Research Article

Common Fixed Points for Multivalued Mappings in Complex Valued Metric Spaces with Applications

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In this paper, we introduce the notion of multivalued contractive mappings in complex valued metric space and prove common fixed point theorems for two multivalued contractive mappings in complex valued metric spaces without using the notion of continuity. Our results improve and extend the results of Azam et al. (2011).

1. Introduction

The Banach fixed point theorem was used to establish the existence of a unique solution for a nonlinear integral equation [1]. Moreover, this theorem plays an important role in several branches of mathematics. For instance, it has been used to show the existence of solutions of nonlinear Volterra integral equations and nonlinear integrodifferential equations in the Banach spaces and to show the convergence of algorithms in computational mathematics. Because of its importance and usefulness for mathematical theory, it has become a very popular tool of mathematical analysis in many directions. Nadler [2] introduced the concept of multivalued contraction mappings and obtained the fixed points results for multivalued mappings. Huang and Zhang [3] introduced the notion of cone metric space which is a generalization of metric spaces. They extended Banach contraction principle to cone metric spaces. Since then, Arshad et al. [4], Azam and Arshad [5], Latif and Shaddad [6], Karapınar [7], and many others obtained fixed point theorems in cone metric spaces (see [8]).

The fixed point results regarding rational contractive conditions cannot be extended in cone metric spaces. Azam et al. [9] introduced the concept of complex valued metric spaces and obtained sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive

type condition involving rational inequalities. In the same way, Rouzkard and Imdad [10] established some common fixed point theorems satisfying certain rational expressions in complex valued metric spaces which generalize, unify, and complement the results of Azam et al. [9]. Recently, Sintunavarat and Kumam [11] obtained common fixed point results by replacing constant of contractive condition with control functions. For more details in the subject, we refer to [12–19].

The aim of this paper is to extend the results of Azam et al. [9] to multivalued mappings in complex valued metric spaces.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \quad \text{iff} \quad \operatorname{Re}(z_1) \leqslant \operatorname{Re}(z_2),$$
$$\operatorname{Im}(z_1) \leqslant \operatorname{Im}(z_2). \tag{1}$$

It follows that

$$z_1 \precsim z_2 \tag{2}$$

if one of the following conditions is satisfied:

$$
\begin{aligned}
&\text{(i)} \quad \operatorname{Re}(z_1) = \operatorname{Re}(z_2), \qquad \operatorname{Im}(z_1) < \operatorname{Im}(z_2), \\
&\text{(ii)} \quad \operatorname{Re}(z_1) < \operatorname{Re}(z_2), \qquad \operatorname{Im}(z_1) = \operatorname{Im}(z_2), \\
&\text{(iii)} \quad \operatorname{Re}(z_1) < \operatorname{Re}(z_2), \qquad \operatorname{Im}(z_1) < \operatorname{Im}(z_2), \\
&\text{(iv)} \quad \operatorname{Re}(z_1) = \operatorname{Re}(z_2), \qquad \operatorname{Im}(z_1) = \operatorname{Im}(z_2).
\end{aligned}
\tag{3}
$$

In particular, we will write $z_1 \precsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$
\begin{aligned}
&0 \preceq z_1 \precsim z_2 \implies |z_1| < |z_2|, \\
&z_1 \preceq z_2, \; z_2 \prec z_3 \implies z_1 \prec z_3.
\end{aligned}
\tag{4}
$$

Definition 1. Let X be a nonempty set. Suppose that a mapping $d : X \times X \to \mathbb{C}$ satisfies

(1) $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a *complex valued metric* on X, and (X, d) is called a *complex valued metric space*. A point $x \in X$ is called *interior point* of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$
B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A.
\tag{5}
$$

A point $x \in X$ is called a *limit point* of A whenever for every $0 \prec r \in \mathbb{C}$,

$$
B(x, r) \bigcap (A \setminus \{x\}) \neq \phi.
\tag{6}
$$

A is called *open* whenever each element of A is an interior point of A. Moreover, a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$
F = \{B(x, r) : x \in X, \; 0 \prec r\}
\tag{7}
$$

is a subbasis for a Hausdorff topology τ on X.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x) \prec c$, then $\{x_n\}$ is said to be *convergent*, $\{x_n\}$ converges to x, and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n \to \infty} x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0, d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a *Cauchy sequence* in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a *complete* complex valued metric space. We require the following lemmas.

Lemma 2 (see [9]). *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$.*

Lemma 3 (see [9]). *Let (X, d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to \infty$.*

3. Main Results

Let (X, d) be a complex valued metric space.

We denote by $N(X)$ (resp., $CB(X)$) the set of nonempty (resp., closed and bounded) subsets of a complex valued metric space. Now, we denote $s(z_1) = \{z_2 \in \mathbb{C} : z_1 \preceq z_2\}$ for $z_1 \in \mathbb{C}$, and $s(a, B) = \cup_{b \in B} s(d(a, b)) = \cup_{b \in B} \{z \in \mathbb{C} : d(a, b) \preceq z\}$ for $a \in X$ and $B \in C(X)$. For $A, B \in C(X)$, we denote

$$
s(A, B) = \left(\bigcap_{a \in A} s(a, B) \right) \bigcap \left(\bigcap_{b \in B} s(b, A) \right).
\tag{8}
$$

Remark 4. Let (X, d) be a complex valued metric space. If $\mathbb{C} = R$, then (X, d) is a metric space. Moreover, for $A, B \in CB(X)$, $H(A, B) = \inf s(A, B)$ is the Hausdorff distance induced by d.

Let (X, d) be a complex valued metric space and let $CB(X)$ be a collection of nonempty closed subsets of X. Let $T : X \to CB(X)$ be a multivalued map. For $x \in X$ and $A \in CB(X)$, define

$$
W_x(A) = \{d(x, a) : a \in A\}.
\tag{9}
$$

Thus, for $x, y \in X$

$$
W_x(Ty) = \{d(x, u) : u \in Ty\}.
\tag{10}
$$

Definition 5. Let (X, d) be a complex valued metric space. A nonempty subset A of X is called bounded from below if there exists some $z \in \mathbb{C}$, such that $z \preceq a$ for all $a \in A$.

Definition 6. Let (X, d) be a complex valued metric space. A multivalued mapping $F : X \to 2^{\mathbb{C}}$ is called bounded from below if for $x \in X$ there exists $z_x \in \mathbb{C}$ such that

$$
z_x \preceq u, \quad \forall u \in Fx.
\tag{11}
$$

Definition 7. Let (X, d) be a complex valued metric space. The multivalued mapping $T : X \to CB(X)$ is said to have lower bound property (l.b. property) on (X, d) if for any $x \in X$, the multivalued mapping $F_x : X \to 2^{\mathbb{C}}$ defined by

$$
F_x(Ty) = W_x(Ty)
\tag{12}
$$

is bounded from below. That is, for $x, y \in X$ there exists an element $l_x(Ty) \in \mathbb{C}$ such that

$$
l_x(Ty) \preceq u
\tag{13}
$$

for all $u \in W_x(Ty)$, where $l_x(Ty)$ is called lower bound of T associated with (x, y).

