$*$" is an isometry. □

$\alpha_q(x)$ provides information about $\mathrm{dist}(x, \mathcal{E}(\mathcal{J})_1)$ as follows.

Theorem 26. *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$ and $x \in \mathcal{J}_1(v) \setminus \mathcal{J}_q^{-1}$. Then* $\mathrm{dist}(x, \mathcal{E}(\mathcal{J})_1) \geq \max\{\alpha_q(x) + 1, \|x\| - 1\}$.

Proof. Since $v \in \mathcal{E}(\mathcal{J})_1$, $\|x - v\| \geq \|x\| - 1$. We assume $\|x - v\| < 1$, then the fact that v is the unit of the JB^*-algebra $\mathcal{J}_1(v)$ gives the invertibility of x in $\mathcal{J}_1(v)$, and so x is positive and invertible in the unitary isotope of $\mathcal{J}_1(v)$ induced by certain unitary element u in $\mathcal{J}_1(v)$; hence, $u \in \mathcal{E}(\mathcal{J})_1$ by [20, Lemma 4]. For any $z \in \mathcal{J}_1(v)^{[u]}$, we have $L(v, v)z = z$ and $L_v(u, u) = I$ on $\mathcal{J}_1(v)$ since u is unitary in $\mathcal{J}_1(v)$, so that $L(u, u)z = L_v(u, u)z = z$ by Theorem 2. Hence, $\mathcal{J}_1(v)^{[u]} \subseteq \mathcal{J}_1(u)$. For the reverse inclusion, recall that $u \in \mathcal{E}(\mathcal{J})_1$, and so $\mathcal{J}_1(u) = P(u)(\mathcal{J})$ (see the above proof of Theorem 4); similarly, $\mathcal{J}_1(v) = P(v)(\mathcal{J})$, and so $u = P(v)w$ for some $w \in \mathcal{J}$. Hence, for any fixed $z \in \mathcal{J}_1(u)$, there exists $s \in \mathcal{J}$ with $z = P(u)s = P(P(v)w)s = P(v)P(w)P(v)s \in P(v)\mathcal{J} = \mathcal{J}_1(v)$. Thus, $\mathcal{J}_1(u) \subseteq \mathcal{J}_1(v)^{[u]}$. Therefore, $\mathcal{J}_1(v)^{[u]} = \mathcal{J}_1(u)$ as sets. Moreover, we observe that $a \circ_{v_u} b = \{au^{*_v}b\}_v = \{au^*b\} = a \circ_u b$ and $a^{*_{v_u}} = \{ua^{*_v}u\}_v = \{ua^*u\} = a^{*_u}$ by Theorem 2. We conclude that $(\mathcal{J}_1(v))^{[u]}$ and $\mathcal{J}_1(u)$ coincide as JB^*-triples and x is positive invertible in the Peirce 1-space $\mathcal{J}_1(u)$. So, $x \in \mathcal{J}_q^{-1}$ by Theorem 10; a contradiction to the hypothesis. Thus, $\|x - v\| \geq 1$.

By setting $y = \|x - v\|^{-1}(x - v) + v$, we get $y \in \mathcal{J}_1(v)$ since $x, v \in \mathcal{J}_1(v)$. Then $\|y - v\| = \|x - v\|^{-1}\|x - v\| = 1$. However, the open unit ball $\mathfrak{B}(v; 1)$ in $\mathcal{J}_1(v)$ about v is included in $(\mathcal{J}_1(v))^{-1}$ (cf. [22, Lemma 2.1]). Therefore, $y \in \overline{(\mathcal{J}_1(v))^{-1}}$. From our above discussion, we also have $(\mathcal{J}_1(v))^{-1} \subseteq \mathcal{J}_q^{-1}$. Thus, $\alpha_q(x) = \inf_{z \in \mathcal{J}_q^{-1}}\|x - z\| \leq \inf_{w \in (\mathcal{J}_1(v))^{-1}}\|x - w\| \leq \|x - y\| = \|x - \|x - v\|^{-1}(x - v) - v\| = \|x - v\|(1 - \|x - v\|^{-1}) = \|x - v\| - 1$ as $\|x - v\| \geq 1$; which means $\alpha_q(x) + 1 \leq \|x - v\|$. However, $\|x\| - 1 \leq \|x - v\|$, as seen above. We conclude that $\mathrm{dist}(x, \mathcal{E}(\mathcal{J})_1) \geq \max\{\alpha_q(x) + 1, \|x\| - 1\}$. □

