Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 879084, 7 pages
http://dx.doi.org/10.1155/2013/879084

Research Article

Common Fixed Points for Weak ψ-Contractive Mappings in Ordered Metric Spaces with Applications

Sumit Chandok[1] and Simona Dinu[2]

[1] *Department of Mathematics, Khalsa College of Engineering & Technology (Punjab Technical University), Ranjit Avenue, Amritsar 143001, India*
[2] *Faculty of Applied Sciences, University Politehnica of Bucharest, 313 Splaiul Independentei, 060042 Bucharest, Romania*

Correspondence should be addressed to Simona Dinu; simogrigo@yahoo.com

Received 7 July 2013; Accepted 29 August 2013

Academic Editor: Lishan Liu

We obtain some new common fixed point theorems satisfying a weak contractive condition in the framework of partially ordered metric spaces. The main result generalizes and extends some known results given by some authors in the literature.

1. Introduction and Preliminaries

Fixed point and common fixed point theorems for different types of nonlinear contractive mappings have been investigated extensively by various researchers (see [1–41]). Fixed point problems involving weak contractions and the mappings satisfying weak contractive type inequalities have been studied by many authors (see [10–20] and references cited therein).

Recently, many researchers have obtained fixed point, common fixed point, coupled fixed point, and coupled common fixed point results in partially ordered metric spaces (see [3, 6–8, 10–12, 29, 30, 32, 36]) and other spaces (see [5, 15, 31, 35, 38, 40, 41]).

Let $(X, \leq)$ be a partially ordered set and f, g two self-mappings on X. A pair (f, g) of self-mappings of X is said to be *weakly increasing* [4] if $fx \leq gfx$ and $gx \leq fgx$ for any $x \in X$. An ordered pair (f, g) is said to be *partially weakly increasing* if $fx \leq gfx$ for all $x \in X$.

Note that a pair (f, g) is weakly increasing if and only if the ordered pairs (f, g) and (g, f) are partially weakly increasing.

Example 1 (see [3]). Let $X = [0, 1]$ be endowed with usual ordering and $f, g : X \to X$ two mappings given by $fx = x^2$ and $gx = \sqrt{x}$. Clearly, the pair (f, g) is partially weakly increasing. But $gx = \sqrt{x} \nleq x = fgx$ for any $x \in (0, 1)$ implies that the pair (g, f) is not partially weakly increasing.

Let $(X, \leq)$ be a partially ordered set. A mapping $f : X \to X$ is called a *weak annihilator* of a mapping $g : X \to X$ if $fgx \leq x$ for all $x \in X$.

Example 2 (see [3]). Let $X = [0, 1]$ be endowed with usual ordering and $f, g : X \to X$ be two mappings given by $fx = x^2$ and $gx = x^3$. It is clear that $fgx = x^6 \leq x$ for $x \in X$ implies that f is a weak annihilator of g.

Let $(X, \leq)$ be a partially ordered set. A mapping f is called a *dominating* if $x \leq fx$ for any $x \in X$.

Example 3 (see [3]). Let $X = [0, 1]$ be endowed with usual ordering and $f : X \to X$ a mapping defined by $fx = x^{1/3}$, since $x \leq x^{1/3} = fx$ for $x \in X$ implies that f is a dominating mapping.

A subset W of a partially ordered set X is said to be *well ordered* if every two elements of W are comparable.

Let M be a nonempty subset of a metric space (X, d). Let S and T be mappings from a metric space (X, d) into itself. A point $x \in M$ is a *common fixed* (*resp., coincidence*) *point* of S and T if $x = Sx = Tx$ (resp., Sx = Tx). The set of fixed points (resp., coincidence points) of S and T is denoted by $F(S,T)$ (resp., $C(S,T)$).

In 1986, Jungck [24] introduced the more generalized commuting mappings in metric spaces, called compatible mappings, which also are more general than the concept of weakly commuting mappings (that is, the mappings $S, T : X \to X$ are said to be *weakly commuting* if $d(STx, TSx) \leq d(Sx, Tx)$ for all $x \in X$) introduced by Sessa [34] as follows.

Definition 4. Let S and T be mappings from a metric space (X, d) into itself. The mappings S and T are said to be *compatible* if

$$\lim_{n\to\infty} d\left(STx_n, TSx_n\right) = 0, \tag{1}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converses are not necessarily true and some examples can be found in [24–26].