Definition 8. Let (X, d) be a complex valued metric space. The multivalued mapping $T : X \to CB(X)$ is said to have the greatest lower bound property (g.l.b. property) on (X, d) if the greatest lower bound of $W_x(Ty)$ exists in $\mathbb{C}$ for all $x, y \in X$. We denote $d(x, Ty)$ by the g.l.b of $W_x(Ty)$. That is,

$$
d(x, Ty) = \inf \{d(x, u) : u \in Ty\}.
\tag{14}
$$

Theorem 9. *Let (X, d) be a complete complex valued metric space and let $S, T : X \rightarrow CB(X)$ be multivalued mappings with g.l.b. property such that*

$$\lambda d(x, y) + \frac{\mu d(x, Sx) d(y, Ty) + \gamma d(y, Sx) d(x, Ty)}{1 + d(x, y)}$$

$$\in s(Sx, Ty) \tag{15}$$

for all $x, y \in X$ and $\lambda, \mu,$ and γ are nonnegative real numbers with $\lambda + \mu + \gamma < 1$. Then S, T have a common fixed point.

Proof. Let x_0 be an arbitrary point in X and $x_1 \in Sx_0$. From (15), we have

$$\lambda d(x_0, x_1)$$

$$+ \frac{\mu d(x_0, Sx_0) d(x_1, Tx_1) + \gamma d(x_1, Sx_0) d(x_0, Tx_1)}{1 + d(x_0, x_1)}$$

$$\in s(Sx_0, Tx_1). \tag{16}$$

This implies that

$$\lambda d(x_0, x_1)$$

$$+ \frac{\mu d(x_0, Sx_0) d(x_1, Tx_1) + \gamma d(x_1, Sx_0) d(x_0, Tx_1)}{1 + d(x_0, x_1)}$$

$$\in \left(\bigcap_{x \in Sx_0} s(x, Tx_1) \right),$$

$$\lambda d(x_0, x_1)$$

$$+ \frac{\mu d(x_0, Sx_0) d(x_1, Tx_1) + \gamma d(x_1, Sx_0) d(x_0, Tx_1)}{1 + d(x_0, x_1)}$$

$$\in s(x, Tx_1) \quad \forall x \in Sx_0. \tag{17}$$

Since $x_1 \in Sx_0$, we have

$$\lambda d(x_0, x_1)$$

$$+ \frac{\mu d(x_0, Sx_0) d(x_1, Tx_1) + \gamma d(x_1, Sx_0) d(x_0, Tx_1)}{1 + d(x_0, x_1)}$$

$$\in s(x_1, Tx_1),$$

$$\lambda d(x_0, x_1)$$

$$+ \frac{\mu d(x_0, Sx_0) d(x_1, Tx_1) + \gamma d(x_1, Sx_0) d(x_0, Tx_1)}{1 + d(x_0, x_1)}$$

$$\in s(x_1, Tx_1) = \bigcup_{x \in Tx_1} s(d(x_1, x)). \tag{18}$$

So there exists some $x_2 \in Tx_1$ such that

$$\lambda d(x_0, x_1)$$

$$+ \frac{\mu d(x_0, Sx_0) d(x_1, Tx_1) + \gamma d(x_1, Sx_0) d(x_0, Tx_1)}{1 + d(x_0, x_1)}$$

$$\in s(d(x_1, x_2)). \tag{19}$$

Therefore,

$$d(x_1, x_2)$$

$$\preceq \lambda d(x_0, x_1)$$

$$+ \frac{\mu d(x_0, Sx_0) d(x_1, Tx_1) + \gamma d(x_1, Sx_0) d(x_0, Tx_1)}{1 + d(x_0, x_1)}. \tag{20}$$

By using the greatest lower bound property (g.l.b property) of S and T, we get

$$d(x_1, x_2) \preceq \lambda d(x_0, x_1)$$

$$+ \frac{\mu d(x_0, x_1) d(x_1, x_2) + \gamma d(x_1, x_1) d(x_0, x_2)}{1 + d(x_0, x_1)}$$

$$= \lambda d(x_0, x_1) + \frac{\mu d(x_0, x_1) d(x_1, x_2)}{1 + d(x_0, x_1)}, \tag{21}$$

which implies that

$$\left| d(x_1, x_2) \right| \leq \lambda \left| d(x_0, x_1) \right| + \frac{\mu \left| d(x_0, x_1) \right| \left| d(x_1, x_2) \right|}{\left| 1 + d(x_0, x_1) \right|}$$

$$= \lambda \left| d(x_0, x_1) \right| + \mu \left| d(x_1, x_2) \right| \left| \frac{d(x_0, x_1)}{1 + d(x_0, x_1)} \right|,$$

$$\left| d(x_1, x_2) \right| \leq \lambda \left| d(x_0, x_1) \right| + \mu \left| d(x_1, x_2) \right|,$$

$$(1 - \mu) \left| d(x_1, x_2) \right| \leq \lambda \left| d(x_0, x_1) \right|,$$

$$\left| d(x_1, x_2) \right| \leq \frac{\lambda}{(1 - \mu)} \left| d(x_0, x_1) \right|$$

$$= h \left| d(x_0, x_1) \right|, \quad \text{where } h = \frac{\lambda}{(1 - \mu)} < 1. \tag{22}$$

Inductively, we can construct a sequence $\{x_n\}$ in X such that for $n = 0, 1, 2, \ldots$

$$\left| d(x_n, x_{n+1}) \right| \leq h^n \left| d(x_0, x_1) \right| \tag{23}$$

with $h = \lambda/(1 - \mu) < 1$, for $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$.

Now for $m > n$, we get

$$
\begin{aligned}
\left|d\left(x_n, x_m\right)\right| &\leq \left|d\left(x_n, x_{n+1}\right)\right| + \left|d\left(x_{n+1}, x_{n+2}\right)\right| \\
&\quad + \cdots + \left|d\left(x_{m-1}, x_m\right)\right| \\
&\leq \left[h^n + h^{n+1} + \cdots + h^{m-1}\right]\left|d\left(x_0, x_1\right)\right| \\
&\leq \left[\frac{h^n}{1-h}\right]\left|d\left(x_0, x_1\right)\right|.
\end{aligned} \tag{24}
$$

Therefore,

$$
\left|d\left(x_n, x_m\right)\right| \leq \frac{h^n}{1-h}\left|d\left(x_0, x_1\right)\right| \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty. \tag{25}
$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. We now show that $u \in Tu$ and $u \in Su$. From (15), we have

$$
\begin{aligned}
&\lambda d\left(x_{2k}, u\right) \\
&\quad + \frac{\mu d\left(x_{2k}, Sx_{2k}\right)d\left(u, Tu\right) + \gamma d\left(u, Sx_{2k}\right)d\left(x_{2k}, Tu\right)}{1 + d\left(x_{2k}, u\right)} \\
&\quad \in s\left(Sx_{2k}, Tu\right).
\end{aligned} \tag{26}
$$

This implies that

$$
\begin{aligned}
&\lambda d\left(x_{2k}, u\right) \\
&\quad + \frac{\mu d\left(x_{2k}, Sx_{2k}\right)d\left(u, Tu\right) + \gamma d\left(u, Sx_{2k}\right)d\left(x_{2k}, Tu\right)}{1 + d\left(x_{2k}, u\right)} \\
&\quad \in \left(\bigcap_{x \in Sx_{2k}} s\left(x, Tu\right)\right), \\
&\lambda d\left(x_{2k}, u\right) \\
&\quad + \frac{\mu d\left(x_{2k}, Sx_{2k}\right)d\left(u, Tu\right) + \gamma d\left(u, Sx_{2k}\right)d\left(x_{2k}, Tu\right)}{1 + d\left(x_{2k}, u\right)} \\
&\quad \in s\left(x, Tu\right) \quad \forall x \in Sx_{2k}.
\end{aligned} \tag{27}
$$

