Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 854893, 7 pages
http://dx.doi.org/10.1155/2013/854893

Research Article

An Implicit Algorithm for the Split Fixed Point and Convex Feasibility Problems

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Received 18 July 2013; Accepted 29 August 2013

Academic Editor: Jen-Chih Yao

We consider an implicit algorithm for the split fixed point and convex feasibility problems. Strong convergence theorem is obtained.

1. Introduction

Due to their broad applicability in many areas, especially in signal processing (e.g., phase retrieval) and image restoration, the split feasibility problems continue to receive great attention; see, for example, [1–6]. The present paper is devoted to this topic. Now we recall that the split feasibility problem originally introduced by Censor and Elfving [7] is to find $x^{\dagger}$ such that

$$x^{\dagger} \in \mathbb{C}, \quad \mathbb{A}x^{\dagger} \in \mathbb{Q}, \tag{1}$$

where $\mathbb{C}$ and $\mathbb{Q}$ are two closed convex subsets of two Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively, and $\mathbb{A} : \mathbb{H}_1 \to \mathbb{H}_2$ is a bounded linear operator. A special case of (1) is when $\mathbb{Q} = \{b\}$ is singleton, and then (1) is reduced to the convexly constrained linear inverse problem

$$x^{\dagger} \in \mathbb{C}, \quad \mathbb{A}x^{\dagger} = b, \tag{2}$$

which has received considerable attention. We can use projected Landweber algorithm to solve (2). The projected Landweber algorithm generates a sequence $\{x^k\}$ in such a way that

$$x^{k+1} = \operatorname{proj}_{\mathbb{C}}\left(x^k + \gamma\mathbb{A}^*\left(b - \mathbb{A}x^k\right)\right), \tag{3}$$

where $\operatorname{proj}_{\mathbb{C}}$ denotes the nearest point projection from $\mathbb{H}_1$ onto $\mathbb{C}$, $\gamma > 0$ is a parameter such that $0 < \gamma < 2/\|\mathbb{A}\|^2$, and $\mathbb{A}^*$ is the transpose of $\mathbb{A}$. When the system (2) is reduced to the unconstrained linear system

$$\mathbb{A}x^* = b, \tag{4}$$

then the projected Landweber algorithm is turned to the Landweber algorithm

$$x^{k+1} = x^k + \gamma\mathbb{A}^*\left(b - \mathbb{A}x^k\right). \tag{5}$$

Note that (1) is equivalent to the fixed point equation

$$x^{\dagger} = \operatorname{proj}_{\mathbb{C}}\left(I - \eta\mathbb{A}^*\left(I - \operatorname{proj}_{\mathbb{Q}}\right)\mathbb{A}\right)x^{\dagger}. \tag{6}$$

Using this relation, we can suggest the following iterative algorithm:

$$x^{k+1} = \operatorname{proj}_{\mathbb{C}}\left(x^k - \zeta\mathbb{A}^*\left(I - \operatorname{proj}_{\mathbb{Q}}\right)\mathbb{A}x^k\right), \quad k \in \mathbb{N}, \tag{7}$$

which is refereed as $\mathbb{CQ}$ algorithm and was devised by Byrne [8]. $\mathbb{CQ}$ algorithm has been extensively studied; see, for instance, [9–11].

The $\mathbb{CQ}$ algorithm (7) is proved to converge weakly but fails to converge in norm in general infinite-dimensional Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$. Tikhonov's regularization method can solve this problem. First, we define a convex function f by

$$f(x) = \frac{1}{2}\left\|\mathbb{A}x - \operatorname{proj}_{\mathbb{Q}}\mathbb{A}x\right\|^2 \tag{8}$$

with its gradient

$$\nabla f(x) = \mathbb{A}^* (I - \mathrm{proj}_{\mathbb{Q}}) \mathbb{A}x \tag{9}$$

and consider the minimization problem

$$\min_{x\in\mathbb{C}} f(x). \tag{10}$$

It is known that $x^* \in \mathbb{C}$ solves (1) if and only if $f(x^*)$. We know that (10) is ill-posed. So regularization is needed. We consider Tikhonov's regularization:

$$\min_{x\in\mathbb{C}} f_\alpha(x) := \frac{1}{2}\left\|(I - \mathrm{proj}_{\mathbb{Q}}) \mathbb{A}x\right\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{11}$$

where $\alpha > 0$ is the regularization parameter. The gradient ∇f_α of f_α is given by

$$\nabla f_\alpha(x) = \nabla f(x) + \alpha x = \mathbb{A}^* (I - \mathrm{proj}_{\mathbb{Q}}) \mathbb{A}x + \alpha x. \tag{12}$$