In [27], Jungck and Rhoades introduced the concept of weakly compatible mappings and proved some common fixed point theorems for these mappings.

Definition 5. The mappings S and T are said to be *weakly compatible* if they commute at coincidence points of S and T.

In Djoudi and Nisse [21], we can find an example to show that there exists weakly compatible mappings which are not compatible mappings in metric spaces.

Let Ψ denote the set of all functions $\psi : [0,\infty)^5 \to [0,\infty)$ such that

(a) ψ is continuous;

(b) ψ is strictly increasing in all the variables;

(c) for all $t \in [0,\infty) \setminus \{0\}$,

$$\begin{gathered} \psi(t,t,t,0,2t) < t, \qquad \psi(t,t,t,2t,0) < t, \\ \psi(0,0,t,t,0) < t, \qquad \psi(0,t,0,0,t) < t, \\ \psi(t,0,0,t,t) < t. \end{gathered} \tag{2}$$

It is easy to verify that the following functions are from the class Ψ, see [18]:

$$\begin{aligned} &\psi\left(t_1,t_2,t_3,t_4,t_5\right) = k\max\left\{t_1,t_2,t_3,\frac{t_4}{2},\frac{t_5}{2}\right\}, \\ &\qquad\qquad \text{for } k \in (0,1); \\ &\psi\left(t_1,t_2,t_3,t_4,t_5\right) = k\max\left\{t_1,t_2,t_3,\frac{t_4+t_5}{2}\right\}, \\ &\qquad\qquad \text{for } k \in (0,1). \end{aligned} \tag{3}$$

Definition 6 (see [18]). Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a metric space. The mapping $f : X \to X$ is said to be a ψ-*contractive* mapping, if

$$\begin{aligned} d\left(fx,fy\right) \leq \psi(&d\left(x,y\right), d\left(x,fx\right), d\left(y,fy\right), \\ &d\left(x,fy\right), d\left(y,fx\right)), \end{aligned} \tag{4}$$

for $x \geq y$.

Recently, Chen introduced ψ-contractive mappings. The purpose of this paper is to extend the results of Chen for four mappings, in the framework of ordered metric spaces.

2. Main Results

Now, we give the main results in this paper.

Theorem 7. *Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T, f, g, and S are self-mappings on X, the pairs (T, f) and (S, g) are partially weakly increasing with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, and the dominating mappings f and g are weak annihilators of T and S, respectively. Further, suppose that for any two comparable elements $x, y \in X$, and $\psi \in \Psi$,*

$$\begin{aligned} d\left(fx,gy\right) \leq \psi(&d\left(Sx,Ty\right), d\left(Sx,fx\right), d\left(Ty,gy\right), \\ &d\left(Sx,gy\right), d\left(Ty,fx\right)), \end{aligned} \tag{5}$$

holds. If, for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all $n \geq 1$, $y_n \to u$ implies that $x_n \leq u$ and either

(a) *f and S are compatible, f or S is continuous, and g, T are weakly compatible or*

(b) *g and T are compatible, g or T is continuous, and f, S are weakly compatible,*

then f, g, S, and T have a common fixed point in X. Moreover, the set of common fixed points of f, g, S, and T is well ordered if and only if f, g, S, and T have one and only one common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n-1} = fx_{2n-2} = Tx_{2n-1}, \qquad y_{2n} = gx_{2n-1} = Sx_{2n}, \tag{6}$$

for each $n \geq 1$. By assumptions, we have

$$\begin{gathered} x_{2n-2} \leq fx_{2n-2} = Tx_{2n-1} \leq fTx_{2n-1} \leq x_{2n-1}, \\ x_{2n-1} \leq gx_{2n-1} = Sx_{2n} \leq gSx_{2n} \leq x_{2n}, \end{gathered} \tag{7}$$

for each $n \geq 1$. Thus, for each $n \geq 1$, we have $x_n \leq x_{n+1}$. Without loss of generality, we assume that $y_{2n} \neq y_{2n+1}$ for each $n \geq 1$.