Since $x_{2k+1} \in Sx_{2k}$, we have

$$
\begin{aligned}
&\lambda d\left(x_{2k}, u\right) \\
&\quad + \frac{\mu d\left(x_{2k}, Sx_{2k}\right)d\left(u, Tu\right) + \gamma d\left(u, Sx_{2k}\right)d\left(x_{2k}, Tu\right)}{1 + d\left(x_{2k}, u\right)} \\
&\quad \in s\left(x_{2k+1}, Tu\right).
\end{aligned} \tag{28}
$$

By definition,

$$
\begin{aligned}
&\lambda d\left(x_{2k}, u\right) \\
&\quad + \frac{\mu d\left(x_{2k}, Sx_{2k}\right)d\left(u, Tu\right) + \gamma d\left(u, Sx_{2k}\right)d\left(x_{2k}, Tu\right)}{1 + d\left(x_{2k}, u\right)} \\
&\quad \in s\left(x_{2k+1}, Tu\right) = \bigcup_{u' \in Tu} s\left(d\left(x_{2k+1}, u'\right)\right).
\end{aligned} \tag{29}
$$

There exists some $u_k \in Tu$ such that

$$
\begin{aligned}
&\lambda d\left(x_{2k}, u\right) \\
&\quad + \frac{\mu d\left(x_{2k}, Sx_{2k}\right)d\left(u, Tu\right) + \gamma d\left(u, Sx_{2k}\right)d\left(x_{2k}, Tu\right)}{1 + d\left(x_{2k}, u\right)} \\
&\quad \in s\left(d\left(x_{2k+1}, u_k\right)\right);
\end{aligned} \tag{30}
$$

that is,

$$
\begin{aligned}
&d\left(x_{2k+1}, u_k\right) \\
&\quad \leq \lambda d\left(x_{2k}, u\right) \\
&\quad + \frac{\mu d\left(x_{2k}, Sx_{2k}\right)d\left(u, Tu\right) + \gamma d\left(u, Sx_{2k}\right)d\left(x_{2k}, Tu\right)}{1 + d\left(x_{2k}, u\right)}.
\end{aligned} \tag{31}
$$

By using the greatest lower bound property (g.l.b. property) of S and T, we have

$$
\begin{aligned}
&d\left(x_{2k+1}, u_k\right) \\
&\quad \leq \lambda d\left(x_{2k}, u\right) + \left(\mu d\left(x_{2k}, x_{2k+1}\right)d\left(u, u_k\right)\right. \\
&\quad \left. + \gamma d\left(u, x_{2k+1}\right)d\left(x_{2k}, u_k\right)\right)\left(1 + d\left(x_{2k}, u\right)\right)^{-1}.
\end{aligned} \tag{32}
$$

Since

$$
d\left(u, u_k\right) \leq d\left(u, x_{2k+1}\right) + d\left(x_{2k+1}, u_k\right), \tag{33}
$$

using (32), we get

$$
\begin{aligned}
&d\left(u, u_k\right) \\
&\quad \leq d\left(u, x_{2k+1}\right) + \lambda d\left(u, x_{2k+1}\right) \\
&\quad + \frac{\mu d\left(x_{2k}, x_{2k+1}\right)d\left(u, u_k\right) + \gamma d\left(u, x_{2k+1}\right)d\left(x_{2k}, u_k\right)}{1 + d\left(x_{2k}, u\right)}, \\
&\left|d\left(u, u_k\right)\right| \\
&\quad \leq \left|d\left(u, x_{2k+1}\right)\right| + \lambda \left|d\left(u, x_{2k+1}\right)\right| \\
&\quad + \left(\mu \left|d\left(x_{2k}, x_{2k+1}\right)\right|\left|d\left(u, u_k\right)\right|\right. \\
&\quad \left. + \gamma \left|d\left(u, x_{2k+1}\right)\right|\left|d\left(x_{2k}, u_k\right)\right|\right) \\
&\quad \times \left|1 + d\left(x_{2k}, u\right)\right|^{-1}.
\end{aligned} \tag{34}
$$

Taking the limit as $k \rightarrow \infty$, we get $|d(u, u_k)| \rightarrow 0$ as $k \rightarrow \infty$. By [9, lemma 2], we have $u_k \rightarrow u$ as $k \rightarrow \infty$. Since Tu is closed, $u \in Tu$. Similarly, it follows that $u \in Su$. Thus, S and T have a common fixed point. $\qquad \square$

By setting $\gamma = 0$ in Theorem 9, we get the following corollary.

Corollary 10. *Let (X, d) be a complete complex valued metric space and let $S, T : X \rightarrow CB(X)$ be multivalued mappings with g.l.b property such that*

$$\lambda d(x, y) + \mu \frac{d(x, Sx) d(y, Ty)}{1 + d(x, y)} \in s(Sx, Ty) \qquad (35)$$

for all $x, y \in X$ and λ, μ are nonnegative real numbers with $\lambda + \mu < 1$. Then S, T have a common fixed point.

By setting $S = T$ in Theorem 9, we get the following Corollary.

Corollary 11. *Let (X, d) be a complete complex valued metric space and let $T : X \rightarrow CB(X)$ be multivalued mapping with g.l.b property such that*

$$\lambda d(x, y) + \frac{\mu d(x, Tx) d(y, Ty) + \gamma d(y, Tx) d(x, Ty)}{1 + d(x, y)}$$

$$\in s(Tx, Ty) \qquad (36)$$

for all $x, y \in X$ and $\lambda, \mu,$ and γ are nonnegative real numbers with $\lambda + \mu + \gamma < 1$. Then T has a fixed point in X.

By Remark 4, we have the following corollaries from Theorem 9.

Corollary 12. *Let (X, d) be a complete metric space and let $S, T : X \rightarrow CB(X)$ be multivalued mappings such that*

$$H(Sx, Ty) \le \lambda d(x, y)$$

$$+ \frac{\mu d(x, Sx) d(y, Ty) + \gamma d(y, Sx) d(x, Ty)}{1 + d(x, y)} \qquad (37)$$

for all $x, y \in X$ and $\lambda, \mu,$ and γ are nonnegative real numbers with $\lambda + \mu + \gamma < 1$. Then S, T have a common fixed point.

Remark 13. By equating λ, μ, γ to 0 in all possible combinations, one can derive a host of corollaries which include the Banach fixed point theorem for multivalued mappings in complete metric space.