Define a Picard iterates

$$x_\alpha^{k+1} = \mathrm{proj}_{\mathbb{C}} (I - \delta (\mathbb{A}^* (I - \mathrm{proj}_{\mathbb{Q}}) \mathbb{A} + \alpha I)) x_\alpha^k, \quad k \in \mathbb{N}. \tag{13}$$

Xu [12] proves that if (1) is solvable, then as $k \to \infty$, $x_\alpha^k \to x_\alpha$ and consequently the strong $\lim_{\alpha\to 0} x_\alpha$ exists and is the minimum-norm solution of (1). Note that (13) is a double-step iteration. Xu [12] introduced a single step regularized method:

$$\begin{aligned} x^{k+1} &= \mathrm{proj}_{\mathbb{C}} \left(I - \delta_k \nabla f_{\xi_k}\right) x^k \\ &= \mathrm{proj}_{\mathbb{C}} \left((1 - \xi_k\delta_k) x^k \right. \\ &\quad \left. -\delta_k \mathbb{A}^* (I - \mathrm{proj}_{\mathbb{Q}}) \mathbb{A}x^k\right), \quad k \in \mathbb{N}. \end{aligned} \tag{14}$$

It is shown that the sequence $\{x^k\}$ generated by (14) converges to the solution of (1) provided that the parameters $\{\xi_k\} \subset (0,1)$ and $\{\delta_k\} \subset (0, \xi_n/(\|A\|^2 + \xi_n))$ satisfy

$$\begin{gathered} \lim_{k\to\infty} \xi_k = 0, \qquad \sum_{k=1}^{\infty} \xi_k\delta_k = \infty, \\ \lim_{k\to\infty} \frac{|\delta_{k+1} - \delta_k| + \delta_k |\xi_{k+1} - \xi_k|}{(\xi_{k+1}\delta_{k+1})^2} = 0. \end{gathered} \tag{15}$$

Inspired by (14), Ceng et al. [3] introduced the following relaxed extragradient method:

$$\begin{aligned} y^k &= \mathrm{proj}_{\mathbb{C}} \left(x^k - \xi_k \left(\nabla f(x_k) + \theta_k x^k\right)\right), \\ x^{k+1} &= \beta_k x^k + \gamma_k y^k + \delta_k \mathrm{proj}_{\mathbb{C}} \\ &\quad \times \left(x^k - \xi_k \left(\nabla f\left(y^k\right) + \theta_k y^k\right)\right), \quad k \in \mathbb{N}, \end{aligned} \tag{16}$$

where the sequences $\{\theta_k\} \subset (0,1), \{\beta_k\} \subset (0,1), \{\gamma_k\} \subset (0,1), \{\delta_k\} \subset (0,1)$, and $\{\xi_k\} \subset (0, \theta_k/(\|A\|^2 + \theta_k)^2)$ satisfy the conditions

$$\begin{gathered} \lim_{k\to\infty} \theta_k = \lim_{k\to\infty} \xi_k = \lim_{k\to\infty} \frac{|\xi_{k+1} - \xi_k| + \xi_k |\theta_{k+1} - \theta_k|}{\theta_{k+1}^3 \xi_{k+1}^2 \delta_{k+1}} = 0, \\ \sum_{k=1}^{\infty} \theta_k^2 \xi_k \delta_k = \infty, \qquad \frac{2\delta_k}{\|A\|^2 + \theta_k} \le \gamma_k \xi_k \end{gathered} \tag{17}$$

for all $k \in \mathbb{N}$. Ceng et al. proved that the sequence $\{x^k\}$ generated by (16) converges to the solution of (1) which is the the minimum-norm element. Recently, Ceng et al. [13] further introduced another regularization for the split feasibility problem and the fixed point problem:

$$\begin{aligned} y^k &= \mathrm{proj}_{\mathbb{C}} \left(x^k - \gamma_k \left(\nabla f(x_k) + \theta_k x^k\right)\right), \\ x^{k+1} &= \vartheta_k x^k + (1 - \vartheta_k) \\ &\quad \times U \mathrm{proj}_{\mathbb{C}} \left(x^k - \gamma_k \left(\nabla f\left(y^k\right) + \theta_k y^k\right)\right), \quad k \in \mathbb{N}. \end{aligned} \tag{18}$$

Ceng et al. proved that algorithm (18) has weak convergence.