Now, we claim that for all $n \in \mathbb{N}$, we have

$$d\left(y_{n+1}, y_{n+2}\right) < d\left(y_n, y_{n+1}\right). \tag{8}$$

Suppose to the contrary that $d(y_{2n}, y_{2n+1}) \leq d(y_{2n+1}, y_{2n+2})$ for some $n \in \mathbb{N}$. Since y_{2n} and y_{2n+1} are comparable, from (5), we have

$$\begin{aligned}
&d(y_{2n+1}, y_{2n+2}) \\
&= d(fx_{2n}, gx_{2n+1}) \\
&\leq \psi(d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Tx_{2n+1}, gx_{2n+1}), \\
&\qquad d(Sx_{2n}, gx_{2n+1}), d(Tx_{2n+1}, fx_{2n})) \\
&= \psi(d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \\
&\qquad d(y_{2n}, y_{2n+2}), d(y_{2n+1}, y_{2n+1})) \\
&= \psi(d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), \\
&\qquad d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+2}), 0) \\
&\leq \psi(d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \\
&\qquad d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}), 0) \\
&\leq \psi(d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}), \\
&\qquad d(y_{2n+1}, y_{2n+2}), 2d(y_{2n+1}, y_{2n+2}), 0) \\
&< d(y_{2n+1}, y_{2n+2}),
\end{aligned} \tag{9}$$

which is a contradiction. Hence $d(y_{2n+1}, y_{2n+2}) \leq d(y_{2n}, y_{2n+1})$ for each $n \geq 1$.

Similarly, we can prove that $d(y_{2n+1}, y_{2n}) \leq d(y_{2n}, y_{2n-1})$ for each $n \geq 1$.

Therefore, we can conclude that (8) holds.

Let us denote $c_n = d(y_{n+1}, y_n)$. Then, from (8), c_n is a nonincreasing sequence and bounded below. Thus, it must converge to some $c \geq 0$. If $c > 0$, then by the above inequalities, we have $c \leq c_{n+1} \leq \psi(c_n, c_n, c_n, 2c_n, 0)$. Taking the limit, as $n \to \infty$, we have $c \leq c \leq \psi(c, c, c, 2c, 0) < c$, which is a contradiction. Hence,

$$d(y_{n+1}, y_n) \longrightarrow 0. \tag{10}$$

Now, we show that $\{y_n\}$ is a Cauchy sequence.

Suppose that $\{y_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon > 0$ for which we can find two sequences of natural numbers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$d(y_{m(k)}, y_{n(k)}) \geq \epsilon, \qquad d(y_{m(k)}, y_{n(k)-1}) < \epsilon. \tag{11}$$

From (11), it follows that

$$\begin{aligned}
\epsilon &\leq d(y_{m(k)}, y_{n(k)}) \\
&\leq d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)}) \\
&< \epsilon + d(y_{n(k)-1}, y_{n(k)}).
\end{aligned} \tag{12}$$

Letting $k \to \infty$ and using (10), we have

$$\lim_{k \to \infty} d(y_{m(k)}, y_{n(k)}) = \epsilon. \tag{13}$$

Again,

$$\begin{aligned}
d(y_{m(k)-1}, y_{n(k)-1}) &\leq d(y_{m(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{n(k)}) \\
&\quad + d(y_{n(k)}, y_{n(k)-1}), \\
d(y_{m(k)}, y_{n(k)}) &\leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{n(k)-1}) \\
&\quad + d(y_{n(k)-1}, y_{n(k)}).
\end{aligned} \tag{14}$$

Letting $k \to \infty$ in the above inequalities and using (10) and (13), we have

$$\lim_{k \to \infty} d(y_{m(k)-1}, y_{n(k)-1}) = \epsilon. \tag{15}$$

Again,

$$\begin{aligned}
d(y_{n(k)-1}, y_{m(k)}) &\leq d(y_{m(k)-1}, y_{n(k)-1}) + d(y_{m(k)-1}, y_{m(k)}), \\
d(y_{m(k)-1}, y_{n(k)-1}) &\leq d(y_{m(k)-1}, y_{m(k)}) + d(y_{n(k)-1}, y_{m(k)}).
\end{aligned} \tag{16}$$

Letting $k \to \infty$ in the above inequalities and using (10) and (15), we have

$$\lim_{k \to \infty} d(y_{n(k)-1}, y_{m(k)}) = \epsilon. \tag{17}$$

Similarly, we have

$$\lim_{k \to \infty} d(y_{m(k)-1}, y_{n(k)}) = \epsilon. \tag{18}$$

Also, again from (10), (15), and the inequality

$$d(y_{m(k)-1}, y_{n(k)+1}) - d(y_{m(k)-1}, y_{n(k)}) \leq d(y_{n(k)}, y_{n(k)+1}), \tag{19}$$

it follows that

$$\lim_{k \to \infty} d(y_{m(k)-1}, y_{n(k)+1}) = \epsilon. \tag{20}$$

Now, we have

$$\begin{aligned}
&d(y_{m(k)}, y_{n(k)+1}) \\
&\quad \leq \psi(d(y_{m(k)-1}, y_{n(k)}), d(y_{m(k)-1}, y_{m(k)}), \\
&\qquad\quad d(y_{n(k)}, y_{n(k)+1}), d(y_{m(k)-1}, y_{n(k)+1}), \\
&\qquad\quad d(y_{n(k)}, y_{m(k)})).
\end{aligned} \tag{21}$$