Example 14. Let $X = [0, 1]$. Define $d : X \times X \rightarrow \mathbb{C}$ by

$$d(x, y) = |x - y| e^{i\theta}, \qquad \theta = \tan^{-1} \left| \frac{y}{x} \right|. \qquad (38)$$

Then (X, d) is a complex valued metric space. Consider the mappings $S, T : X \rightarrow CB(X)$ defined by

$$Sx = \left[0, \frac{1}{5} x \right], \qquad Tx = \left[0, \frac{1}{10} x \right]. \qquad (39)$$

The contractive condition of main theorem is trivial for the case when $x = y = 0$. Suppose without any loss of generality that all x, y are nonzero and $x < y$. Then

$$d(x, y) = |y - x| e^{i\theta},$$

$$d(x, Sx) = \left| x - \frac{x}{5} \right| e^{i\theta},$$

$$d(y, Ty) = \left| y - \frac{y}{10} \right| e^{i\theta},$$

$$d(y, Sx) = \left| y - \frac{x}{5} \right| e^{i\theta}, \qquad (40)$$

$$d(x, Ty) = \left| x - \frac{y}{10} \right| e^{i\theta},$$

$$s(Sx, Ty) = s\left(\left| \frac{x}{5} - \frac{y}{10} \right| e^{i\theta} \right).$$

Consider,

$$\lambda |d(x, y)| + \frac{\mu |d(x, Sx)| |d(y, Ty)| + \gamma |d(y, Sx)| |d(x, Ty)|}{|1 + d(x, y)|}$$

$$= \lambda |y - x|$$

$$+ \frac{\mu |x - (x/5)| |y - (y/10)| + \gamma |y - (x/5)| |x - (y/10)|}{|1 + d(x, y)|}$$

$$= \lambda |y - x|$$

$$+ \frac{\mu |x - (x/5)| |y - (y/10)| + \gamma |y - (x/5)| |x - (y/10)|}{|1 + d(x, y)|}. \qquad (41)$$

Clearly, for any value of μ and γ and $\lambda = 1/5$, we have

$$\left| \frac{x}{5} - \frac{y}{10} \right|$$

$$\le \frac{1}{5} |y - x|$$

$$+ \frac{\mu |x - (x/5)| |y - (y/10)| + \gamma |y - (x/5)| |x - (y/10)|}{|1 + d(x, y)|}. \qquad (42)$$

Thus,

$$\lambda d(x, y) + \frac{\mu d(x, Sx) d(y, Ty) + \gamma d(y, Sx) d(x, Ty)}{1 + d(x, y)}$$

$$\in s(Sx, Ty). \qquad (43)$$

Hence, all the conditions of our main theorem are satisfied and 0 is a common fixed point of S and T.

In the following results, we considered the Kannan type contractive condition involving rational expressions.

Theorem 15. *Let (X, d) be a complete complex valued metric space and let $S, T : X \rightarrow CB(X)$ be multivalued mappings with g.l.b property such that*

$$ad(x, Sx) + bd(y, Ty) + c\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} \in s(Sx, Ty) \tag{44}$$

for all $x, y \in X$ and $a + b + c < 1$. Then S and T have a common fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Sx_0$. From (44), we get

$$ad(x_0, Sx_0) + bd(x_1, Tx_1) + c\frac{d(x_0, Sx_0)d(x_1, Tx_1)}{1 + d(x_0, x_1)} \tag{45}$$
$$\in s(Sx_0, Tx_1).$$

This implies that

$$ad(x_0, Sx_0) + bd(x_1, Tx_1) + c\frac{d(x_0, Sx_0)d(x_1, Tx_1)}{1 + d(x_0, x_1)}$$
$$\in \left(\bigcap_{x \in Sx_0} s(x, Tx_1)\right),$$
$$ad(x_0, Sx_0) + bd(x_1, Tx_1) + c\frac{d(x_0, Sx_0)d(x_1, Tx_1)}{1 + d(x_0, x_1)}$$
$$\in s(x, Tx_1) \quad \forall x \in Sx_0. \tag{46}$$

Since $x_1 \in Sx_0$, we have

$$ad(x_0, Sx_0) + bd(x_1, Tx_1) + c\frac{d(x_0, Sx_0)d(x_1, Tx_1)}{1 + d(x_0, x_1)}$$
$$\in s(x_1, Tx_1),$$
$$ad(x_0, Sx_0) + bd(x_1, Tx_1) + c\frac{d(x_0, Sx_0)d(x_1, Tx_1)}{1 + d(x_0, x_1)}$$
$$\in s(x_1, Tx_1) = \bigcup_{x \in Tx_1} s(d(x_1, x)). \tag{47}$$

So there exists some $x_2 \in Tx_1$, such that

$$ad(x_0, Sx_0) + bd(x_1, Tx_1) + c\frac{d(x_0, Sx_0)d(x_1, Tx_1)}{1 + d(x_0, x_1)}$$
$$\in s(d(x_1, x_2)). \tag{48}$$

That is,

$$d(x_1, x_2) \preceq ad(x_0, Sx_0) + bd(x_1, Tx_1)$$
$$+ c\frac{d(x_0, Sx_0)d(x_1, Tx_1)}{1 + d(x_0, x_1)}. \tag{49}$$

By using the greatest lower bound property (g.l.b property) of S and T, we get

$$d(x_1, x_2) \preceq ad(x_0, x_1) + bd(x_1, x_2) + c\frac{d(x_0, x_1)d(x_1, x_2)}{1 + d(x_0, x_1)} \tag{50}$$

which implies that

$$|d(x_1, x_2)| \le a|d(x_0, x_1)| + b|d(x_1, x_2)|$$
$$+ c\frac{|d(x_0, x_1)||d(x_1, x_2)|}{|1 + d(x_0, x_1)|}. \tag{51}$$

Since $|d(x_0, x_1)| < |1 + d(x_0, x_1)|$, we have

$$|d(x_1, x_2)| \le a|d(x_0, x_1)| + b|d(x_1, x_2)| + c|d(x_1, x_2)|,$$
$$(1 - b - c)|d(x_1, x_2)| \le a|d(x_0, x_1)|$$
$$|d(x_1, x_2)| \le \frac{a}{(1 - b - c)}|d(x_0, x_1)|. \tag{52}$$

Thus,

$$|d(x_1, x_2)| \le \frac{a}{(1 - b - c)}|d(x_0, x_1)|$$
$$= l|d(x_0, x_1)|, \quad \text{where } l = \frac{a}{(1 - b - c)} < 1. \tag{53}$$

Inductively, we can construct a sequence $\{x_n\}$ in X such that for $n = 0, 1, 2, \ldots$

$$|d(x_n, x_{n+1})| \le l^n|d(x_0, x_1)| \tag{54}$$

with $l = a/(1 - b - c) < 1$, for $x_{2n+1} \in Sx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$. Now for $m > n$, we get

$$|d(x_n, x_m)| \le |d(x_n, x_{n+1})| + |d(x_{n+1}, x_{n+2})|$$
$$+ \cdots + |d(x_{m-1}, x_m)|$$
$$\le [l^n + l^{n+1} + \cdots + l^{m-1}]|d(x_0, x_1)| \tag{55}$$
$$\le \left[\frac{l^n}{1 - l}\right]|d(x_0, x_1)|,$$

and so

$$|d(x_n, x_m)| \le \frac{l^n}{1 - l}|d(x_0, x_1)| \longrightarrow 0 \quad \text{as } m, n \longrightarrow \infty. \tag{56}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that $x_n \rightarrow v$ as $n \rightarrow \infty$. We now show that $v \in Tv$ and $v \in Sv$. So from (44), we get

$$ad(x_{2n}, Sx_{2n}) + bd(v, Tv) + c\frac{d(x_{2n}, Sx_{2n})d(v, Tv)}{1 + d(x_{2n}, v)} \tag{57}$$
$$\in s(Sx_{2n}, Tv).$$