At present, we do not know whether inequality in the above theorem can be replaced with equality. To give some partial affirmative answer to this question, we need the following result.

Theorem 27. *If $\mathcal{J}$ is a JB^*-triple, then* $\operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1) \leq \max\{1, \|x\| - 1\}$ *for all* $x \in \overline{\mathcal{J}_q^{-1}}$.

Proof. Let $x \in \mathcal{J}_q^{-1}$. Then by Theorem 6, x is positive invertible in the JB^*-algebra $\mathcal{J}_1(v)$ for some $v \in \mathcal{E}(\mathcal{J})_1$, which is a JB^*-algebra with unit v. Hence, x being positive satisfies $\|x - v\| = \operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1)$ by [26, Proposition 3.2]. By the functional calculus for positive elements in JB^*-algebras,

$$\|x - v\| \leq \begin{cases} 1 & \text{if } \|x\| \leq 2; \\ \|x\| - 1 & \text{if } \|x\| \geq 2. \end{cases} \tag{2}$$

Hence, $\operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1) \leq \max\{1, \|x\| - 1\}$ for all $x \in \mathcal{J}_q^{-1}$. Next, suppose (x_n) is a sequence in $\mathcal{J}_q^{-1}$ such that $\|x_n - x\| < (1/n)$ for all $n \in \mathbb{N}$. By Theorem 6, there exists a unique $v_n \in \mathcal{E}(\mathcal{J})_1$ corresponding to each x_n. Hence, $\operatorname{dist}(x_n, \mathcal{E}(\mathcal{J})_1) \leq \|x_n - v_n\|$ for all n. By (2), $\|x_n - v_n\| \leq \max\{1, \|x_n\| - 1\}$ for all n. Hence, for large n, we get $\operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1) \leq \|x - v_n\| = \|x_n - v_n\| + \|x - x_n\| \leq \max\{1, \|x_n\| - 1\} + (1/n)$ by using (2). It follows that $\operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1) \leq \max\{1, \|x\| - 1\}$. □

The following result gives some cases in which the inequality appearing in Theorem 26 becomes equality.

Theorem 28. *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$, and $x \in \mathcal{J}_1(v) \setminus \mathcal{J}_q^{-1}$.*

(i) *If $\alpha_q(x) = \|x\|$ then* $\operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1) = \alpha_q(x) + 1$. *In particular, $\alpha_q(x) = \|x\| = 1$ implies* $\operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1) = 2$.

(ii) *If $\alpha_q(x) < \|x\| = 1$, then* $\operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1) = \alpha_q(x) + 1$.

Proof. (i) Since $0 \in \overline{\mathcal{J}_q^{-1}}$, $\operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1) \leq \|x - 0\| + \operatorname{dist}(0, \mathcal{E}(\mathcal{J})_1) \leq \alpha_q(x) + \max\{1, \|0\| - 1\} = \alpha_q(x) + 1$ by Theorem 27. On the other hand, $\operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1) \geq \alpha_q(x) + 1$ by Theorem 26. Hence, the part (i) is proved.