Letting $k \to \infty$, we get

$$\begin{aligned}
\epsilon &\leq \psi(\epsilon, 0, 0, \epsilon, \epsilon) \\
&< \epsilon,
\end{aligned} \tag{22}$$

which is a contradiction. Thus $\{y_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $z \in X$ such that $y_n \to z$. Therefore, we have

$$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n} = z,$$
$$\lim_{n\to\infty} y_{2n+2} = \lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} gx_{2n+1} = z. \tag{23}$$

Assume that S is continuous. Since f and S are compatible, we have

$$\lim_{n\to\infty} fSx_{2n+2} = \lim_{n\to\infty} Sfx_{2n+2} = Sz. \tag{24}$$

Also, $x_{2n+1} \leq gx_{2n+1} = Sx_{2n+2}$. Now, we have

$$\begin{aligned} d&(fSx_{2n}, gx_{2n+1}) \\ &\leq \psi\big(d\big(S^2x_{2n}, Tx_{2n+1}\big), d\big(S^2x_{2n}, fSx_{2n}\big), \\ &\quad d(Tx_{2n+1}, gx_{2n+1}), d\big(S^2x_{2n}, gx_{2n+1}\big), \\ &\quad d(Tx_{2n+1}, fSx_{2n})\big). \end{aligned} \tag{25}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(Sz, z) &\leq \psi(d(Sz, z), 0, 0, d(Sz, z), d(z, Sz)) \\ &< d(Sz, z), \end{aligned} \tag{26}$$

which implies that $Sz = z$.

Now, it follows that $x_{2n+1} \leq gx_{2n+1}$ and $gx_{2n+1} \to z$, $x_{2n+1} \leq z$. From (5), we have

$$\begin{aligned} d&(fz, gx_{2n+1}) \\ &\leq \psi(d(Sz, Tx_{2n+1}), d(Sz, fz), d(Tx_{2n+1}, gx_{2n+1}), \\ &\quad d(Sz, gx_{2n+1}), d(Tx_{2n+1}, fz)). \end{aligned} \tag{27}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(fz, z) &\leq \psi(0, d(fz, z), 0, 0, d(z, fz)) \\ &< d(fz, z), \end{aligned} \tag{28}$$

which implies that $fz = z$. Since $f(X) \subseteq T(X)$, there exists $w \in X$ such that $Sz = z = fz = Tw$. Suppose that $gw \neq Tw$. Since $z \leq fz = Tw \leq fTw \leq w$ implies $z \leq w$, from (5), we obtain

$$\begin{aligned} d(Tw, gw) &= d(fz, gw) \\ &\leq \psi(d(Sz, Tw), d(Sz, fz), d(Tw, gw), \\ &\quad d(Sz, gw), d(Tw, fz)) \\ &= \psi_2(0, 0, d(Tw, gw), d(Tw, gw), 0) \\ &< d(Tw, gw), \end{aligned} \tag{29}$$

which implies that $Tw = gw$. Since g and T are weakly compatible, $gz = gfz = gTw = Tgw = Tfz = Tz$. Thus, z is a coincidence point of T and g.

Now, $x_{2n} \leq fx_{2n}$ and $fx_{2n} \to z$ implies $x_{2n} \leq z$. Thus, from (5), we obtain

$$\begin{aligned} d(fx_{2n}, gz) \leq \psi(&d(Sx_{2n}, Tz), d(Sx_{2n}, fx_{2n}), \\ &d(Tz, gz), d(Sx_{2n}, gz), d(Tz, fx_{2n})). \end{aligned} \tag{30}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(z, gz) &\leq \psi(d(z, gz), 0, 0, d(z, gz), d(z, gz)) \\ &< d(z, gz), \end{aligned} \tag{31}$$

which implies that $gz = z$. Therefore, we have $fz = gz = Sz = Tz = z$.

If f is continuous, then, following the similar arguments, also we get the result.