This implies that

$$ad\left(x_{2n}, Sx_{2n}\right) + bd\left(v, Tv\right) + c\frac{d\left(x_{2n}, Sx_{2n}\right)d\left(v, Tv\right)}{1 + d\left(x_{2n}, v\right)}$$

$$\in \left(\bigcap_{x \in Sx_{2n}} s\left(x, Tv\right)\right),$$

$$ad\left(x_{2n}, Sx_{2n}\right) + bd\left(v, Tv\right) + c\frac{d\left(x_{2n}, Sx_{2n}\right)d\left(v, Tv\right)}{1 + d\left(x_{2n}, v\right)}$$

$$\in s\left(x, Tv\right) \quad \forall x \in Sx_{2n}.$$

$$(58)$$

Since $x_{2n+1} \in Sx_{2n}$, we have

$$ad\left(x_{2n}, Sx_{2n}\right) + bd\left(v, Tv\right) + c\frac{d\left(x_{2n}, Sx_{2n}\right)d\left(v, Tv\right)}{1 + d\left(x_{2n}, v\right)}$$

$$\in s\left(x_{2n+1}, Tv\right).$$

$$(59)$$

By definition,

$$ad\left(x_{2n}, Sx_{2n}\right) + bd\left(v, Tv\right) + c\frac{d\left(x_{2n}, Sx_{2n}\right)d\left(v, Tv\right)}{1 + d\left(x_{2n}, v\right)}$$

$$\in s\left(x_{2n+1}, Tv\right) = \bigcup_{v' \in Tv} s\left(d\left(x_{2n+1}, v'\right)\right).$$

$$(60)$$

There exists some $v_n \in Tv$ such that

$$ad\left(x_{2n}, Sx_{2n}\right) + bd\left(v, Tv\right) + c\frac{d\left(x_{2n}, Sx_{2n}\right)d\left(v, Tv\right)}{1 + d\left(x_{2n}, v\right)}$$

$$\in s\left(d\left(x_{2n+1}, v_n\right)\right);$$

$$(61)$$

that is,

$$d\left(x_{2n+1}, v_n\right) \preceq ad\left(x_{2n}, Sx_{2n}\right) + bd\left(v, Tv\right)$$

$$+ c\frac{d\left(x_{2n}, Sx_{2n}\right)d\left(v, Tv\right)}{1 + d\left(x_{2n}, v\right)}.$$

$$(62)$$

By using the greatest lower bound property (g.l.b property) of S and T, we get

$$d\left(x_{2n+1}, v_n\right) \preceq ad\left(x_{2n}, x_{2n+1}\right) + bd\left(v, v_n\right)$$

$$+ c\frac{d\left(x_{2n}, x_{2n+1}\right)d\left(v, v_n\right)}{1 + d\left(x_{2n}, v\right)}.$$

$$(63)$$

Now by using the triangular inequality, we get

$$d\left(v, v_n\right) \preceq d\left(v, x_{2n+1}\right) + d\left(x_{2n+1}, v_n\right)$$

$$\preceq d\left(v, x_{2n+1}\right) + ad\left(x_{2n}, x_{2n+1}\right)$$

$$+ bd\left(v, v_n\right) + c\frac{d\left(x_{2n}, x_{2n+1}\right)d\left(v, v_n\right)}{1 + d\left(x_{2n}, v\right)}.$$

$$(64)$$

That is,

$$(1 - b)\left|d\left(v, v_n\right)\right| \leq \left|d\left(v, x_{2n+1}\right)\right| + a\left|d\left(x_{2n}, x_{2n+1}\right)\right|$$

$$+ c\left|\frac{d\left(x_{2n}, x_{2n+1}\right)d\left(v, v_n\right)}{1 + d\left(x_{2n}, v\right)}\right|.$$

$$(65)$$

It follows that

$$\left|d\left(v, v_n\right)\right| \leq \frac{1}{(1 - b)}\left|d\left(v, x_{2n+1}\right)\right| + \frac{a}{(1 - b)}\left|d\left(x_{2n}, x_{2n+1}\right)\right|$$

$$+ \frac{c}{(1 - b)}\frac{\left|d\left(x_{2n}, x_{2n+1}\right)\right|\left|d\left(v, v_n\right)\right|}{\left|1 + d\left(x_{2n}, v\right)\right|}.$$

$$(66)$$

By letting $n \to \infty$ in the above inequality, we get

$$\left|d\left(v, v_n\right)\right| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

$$(67)$$

By [9, Lemma 2], we have $v_n \to v$ as $n \to \infty$. Since Tv is closed, $v \in Tv$. Similarly, it follows that $v \in Sv$. Thus, S and T have a common fixed point. □

Corollary 16. *Let (X, d) be a complete complex valued metric space and let $S, T : X \to CB(X)$ be multivalued mappings with g.l.b property such that*

$$\alpha\left(d\left(x, Sx\right) + d\left(y, Ty\right)\right) \in s\left(Sx, Ty\right) \quad (68)$$

for all $x, y \in X$ and $0 \leq \alpha < 1/2$. Then S and T have a common fixed point.

By setting $S = T$ in Theorem 15, we get the following corollary.

Corollary 17. *Let (X, d) be a complete complex valued metric space and let $T : X \to CB(X)$ be multivalued mappings with g.l.b property such that*

$$ad\left(x, Tx\right) + bd\left(y, Ty\right) + c\frac{d\left(x, Tx\right)d\left(y, Ty\right)}{1 + d\left(x, y\right)} \in s\left(Tx, Ty\right) \quad (69)$$

for all $x, y \in X$ and $a + b + c < 1$. Then T has a fixed point.

By Remark 4 we have the following corollaries.

Corollary 18. *Let (X, d) be a complete metric space and let $S, T : X \to CB(X)$ be multivalued mappings such that*

$$H\left(Sx, Ty\right) \leq ad\left(x, Sx\right) + bd\left(y, Ty\right) + c\frac{d\left(x, Sx\right)d\left(y, Ty\right)}{1 + d\left(x, y\right)} \quad (70)$$

for all $x, y \in X$ and a, b, and c are nonnegative real numbers with $a + b + c < 1$. Then S, T have a common fixed point.

Remark 19. By writing a, b, c in all possible combinations, one can derive a host of corollaries which include the Kannan fixed point theorem for multivalued mappings in complete metric space.

In the following results, we considered the Chatterjea type locally contractive condition involving rational expressions.

Theorem 20. *Let* $S, T : X \rightarrow CB(X)$ *be multivalued mappings with the g.l.b property on complete complex valued metric space* (X, d), $x_0 \in X$, *and* $0 \prec r \in \mathbb{C}$. *If* S *and* T *satisfy*

$$ad(y, Sx) + bd(x, Ty) + c\left[\frac{d(y, Sx)\, d(x, Ty)}{1 + d(x, y)}\right] \in s(Sx, Ty)$$

$$(71)$$

for all $x, y \in \overline{B}(x_0, r)$ *and*

$$(1 - k)\, r \in s(x_0, Sx_0), \tag{72}$$

where $a, b,$ *and* c *are nonnegative real numbers with* $k = b/(1 - b) < 1$, *then* S *and* T *have a common fixed point in* $\overline{B}(x_0, r)$.