(ii) Let $\epsilon > 0$ with $\alpha_q(x) + \epsilon < 1$. Then there exists $y \in \mathcal{J}_q^{-1}$ such that $\|y - x\| < \alpha_q(x) + \epsilon$. So, $\|y\| \leq \|y - x\| + \|x\| < \alpha_q(x) + \epsilon + 1 < 2$. Since $y \in \mathcal{J}_q^{-1}$, y is positive and invertible in $\mathcal{J}_1(v)$ for some $v \in \mathcal{E}(\mathcal{J})_1$ by Theorem 6; $\mathcal{J}_1(v)$ is a JB^*-algebra with unit v. By the continuous functional calculus, we get $\|y - v\| \leq \|y\| - 1 < 2 - 1 = 1$ since $\|y\| < 2$. Therefore, $\|x - v\| \leq \|x - y\| + \|y - v\| < \alpha_q(x) + \epsilon + 1$. Hence, $\operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1) \leq \alpha_q(x) + 1$. Moreover, $\operatorname{dist}(x, \mathcal{E}(\mathcal{J})_1) \geq \alpha_q(x) + 1$ by Theorem 26. □

The next result gives some conditions sufficient for the existence of extreme point approximants.

Corollary 29. *Let $\mathcal{J}$ be a JB^*-triple, $v \in \mathcal{E}(\mathcal{J})_1$ and $x \in \mathcal{J}$ with $\|x\| \geq 2$.*

(i) *If x is positive in $\mathcal{J}_1(v)$, then $\|x - v\| = \|x\| - 1$.*

(ii) *If $x \in \mathcal{J}_q^{-1}$, then there is $u \in \mathcal{E}(\mathcal{J})_1$ such that $\|x - u\| = \|x\| - 1$.*

Proof. (i) Since v being an extreme point of the closed unit ball, v is a tripotent in $\mathcal{J}$, $v = P_v v = \{vv^*v\}$. Hence, $\|v\| = \|\{vv^*v\}\| = \|v\|^3$, so that $\|v\| = 1$. Now, since x is positive in $\mathcal{J}_1(v)$ and since $\|x\| \geq 2$, we get by the functional calculus for positive elements that $\|x - v\| = \|x\| - 1$.

(ii) Suppose $x \in \mathcal{J}_q^{-1}$. By Theorem 6, x is positive in $\mathcal{J}_1(v)$ for some $v \in \mathcal{E}(\mathcal{J})_1$. Hence, $\|x - u\| = \|x\| - 1$ by Part (i). □

Acknowledgments

This work was supported by King Saud University, Deanship of Scientific Research, College of Science, Research Center. The authors thank the anonymous referees for their helpful comments for improving the presentation of this work.

References

[1] L. G. Brown and G. K. Pedersen, "On the geometry of the unit ball of a C^*-algebra," *Journal für die Reine und Angewandte Mathematik*, vol. 469, pp. 113–147, 1995.

[2] L. G. Brown and G. K. Pedersen, "Approximation and convex decomposition by extremals in a C^*-algebra," *Mathematica Scandinavica*, vol. 81, no. 1, pp. 69–85, 1997.

[3] H. Upmeier, *Symmetric Banach manifolds and Jordan C^*-algebras*, vol. 104 of *North-Holland Mathematics Studies*, North-Holland Publishing, Amsterdam, The Netherlands, 1985.

[4] H. M. Tahlawi and A. A. Siddiqui, "Analogue of Brown-Pedersen' quasi invertibility for JB^*-triples," *International Journal of Mathematical Analysis*, vol. 5, no. 30, pp. 1469–1476, 2011.

[5] H. M. Tahlawi and A. A. Siddiqui, "On non-degenerate Jordan triple systems," *International Journal of Algebra*, vol. 5, no. 21-24, pp. 1099–1105, 2011.

[6] J. D. M. Wright and M. A. Youngson, "A Russo-Dye theorem for Jordan C^*-algebras," in *Functional Analysis: Surveys and Recent Results*, vol. 27 of *Mathematical Studies*, pp. 279–282, North-Holland, Amsterdam, The Netherlands, 1977.

[7] A. A. Siddiqui, "A proof of the Russo-Dye theorem for JB^*-algebras," *New York Journal of Mathematics*, vol. 16, pp. 53–60, 2010.