Similarly, the result follows when (b) holds.

Now, suppose that the set of common fixed points of T, S, f, and g is well ordered.

We claim that common fixed points of T, S, f, and g are unique.

Assume that $Tu = Su = fu = gu = u$ and $Tv = Sv = fv = gv = v$, but $u \neq v$. Then, from (5), we have

$$\begin{aligned} d(u, v) &= d(fu, gv) \\ &\leq \psi(d(Su, Tv), d(Su, fu), d(Tv, gv), \\ &\quad d(Su, gv), d(Tv, fu)) \\ &= \psi(d(u, v), 0, 0, d(u, v), d(v, u)) \\ &< d(u, v). \end{aligned} \tag{32}$$

This implies that $d(u, v) = 0$, and hence $u = v$.

Conversely, if T, S, f, and g have only one common fixed point, then the set of common fixed point of f, g, S, and T being singleton is well ordered. This completes the proof. □

Example 8. Consider $X = [0, 1] \cup \{2, 3, 4, \ldots\}$ with usual ordering and

$$d(x, y) = \begin{cases} |x - y| & \text{if } x, y \in [0, 1],\ x \neq y; \\ x + y & \text{if at least one of } x \text{ or } y \notin [0, 1], \\ & x \neq y; \\ 0 & \text{if } x = y. \end{cases} \tag{33}$$

Then $(X, \leq, d)$ is a complete partially ordered metric space.

Let f, g, S, and T be self-mappings on X defined as

$$f(x) = \begin{cases} 0 & \text{if } x = 0; \\ \frac{1}{2} & \text{if } x \in \left(0, \frac{1}{2}\right]; \\ 1 & \text{if } x \in \left(\frac{1}{2}, 1\right]; \\ x & \text{if } x \in \{2, 3, 4, \ldots\}; \end{cases}$$

$$g(x) = \begin{cases} 0 & \text{if } x = 0; \\ \frac{1}{2} & \text{if } x \in \left(0, \frac{1}{2}\right]; \\ x & \text{if } x \in \left(\frac{1}{2}, 1\right] \cup \{2, 3, 4, \ldots\}; \end{cases} \quad (34)$$

$$T(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2}; \\ \frac{1}{2} & \text{if } x \in \left(\frac{1}{2}, 1\right]; \\ x - 1 & \text{if } x \in \{2, 3, 4, \ldots\}; \end{cases}$$

$$S(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2}; \\ 2x - 1 & \text{if } x \in \left(\frac{1}{2}, 1\right]; \\ x & \text{if } x \in \{2, 3, 4, \ldots\}. \end{cases}$$

Define function $\psi : [0, \infty)^5 \to [0, \infty)$ by the formula

$$\psi(t_1, t_2, t_3, t_4, t_5) = \frac{6}{7} \max\left\{t_1, t_2, t_3, \frac{t_4 + t_5}{2}\right\}. \quad (35)$$

Note that f, g, S, and T satisfy all the conditions given in Theorem 7. Moreover, 0 is a common fixed point of f, g, S, and T.

If $f = g$, then we have the following result.

Corollary 9. *Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T, f, and S are self-mappings on X, the pairs (T, f) and (S, f) are partially weakly increasing with $f(X) \subseteq T(X)$ and $f(X) \subseteq S(X)$, and the dominating mapping f is a weak annihilator of T and S. Further, suppose that there exists the function $\psi \in \Psi$ such that, for any two comparable elements $x, y \in X$,*

$$d(fx, fy) \leq \psi(d(Sx, Ty), d(Sx, fx), d(Ty, fy), d(Sx, fy), d(Ty, fx)) \quad (36)$$

holds. If, for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all $n \geq 1$, $y_n \to u$ implies that $x_n \leq u$ and either

(a) *f, S are compatible, f or S is continuous, and f, T are weakly compatible or*

(b) *f, T are compatible, f or T is continuous, and f, S are weakly compatible,*

then f, S, and T have a common fixed point in X. Moreover, the set of common fixed points of f, S, and T is well ordered if and only if f, S, and T have one and only one common fixed point in X.