Proof. Let x_0 be an arbitrary point in X. From (72), one can easily prove that

$$\left| d(x_0, x_1) \right| \le (1 - k)\, |r|. \tag{73}$$

Thus, we have $x_1 \in \overline{B}(x_0, r)$. From (71), we get

$$ad(x_1, Sx_0) + bd(x_0, Tx_1) + c\left[\frac{d(x_1, Sx_0)\, d(x_0, Tx_1)}{1 + d(x_0, x_1)}\right]$$

$$\in s(Sx_0, Tx_1).$$

$$(74)$$

This implies that

$$ad(x_1, Sx_0) + bd(x_0, Tx_1) + c\left[\frac{d(x_1, Sx_0)\, d(x_0, Tx_1)}{1 + d(x_0, x_1)}\right]$$

$$\in s(x_1, Tx_1).$$

$$(75)$$

By the definition, we can take $x_2 \in Tx_1$ such that

$$ad(x_1, Sx_0) + bd(x_0, Tx_1) + c\left[\frac{d(x_1, Sx_0)\, d(x_0, Tx_1)}{1 + d(x_0, x_1)}\right]$$

$$\in s(d(x_1, x_2)).$$

$$(76)$$

By the definition, we get

$$d(x_1, x_2) \preceq ad(x_1, Sx_0) + bd(x_0, Tx_1)$$

$$+ c\left[\frac{d(x_1, Sx_0)\, d(x_0, Tx_1)}{1 + d(x_0, x_1)}\right]. \tag{77}$$

By using the g.l.b property of S and T, we get

$$d(x_1, x_2) \preceq ad(x_1, x_1) + bd(x_0, x_2)$$

$$+ c\left[\frac{d(x_1, x_1)\, d(x_0, x_2)}{1 + d(x_0, x_1)}\right]. \tag{78}$$

Hence, we have

$$d(x_1, x_2) \preceq bd(x_0, x_2) \tag{79}$$

which implies that

$$\left| d(x_1, x_2) \right| \le k\left| d(x_0, x_1)\right|, \tag{80}$$

where $k = b/(1 - b) < 1$. From (73), we get

$$\left| d(x_1, x_2) \right| \le k(1 - k)\, |r|. \tag{81}$$

Note that

$$\begin{aligned} \left| d(x_0, x_2) \right| &\le \left| d(x_0, x_1) \right| + \left| d(x_1, x_2) \right| \\ &\le (1 - k)\, |r| + k(1 - k)\, |r| \\ &= (1 - k)(1 + k)\, |r| \\ &\le \left(1 - k^2\right)|r|. \end{aligned} \tag{82}$$

Thus, we have $x_2 \in \overline{B}(x_0, r)$. From (71), we get

$$ad(x_2, Tx_1) + bd(x_1, Sx_2) + \left[c\frac{d(x_2, Tx_1)\, d(x_1, Sx_2)}{1 + d(x_1, x_2)}\right]$$

$$\in s(Tx_1, Sx_2).$$

$$(83)$$

This implies that

$$ad(x_2, Tx_1) + bd(x_1, Sx_2) + \left[c\frac{d(x_2, Tx_1)\, d(x_1, Sx_2)}{1 + d(x_1, x_2)}\right]$$

$$\in s(x_2, Sx_2).$$

$$(84)$$

By the definition, there exists $x_3 \in Sx_2$ such that

$$ad(x_2, Tx_1) + bd(x_1, Sx_2) + \left[c\frac{d(x_2, Tx_1)\, d(x_1, Sx_2)}{1 + d(x_1, x_2)}\right]$$

$$\in s(d(x_2, x_3)).$$

$$(85)$$

By the definition, we get

$$d(x_2, x_3) \preceq ad(x_2, Tx_1) + bd(x_1, Sx_2)$$

$$+ \left[c\frac{d(x_2, Tx_1)\, d(x_1, Sx_2)}{1 + d(x_1, x_2)}\right]. \tag{86}$$

By using the g.l.b property of S and T, we get

$$d(x_2, x_3) \preceq ad(x_2, x_2) + bd(x_1, x_3)$$

$$+ \left[c\frac{d(x_2, x_2)\, d(x_1, x_3)}{1 + d(x_1, x_2)}\right]. \tag{87}$$

Hence, we have

$$d(x_2, x_3) \preceq bd(x_1, x_3) \tag{88}$$

which implies that

$$|d(x_2, x_3)| \leq k|d(x_1, x_2)|, \tag{89}$$

where $k = b/(1-b) < 1$. Consider that

$$
\begin{aligned}
|d(x_0, x_3)| &\leq |d(x_0, x_1)| + |d(x_1, x_2)| + |d(x_2, x_3)| \\
&\leq (1-k)|r| + k(1-k)|r| + k^2(1-k)|r| \\
&= (1-k^3)|r| \leq |r|.
\end{aligned}
\tag{90}
$$

So $x_3 \in \overline{B}(x_0, r)$. Continuing in this way, we can construct a sequence $\{x_n\}$ in $\overline{B}(x_0, r)$ such that for each $n \geq 1$,

$$
\begin{aligned}
|d(x_{2n}, x_{2n+1})| &\leq k^{2n}|d(x_0, x_1)|, \\
|d(x_{2n+1}, x_{2n+2})| &\leq k^{2n+1}|d(x_0, x_1)|,
\end{aligned}
\tag{91}
$$

where $k = b/(1-b) < 1$, $x_{2n+1} \in Sx_{2n}$, and $x_{2n+2} \in Tx_{2n+1}$. Now, inductively, we can construct a sequence $\{x_n\}$ in X such that for each $n \geq 0$,

$$|d(x_n, x_{n+1})| \leq k^n|d(x_0, x_1)|. \tag{92}$$

Thus, as in the proof of Theorem 15, $\{x_n\}$ is a Cauchy sequence in $\overline{B}(x_0, r)$. Since X is complete and $\overline{B}(x_0, r)$ is a closed subspace of X, there exists $v \in \overline{B}(x_0, r)$ such that $x_n \to u$ as $n \to \infty$. Now, we show that $v \in Tv$ and $v \in Sv$. From (71), we get

$$
\begin{aligned}
&ad(v, Sx_{2n}) + bd(x_{2n}, Tv) + c\left[\frac{d(v, Sx_{2n})d(x_{2n}, Tv)}{1 + d(x_{2n}, v)}\right] \\
&\quad \in s(Sx_{2n}, Tv).
\end{aligned}
\tag{93}
$$

This implies that

$$
\begin{aligned}
&ad(v, Sx_{2n}) + bd(x_{2n}, Tv) + c\left[\frac{d(v, Sx_{2n})d(x_{2n}, Tv)}{1 + d(x_{2n}, v)}\right] \\
&\quad \in s(x_{2n+1}, Tv).
\end{aligned}
\tag{94}
$$

By the definition, there exists $v_n \in Tv$ such that

$$
\begin{aligned}
&ad(v, Sx_{2n}) + bd(x_{2n}, Tv) + c\left[\frac{d(v, Sx_{2n})d(x_{2n}, Tv)}{1 + d(x_{2n}, v)}\right] \\
&\quad \in s(d(x_{2n+1}, v_n)).
\end{aligned}
\tag{95}
$$