[8] W. Kaup, "A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces," *Mathematische Zeitschrift*, vol. 183, no. 4, pp. 503–529, 1983.

[9] M. Burgos, F. J. Fernández-Polo, J. J. Garcés, and A. M. Peralta, "Orthogonality preservers revisited," *Asian-European Journal of Mathematics*, vol. 2, no. 3, pp. 387–405, 2009.

[10] W. Kaup, "On Grassmannians associated with JB^*-triples," *Mathematische Zeitschrift*, vol. 236, no. 3, pp. 567–584, 2001.

[11] W. Kaup, "On spectral and singular values in JB^*-triples," *Proceedings of the Royal Irish Academy. Section A*, vol. 96, no. 1, pp. 95–103, 1996.

[12] O. Loos, *Jordan Pairs*, Lecture Notes in Mathematics, Springer, New York, NY, USA, 1975.

[13] K. Meyberg, "von Neumann regularity in Jordan triple systems," *Archiv der Mathematik*, vol. 23, pp. 589–593, 1972.

[14] W. Kaup, "On the CR-structure of certain linear group orbits in infinite dimensions," *Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V*, vol. 3, no. 3, pp. 535–554, 2004.

[15] A. Fernández López, E. García Rus, E. Sánchez Campos, and M. Siles Molina, "Strong regularity and generalized inverses in

Jordan systems," *Communications in Algebra*, vol. 20, no. 7, pp. 1917–1936, 1992.

[16] L. J. Bunce, C. H. Chu, and B. Zalar, "Structure spaces and decomposition in JB^*-triples," *Mathematica Scandinavica*, vol. 86, no. 1, pp. 17–35, 2000.

[17] C. M. Edwards and G. T. Rüttimann, "Compact tripotents in bi-dual JB^*-triples," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 120, no. 1, pp. 155–173, 1996.

[18] L. L. Stachó, "On the spectrum of inner derivations in partial Jordan triples," *Mathematica Scandinavica*, vol. 66, no. 2, pp. 242–248, 1990.

[19] M. Burgos, E. A. Kaidi, A. M. Campoy, A. M. Peralta, and M. Ramírez, "Von Neumann regularity and quadratic conorms in JB^*-triples and C^*-algebras," *Acta Mathematica Sinica*, vol. 24, no. 2, pp. 185–200, 2008.

[20] A. A. Siddiqui, "Average of two extreme points in JBW^*-triples," *Japan Academy. Proceedings. Series A*, vol. 83, no. 9-10, pp. 176–178, 2007.

[21] M. A. Rakha, "On the Moore-Penrose generalized inverse matrix," *Applied Mathematics and Computation*, vol. 158, no. 1, pp. 185–200, 2004.

[22] A. A. Siddiqui, "JB^*-algebras of topological stable rank 1," *International Journal of Mathematics and Mathematical Sciences*, Article ID 37186, 24 pages, 2007.

[23] W. Kaup and H. Upmeier, "Jordan algebras and symmetric Siegel domains in Banach spaces," *Mathematische Zeitschrift*, vol. 157, no. 2, pp. 179–200, 1977.

[24] A. A. Siddiqui, "Self-adjointness in unitary isotopes of JB^*-algebras," *Archiv der Mathematik*, vol. 87, no. 4, pp. 350–358, 2006.

[25] J. M. Isidro, "A glimpse at the theory of Jordan-Banach triple systems," *Revista Matemática de la Universidad Complutense de Madrid*, vol. 2, pp. 145–156, 1989.

[26] A. A. Siddiqui, "Convex combinations of unitaries in JB^*-algebras," *New York Journal of Mathematics*, vol. 17, pp. 127–137, 2011.

[27] M. Rørdam, "Advances in the theory of unitary rank and regular approximation," *Annals of Mathematics*, vol. 128, no. 1, pp. 153–172, 1988.

[28] A. A. Siddiqui, "The λ_u-function in JB^*-algebras," *New York Journal of Mathematics*, vol. 17, pp. 139–147, 2011.