Corollary 10. *Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T, f, and g are self-mappings on X, the pairs (T, f) and (T, g) are partially weakly increasing with $f(X) \subseteq T(X)$ and $g(X) \subseteq T(X)$, and the dominating mappings f and g are weak annihilators of T. Further, suppose that there exists the function $\psi \in \Psi$ such that, for any two comparable elements $x, y \in X$,*

$$d(fx, gy) \leq \psi(d(Tx, Ty), d(Tx, fx), d(Ty, gy), d(Tx, gy), d(Ty, fx)) \quad (37)$$

holds. If, for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all $n \geq 1$, $y_n \to u$ implies that $x_n \leq u$ and either

(a) *f, T are compatible, f or T is continuous, and g, T are weakly compatible or*

(b) *g, T are compatible, g or T is continuous, and f, T are weakly compatible,*

then f, g, and T have a common fixed point in X. Moreover, the set of common fixed points of f, g, and T is well ordered if and only if f, g, and T have one and only one common fixed point in X.

Corollary 11. *Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T and f are self-mappings on X, the pair (T, f) is partially weakly increasing with $f(X) \subseteq T(X)$, and the dominating mapping f is a weak annihilator of T. Further, suppose that there exists the function $\psi \in \Psi$ such that, for any two comparable elements $x, y \in X$,*

$$d(fx, fy) \leq \psi(d(Tx, Ty), d(Tx, fx), d(Ty, fy), d(Tx, fy), d(Ty, fx)) \quad (38)$$

holds. If, for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all $n \geq 1$, $y_n \to u$ implies that $x_n \leq u$ and, further, f, T are compatible, f or T is continuous, and f, T are weakly compatible, then f and T have a common fixed point in X. Moreover, the set of common fixed points of f and T is well ordered if and only if f and T have one and only one common fixed point in X.

3. Applications

The aim of the section is to apply our new results to mappings involving contractions of integral type. For this purpose, denote by Λ the set of functions $\mu : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:

(h1) μ is a Lebesgue-integrable mapping on each compact of $[0, \infty)$;

(h2) for any $\epsilon > 0$, we have $\int_0^{\epsilon} \mu(t) > 0$.

Corollary 12. *Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T, f, g, and S are self-mappings on X, the pairs (T, f) and (S, g) are partially weakly increasing with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, and the dominating mappings f and g are weak annihilators of T and S, respectively. Further, suppose that there exists the function $\psi \in \Psi$ such that, for any two comparable elements $x, y \in X$,*

$$\int_0^{d(fx,gy)} \alpha(s)\, ds \leq \int_0^{\psi(d(Sx,Ty),d(Sx,fx),d(Ty,gy),d(Sx,gy),d(Ty,fx))} \alpha(s)\, ds \tag{39}$$

holds, where $\alpha \in \Lambda$. If, for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all $n \geq 1$, $y_n \to u$ implies that $x_n \leq u$ and either

(a) *f, S are compatible, f or S is continuous, and g, T are weakly compatible or*

(b) *g, T are compatible, g or T is continuous, and f, S are weakly compatible,*

then f, g, S, and T have a common fixed point in X. Moreover, the set of common fixed points of f, g, S, and T is well ordered if and only if f, g, S, and T have one and only one common fixed point in X.

Corollary 13. *Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T, f, and g are self-mappings on X, the pairs (T, f) and (T, g) are partially weakly increasing with $f(X) \subseteq T(X)$ and $g(X) \subseteq T(X)$, and the dominating mappings f and g are weak annihilators of T. Further, suppose that there exists the function $\psi \in \Psi$ such that, for any two comparable elements $x, y \in X$,*

$$\int_0^{d(fx,gy)} \alpha(s)\, ds \leq \int_0^{\psi(d(Tx,Ty),d(Tx,fx),d(Ty,gy),d(Tx,gy),d(Ty,fx))} \alpha(s)\, ds \tag{40}$$

holds, where $\alpha \in \Lambda$. If, for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all $n \geq 1$, $y_n \to u$ implies that $x_n \leq u$ and either

(a) *f, T are compatible, f or T is continuous, and g, T are weakly compatible or*

(b) *g, T are compatible, g or T is continuous, and f, T are weakly compatible,*

then f, g, and T have a common fixed point in X. Moreover, the set of common fixed points of f, g, and T is well ordered if and only if f, g, and T have one and only one common fixed point in X.