By the definition, we get

$$
\begin{aligned}
d(x_{2n+1}, v_n) &\preceq ad(v, Sx_{2n}) + bd(x_{2n}, Tv) \\
&\quad + c\left[\frac{d(v, Sx_{2n})d(x_{2n}, Tv)}{1 + d(x_{2n}, v)}\right].
\end{aligned}
\tag{96}
$$

By using the g.l.b property of S and T, we get

$$
\begin{aligned}
d(x_{2n+1}, v_n) &\preceq ad(v, x_{2n+1}) + bd(x_{2n}, v_n) \\
&\quad + c\left[\frac{d(v, x_{2n+1})d(x_{2n}, v_n)}{1 + d(x_{2n}, v)}\right].
\end{aligned}
\tag{97}
$$

Hence, we have

$$
\begin{aligned}
0 \prec d(x_{2n+1}, v_n) &\preceq ad(v, x_{2n+1}) + bd(x_{2n}, v_n) \\
&\quad + c\left[\frac{d(v, x_{2n+1})d(x_{2n}, v_n)}{1 + d(x_{2n}, v)}\right].
\end{aligned}
\tag{98}
$$

Now, by using the triangular inequality, we get

$$
\begin{aligned}
d(v, v_n) &\preceq d(v, x_{2n+1}) + d(x_{2n+1}, v_n) \\
&\preceq d(v, x_{2n+1}) + ad(v, x_{2n+1}) + bd(x_{2n}, v_n) \\
&\quad + c\left[\frac{d(v, x_{2n+1})d(x_{2n}, v_n)}{1 + d(x_{2n}, v)}\right],
\end{aligned}
\tag{99}
$$

and so

$$
\begin{aligned}
|d(v, v_n)| &\leq |d(v, x_{2n+1})| + a|d(v, x_{2n+1})| \\
&\quad + b|d(x_{2n}, v_n)| + c\left[\left|\frac{d(v, x_{2n+1})d(x_{2n}, v_n)}{1 + d(x_{2n}, v)}\right|\right].
\end{aligned}
\tag{100}
$$

By letting $n \to \infty$ in the above inequality, we get $|d(v, v_n)| \to 0$ as $n \to \infty$. By [9, Lemma 2.2], it follows that $v_n \to v$ as $n \to \infty$. Since Tv is closed, $v \in Tv$. Similarly, it follows that $v \in Sv$. Thus, S and T have a common fixed point in X. This completes the proof. $\square$

By taking $c = 0$ in Theorem 20, we get the following Corollary.

Corollary 21. *Let $S, T : X \to CB(X)$ be multivalued mappings with the g.l.b property in complete complex valued metric space (X, d), $x_0 \in X$, and $0 \prec r \in \mathbb{C}$. If S and T satisfy*

$$ad(y, Sx) + bd(x, Ty) \in s(Sx, Ty) \tag{101}$$

for all $x, y \in \overline{B}(x_0, r)$ and

$$(1-k)r \in s(x_0, Sx_0), \tag{102}$$

where a and b are nonnegative real numbers with $k = b/(1-b) < 1$, then S and T have a common fixed point in $\overline{B}(x_0, r)$.

By taking $S = T$ in Theorem 20, we get the following.

Corollary 22. *Let $T : X \to CB(X)$ be multivalued mapping with the g.l.b property in complete complex valued metric space (X, d), $x_0 \in X$, and $0 \prec r \in \mathbb{C}$. If T satisfies*

$$ad(y, Tx) + bd(x, Ty) + c\left[\frac{d(y, Tx)d(x, Ty)}{1 + d(x, y)}\right] \in s(Tx, Ty) \tag{103}$$

for all $x, y \in \overline{B}(x_0, r)$ *and*

$$(1 - k) r \in s(x_0, Tx_0), \tag{104}$$

where $a, b,$ *and* c *are nonnegative real numbers with* $k = b/(1 - b) < 1,$ *then* T *has a fixed point in* $\overline{B}(x_0, r).$

4. Applications

As an application of the main result (Corollary 22), we prove the following homotopy result.

Theorem 23. *Let* (X, d) *be a complete complex valued metric space and let* U *be an open subset of* X. *Let* $F : [0, 1] \times \overline{U} \to CB(X)$ *be multivalued mapping with the g.l.b property. Suppose that there exist* $\mathring{\zeta} \in X$ *and* $0 \prec r \in \mathbb{C}$ *such that*

(a) $\zeta \notin [F(t, \zeta)]$ *for all* $\zeta \in \partial U$ *and* $t \in [0, 1];$

(b) $F(t, \cdot) : \overline{U} \to CB(X)$ *is a multivalued mapping satisfying*

$$ad\left(\acute{\zeta}, F(t, \zeta)\right) + bd\left(\zeta, F\left(\acute{t}, \mathring{\zeta}\right)\right)$$
$$+ c\frac{d\left(\acute{\zeta}, F(t, \zeta)\right) d\left(\zeta, F\left(\acute{t}, \mathring{\zeta}\right)\right)}{1 + d\left(\zeta, \acute{\zeta}\right)} \tag{105}$$
$$\in s\left(F(t, \zeta), F\left(\acute{t}, \mathring{\zeta}\right)\right),$$
$$(1 - k) r \in s\left(\mathring{\zeta}, F\left(\acute{t}, \mathring{\zeta}\right)\right),$$

where $k = b/(1 - b) < 1;$

(c) *there exists a continuous increasing function* $\varphi : (0, 1] \to P \cup \{0\}$ *such that*

$$\varphi(s) - \varphi(t) \in s\left(F(s, \zeta), F\left(t, \mathring{\zeta}\right)\right), \quad \varphi(s) \in \varphi(t) \tag{106}$$

for all $s, t \in [0, 1]$ *and* $\zeta \in \overline{U}$, *where* $P = \{z \in \mathbb{C} : 0 \prec z\}.$

Then $F(0, \cdot)$ *has a fixed point if and only if* $F(1, \cdot)$ *has a fixed point.*

Proof. Suppose that $F(0, \cdot)$ has a fixed point z; so $z \in F(0, z)$. From (a), $z \in U$. Define the following set:

$$\mathcal{Q} := \{(t, \zeta) \in [0, 1] \times U : \zeta \in F(t, \zeta)\}. \tag{107}$$

Clearly, $\mathcal{Q} \neq \emptyset$. We define the partial ordering $\preceq$ in $\mathcal{Q}$ as follows:

$$(t, \zeta) \preceq \left(s, \acute{\zeta}\right) \iff t \le s,$$
$$d\left(\zeta, \acute{\zeta}\right) \preceq \frac{2}{1 - k}(\varphi(s) - \varphi(t)). \tag{108}$$

Let $\mathcal{M}$ be a totally ordered subset of $\mathcal{Q}$ and $\mathring{t} = \sup\{t : (t, \zeta) \in \mathcal{M}\}$. Consider a sequence $\{(t_n, \zeta_n)\}$ in $\mathcal{M}$ such that