Corollary 14. *Let $(X, \leq)$ be a partially ordered set, and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that T and f are self-mappings on X, the pair (T, f) is a partially weakly increasing with $f(X) \subseteq T(X)$, and the dominating mapping f is a weak annihilator of T. Further, suppose that there exists the function $\psi \in \Psi$ such that, for any two comparable elements $x, y \in X$,*

$$\int_0^{d(fx,fy)} \alpha(s)\, ds \leq \int_0^{\psi(d(Tx,Ty),d(Tx,fx),d(Ty,fy),d(Tx,fy),d(Ty,fx))} \alpha(s)\, ds \tag{41}$$

holds, where $\alpha \in \Lambda$. If, for a nondecreasing sequence $\{x_n\}$ with $x_n \leq y_n$ for all $n \geq 1$, $y_n \to u$ implies that $x_n \leq u$ and, further, f, T are compatible, f or T is continuous, and f, T are weakly compatible, then f and T have a common fixed point in X. Moreover, the set of common fixed points of f and T is well ordered if and only if f and T have one and only one common fixed point in X.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] M. Abbas, Y. J. Cho, and T. Nazir, "Common fixed point theorems for four mappings in TVS-valued cone metric spaces," *Journal of Mathematical Inequalities*, vol. 5, no. 2, pp. 287–299, 2011.

[2] M. Abbas and M. A. Khan, "Common fixed point theorem of two mappings satisfying a generalized weak contractive condition," *International Journal of Mathematics and Mathematical Sciences*, vol. 2009, Article ID 131068, 9 pages, 2009.

[3] M. Abbas, T. Nazir, and S. Radenović, "Common fixed points of four maps in partially ordered metric spaces," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1520–1526, 2011.

[4] I. Altun, B. Damjanović, and D. Djorić, "Fixed point and common fixed point theorems on ordered cone metric spaces," *Applied Mathematics Letters*, vol. 23, no. 3, pp. 310–316, 2010.

[5] H. Aydi, M. Postolache, and W. Shatanawi, "Coupled fixed point results for (ψ, ϕ)-metric spaces," *Computers & Mathematics with Applications*, vol. 63, no. 1, pp. 298–309, 2012.

[6] H. Aydi, E. Karapinar, and M. Postolache, "Tripled coincidence point theorems for weak ϕ-contractions in partially ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 44, 2012.

[7] H. Aydi, W. Shatanawi, M. Postolache, Z. Mustafa, and N. Tahat, "Theorems for Boyd-Wong-type contractions in ordered metric spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 359054, 14 pages, 2012.

[8] T. Gnana Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[9] A. Branciari, “A fixed point theorem for mappings satisfying a general contractive condition of integral type,” *International Journal of Mathematics and Mathematical Sciences*, vol. 29, no. 9, pp. 531–536, 2002.

[10] S. Chandok, “Some common fixed point theorems for generalized f-weakly contractive mappings,” *Journal of Applied Mathematics & Informatics*, vol. 29, no. 1-2, pp. 257–265, 2011.

[11] S. Chandok, “Some common fixed point theorems for generalized nonlinear contractive mappings,” *Computers & Mathematics with Applications*, vol. 62, no. 10, pp. 3692–3699, 2011.

[12] S. Chandok, “Common fixed points, invariant approximation and generalized weak contractions,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 102980, 11 pages, 2012.

[13] S. Chandok, “Some common fixed point results for generalized weak contractive mappings in partially ordered metric spaces,” *Journal of Nonlinear Analysis and Optimization*, vol. 4, no. 1, pp. 45–52, 2013.

[14] S. Chandok, M. S. Khan, and K. P. R. Rao, “Some coupled common fixed point theorems for a pair of mappings satisfying a contractive condition of rational type without monotonicity,” *International Journal of Mathematical Analysis*, vol. 7, no. 9-12, pp. 433–440, 2013.

[15] S. Chandok, Z. Mustafa, and M. Postolache, “Coupled common fixed point theorems for mixed g-monotone mappings in partially ordered G-metric spaces,” *UPB Scientific Bulletin A*. In press.

[16] S. Chandok and M. Postolache, “Fixed point theorem for weakly Chatterjea-type cyclic contractions,” *Fixed Point Theory and Applications*, vol. 2012, article 28, 2013.

[17] S. Chandok, W. Sintunavarat, and P. Kumam, “Some coupled common fixed points for a pair of mappings in partially ordered G-metric spaces,” *Mathematical Sciences*, vol. 7, article 24, 2013.

[18] C.-M. Chen, “Fixed point theorems for ψ-contractive mappings in ordered metric spaces,” *Journal of Applied Mathematics*, vol. 2012, Article ID 756453, 10 pages, 2012.

[19] B. S. Choudhury, N. Metiya, and M. Postolache, “A generalized weak contraction principle with applications to coupled coincidence point problems,” *Fixed Point Theory and Applications*, vol. 2013, article 152, 2013.

[20] L. Ćirić, “A generalization of Banach’s contraction principle,” *Proceedings of the American Mathematical Society*, vol. 45, no. 2, pp. 26–273, 1974.

[21] A. Djoudi and L. Nisse, “Greguš type fixed points for weakly compatible maps,” *Bulletin of the Belgian Mathematical Society*, vol. 10, no. 3, pp. 369–378, 2003.

[22] E. Graily, S. M. Vaezpour, R. Saadati, and Y. J. Cho, “Generalization of fixed point theorems in ordered metric spaces concerning generalized distance,” *Fixed Point Theory and Applications*, vol. 2011, article 30, 2011.

[23] R. H. Haghi, M. Postolache, and Sh. Rezapour, “On T-stability of the Picard iteration for generalized φ-contraction mappings,” *Abstract and Applied Analysis*, vol. 2012, Article ID 658971, 7 pages, 2012.

[24] G. Jungck, “Compatible mappings and common fixed points,” *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 4, pp. 771–779, 1986.

[25] G. Jungck, “Compatible mappings and common fixed points. II,” *International Journal of Mathematics and Mathematical Sciences*, vol. 11, no. 2, pp. 285–288, 1988.

[26] G. Jungck, “Common fixed points for commuting and compatible maps on compacta,” *Proceedings of the American Mathematical Society*, vol. 103, no. 3, pp. 977–983, 1988.

[27] G. Jungck and B. E. Rhoades, “Fixed points for set valued functions without continuity,” *Indian Journal of Pure and Applied Mathematics*, vol. 29, no. 3, pp. 227–238, 1998.

[28] G. Jungck and B. E. Rhoades, “Fixed point theorems for occasionally weakly compatible mappings,” *Fixed Point Theory*, vol. 7, no. 2, pp. 287–296, 2006.

[29] J. J. Nieto and R. Rodríguez-López, “Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,” *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[30] J. J. Nieto and R. Rodríguez-López, “Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations,” *Acta Mathematica Sinica*, vol. 23, no. 12, pp. 2205–2212, 2007.

[31] M. O. Olatinwo and M. Postolache, “Stability results of Jungck-type iterative processes in convex metric spaces,” *Applied Mathematics and Computation*, vol. 218, no. 12, pp. 6727–6732, 2012.

[32] A. C. M. Ran and M. C. B. Reurings, “A fixed point theorem in partially ordered sets and some applications to matrix equations,” *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp. 1435–1443, 2004.

[33] B. Samet and H. Yazidi, “Fixed point theorems with respect to a contractive condition of integral type,” *Rendiconti del Circolo Matematico di Palermo*, vol. 60, no. 1-2, pp. 181–190, 2011.

[34] S. Sessa, “On a weak commutativity condition of mappings in fixed point considerations,” *Publications de l’Institut Mathématique*, vol. 32, no. 46, pp. 149–153, 1982.

[35] W. Shatanawi and M. Postolache, “Some fixed-point results for a G-weak contraction in G-metric spaces,” *Abstract and Applied Analysis*, vol. 2012, Article ID 815870, 19 pages, 2012.

[36] W. Shatanawi and M. Postolache, “Common fixed point theorems for dominating and weak annihilator mappings in orderedmetric spaces,” *Fixed Point Theory and Applications*. In press.

[37] W. Shatanawi and M. Postolache, “Common fixed point results of mappings for nonlinear contractions of cyclic form in ordered metric spaces,” *Fixed Point Theory and Applications*, vol. 2013, article 60, 2013.

[38] W. Shatanawi and M. Postolache, “Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces,” *Fixed Point Theory and Applications*, vol. 2013, article 54, 2013.

[39] W. Shatanawi and A. Pitea, “Omega-distance and coupled fixed point in G-metric spaces,” *Fixed Point Theory and Applications*, vol. 2013, article 208, 2013.

[40] W. Shatanawi and A. Pitea, “Some coupled fixed point theorems in quasi-partial metric spaces,” *Fixed Point Theory and Applications*, vol. 2013, article 153, 2013.

[41] W. Shatanawi, S. Chauhan, M. Postolache, M. Abbas, and S. Radenović, “Common fixed points for contractive mappings of integral type in G-metric spaces,” *Journal of Advanced Mathematical Studies*, vol. 6, no. 1, pp. 53–72, 2013.