$(t_n, \zeta_n) \preceq (t_{n+1}, \zeta_{n+1})$ and $t_n \to \mathring{t}$ as $n \to \infty$. Then, for any $n \ge 1$ with $m > n$, we have

$$d(\zeta_m, \zeta_n) \preceq \frac{2}{1 - k}(\varphi(t_m) - \varphi(t_n)) \longrightarrow 0 \tag{109}$$

as $n, m \to \infty$, which implies that $\{\zeta_n\}$ is a Cauchy sequence. Since (X, d) is a complete complex valued metric space, there exists $\mathring{\zeta} \in X$ such that $\zeta_n \to \mathring{\zeta}$. From (a), choose $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$,

$$ad\left(\mathring{\zeta}, F(t_n, \zeta_n)\right) + bd\left(\zeta_n, F\left(\mathring{t}, \mathring{\zeta}\right)\right)$$
$$+ c\frac{d\left(\mathring{\zeta}, F(t_n, \zeta_n)\right) d\left(\zeta_n, F\left(\mathring{t}, \mathring{\zeta}\right)\right)}{1 + d\left(\zeta_n, \mathring{\zeta}\right)}$$
$$\in s\left(F(t_n, \zeta_n), F\left(\mathring{t}, \mathring{\zeta}\right)\right),$$
$$ad\left(\mathring{\zeta}, F(t_n, \zeta_n)\right) + bd\left(\zeta_n, F\left(\mathring{t}, \mathring{\zeta}\right)\right) \tag{110}$$
$$+ \frac{cd\left(\mathring{\zeta}, F(t_n, \zeta_n)\right) d\left(\zeta_n, F\left(\mathring{t}, \mathring{\zeta}\right)\right)}{1 + d\left(\zeta_n, \mathring{\zeta}\right)}$$
$$\in s\left(\zeta_n, F\left(\mathring{t}, \mathring{\zeta}\right)\right),$$

since $\zeta_n \in F(t_n, \zeta_n)$. So there exists $\zeta_k \in F(\mathring{t}, \mathring{\zeta})$ such that

$$d(\zeta_n, \zeta_k) \preceq ad\left(\mathring{\zeta}, F(t_n, \zeta_n)\right) + bd\left(\zeta_n, F\left(\mathring{t}, \mathring{\zeta}\right)\right)$$
$$+ \frac{cd\left(\mathring{\zeta}, F(t_n, \zeta_n)\right) d\left(\zeta_n, F\left(\mathring{t}, \mathring{\zeta}\right)\right)}{1 + d\left(\zeta_n, \mathring{\zeta}\right)}. \tag{111}$$

By using the g.l.b property of F, we get

$$d(\zeta_n, \zeta_k) \preceq ad\left(\mathring{\zeta}, \zeta_n\right) + bd(\zeta_n, \zeta_k) + \frac{cd\left(\mathring{\zeta}, \zeta_n\right) d(\zeta_n, \zeta_k)}{1 + d\left(\zeta_n, \mathring{\zeta}\right)}, \tag{112}$$

which implies that

$$|d(\zeta_n, \zeta_k)| \le a\left|d\left(\mathring{\zeta}, \zeta_n\right)\right| + b|d(\zeta_n, \zeta_k)|$$
$$+ c\left|\frac{d\left(\mathring{\zeta}, \zeta_n\right)}{1 + d\left(\zeta_n, \mathring{\zeta}\right)}\right| |d(\zeta_n, \zeta_k)|. \tag{113}$$

Since $|1 + d(\zeta_n, \mathring{\zeta})| > |d(\zeta_n, \mathring{\zeta})|$, so we have

$$|d(\zeta_n, \zeta_k)| \le a\left|d\left(\mathring{\zeta}, \zeta_n\right)\right| + b|d(\zeta_n, \zeta_k)| + c|d(\zeta_n, \zeta_k)|,$$
$$|d(\zeta_n, \zeta_k)| \le \frac{a}{1 - b - c}\left|d\left(\mathring{\zeta}, \zeta_n\right)\right|. \tag{114}$$

Note that

$$
\begin{aligned}
\left| d\left(\mathring{\zeta}, \zeta_k \right) \right| &\leq \left| d\left(\mathring{\zeta}, \zeta_n \right) \right| + \left| d\left(\zeta_n, \zeta_k \right) \right| \\
&\leq \left| d\left(\mathring{\zeta}, \zeta_n \right) \right| + \frac{a}{1 - b - c} \left| d\left(\zeta_n, \mathring{\zeta} \right) \right| \longrightarrow 0
\end{aligned}
\tag{115}
$$

for all $n \geq n_0$. Thus $\zeta_k \rightarrow \mathring{\zeta} \in F(\mathring{t}, \mathring{\zeta})$ and hence $\mathring{\zeta} \in U$, which implies that $(\mathring{t}, \mathring{\zeta}) \in \mathcal{Q}$. Thus, $(t, \zeta) \preceq (\mathring{t}, \mathring{\zeta})$ for all $(t, \zeta) \in \mathcal{M}$; that is, $(\mathring{t}, \mathring{\zeta})$ is an upper bound of $\mathcal{M}$. Hence, by Zorn's Lemma, $\mathcal{Q}$ has the maximal element $(\mathring{t}, \mathring{\zeta})$.

Now, we claim that $\mathring{t} = 1$. Suppose that $\mathring{t} \leq 1$. Choose $0 \prec r \in \mathbb{C}$ and $\mathring{t} \leq t$ such that $\overline{B}(\mathring{\zeta}, r) \subset U$, where $r = (2/(1 - k))(\varphi(t) - \varphi(\mathring{t}))$. Using (c), we have

$$
\begin{aligned}
\varphi(t) - \varphi\left(\mathring{t} \right) &\in s\left(F(t, \zeta), F\left(\mathring{t}, \mathring{\zeta} \right) \right), \\
\varphi(t) - \varphi\left(\mathring{t} \right) &\in s\left(\mathring{\zeta}, F(t, \zeta) \right)
\end{aligned}
\tag{116}
$$

for all $\mathring{\zeta} \in F(\mathring{t}, \mathring{\zeta})$. So there exists $\zeta \in F(t, \zeta)$ such that $\varphi(t) - \varphi(\mathring{t}) \in s(d(\mathring{\zeta}, \zeta))$ and so

$$
d\left(\zeta, \mathring{\zeta} \right) \preceq \varphi(t) - \varphi\left(\mathring{t} \right) \preceq \frac{(1 - k) r}{2} \prec (1 - k) r,
\tag{117}
$$

which implies that

$$
\left| d\left(\zeta, \mathring{\zeta} \right) \right| \leq (1 - k) |r|.
\tag{118}
$$

Also, by using (b), we conclude that the mapping $F(t, \cdot) : \overline{B}(\mathring{\zeta}, r) \rightarrow CB(X)$ satisfies all the assumptions of Corollary 22 for all $t \in [0, 1]$. Hence, for all $t \in [0, 1]$, there exists $\zeta \in \overline{B}(\mathring{\zeta}, r)$ such that $\zeta \in F(t, \zeta)$. Thus, $(\zeta, t) \in \mathcal{Q}$.

Consider

$$
d\left(\zeta, \mathring{\zeta} \right) \prec r = \frac{2}{1 - k} \left(\varphi(t) - \varphi\left(\mathring{t} \right) \right),
\tag{119}
$$

which implies that $(\mathring{t}, \mathring{\zeta}) \preceq (t, \zeta)$, which is a contradiction. Thus, we have $\mathring{t} = 1$. Therefore, $F(\cdot, 1)$ has a fixed point.

Conversely, if $F(1, \cdot)$ has a fixed point, then, in the same way, we can prove that $F(0, \cdot)$ has a fixed point. This completes the proof. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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