Motivated by the above works, in this paper, our main purpose is to introduce an implicit algorithm for solving the split fixed point and convex feasibility problems. We show that the implicit algorithm converges strongly to the solution of the split fixed point and convex feasibility problems.

2. Preliminaries

Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $\mathbb{C}$ be a nonempty closed convex subset of $\mathbb{H}$.

Definition 1. A mapping $\mathbb{U} : \mathbb{C} \to \mathbb{C}$ is called nonexpansive if

$$\|\mathbb{U}p - \mathbb{U}q\| \le \|p - q\| \tag{19}$$

for all $p, q \in \mathbb{C}$.

We will use $\mathrm{Fix}(\mathbb{U})$ to denote the set of fixed points of $\mathbb{U}$; that is, $\mathrm{Fix}(\mathbb{U}) = \{x^\ddagger \in C : x^\ddagger = \mathbb{U}x^\ddagger\}$.

Definition 2. A mapping $\mathscr{C} : \mathbb{C} \to \mathbb{C}$ is called contractive if

$$\|\mathscr{C}(p) - \mathscr{C}(q)\| \le \kappa \|p - q\| \tag{20}$$

for all $p, q \in \mathbb{C}$ and for some constant $\kappa \in (0,1)$. In this case, we call $\mathscr{C}$ a κ-contraction.

Definition 3. A linear bounded operator $\mathbb{B} : \mathbb{H} \to \mathbb{H}$ is called *strongly positive* if there exists a constant $\overline{\gamma} > 0$ such that

$$\left\langle \mathbb{B}x^\ddagger, x^\ddagger \right\rangle \ge \overline{\gamma} \left\|x^\ddagger\right\|^2 \tag{21}$$

for all $x^\ddagger \in \mathbb{H}$.

Definition 4. We call that $\text{proj}_{\mathbb{C}} : \mathbb{H} \to \mathbb{C}$ is the *metric projection* if for each $x^{\ddagger} \in \mathbb{H}$

$$\left\| x^{\ddagger} - \text{proj}_{\mathbb{C}}\left(x^{\ddagger}\right) \right\| = \inf\left\{ \left\| x^{\ddagger} - x \right\| : x \in \mathbb{C} \right\}. \quad (22)$$

It is well known that the metric projection $\text{proj}_{\mathbb{C}} : \mathbb{H} \to \mathbb{C}$ is characterized by

$$\left\langle x^{\ddagger} - \text{proj}_{\mathbb{C}}\left(x^{\ddagger}\right), x - \text{proj}_{\mathbb{C}}\left(x^{\ddagger}\right) \right\rangle \le 0 \quad (23)$$

for all $x^{\ddagger} \in \mathbb{H}$, $x \in \mathbb{C}$. From this, we can deduce that $\text{proj}_{\mathbb{C}}$ is firmly nonexpansive; that is,

$$\begin{aligned} &\left\| \text{proj}_{\mathbb{C}}\left(x^{\ddagger}\right) - \text{proj}_{\mathbb{C}}(x) \right\|^2 \\ &\quad \le \left\langle x^{\ddagger} - x, \text{proj}_{\mathbb{C}}\left(x^{\ddagger}\right) - \text{proj}_{\mathbb{C}}(x) \right\rangle \end{aligned} \quad (24)$$

for all $x^{\ddagger}, x \in \mathbb{H}$. Hence $\text{proj}_{\mathbb{C}}$ is nonexpansive.

Lemma 5 (see [14]). *Let $\mathbb{C}$ be a closed convex subset of a real Hilbert space $\mathbb{H}$, and let $\mathbb{U} : \mathbb{C} \to \mathbb{C}$ be a nonexpansive mapping. Then, the mapping $I - \mathbb{U}$ is demiclosed. That is, if $\{x^k\}$ is a sequence in C such that $x^k \to x^{\$}$ weakly and $(I - \mathbb{U})x^k \to y$ strongly, then $(I - \mathbb{U})x^{\$} = y$.*

3. Main Result

In this section, we first introduce our algorithm for solving this problem and consequently we give convergence analysis.

Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two Hilbert spaces and $\mathbb{C} \subset \mathbb{H}_1$ and $\mathbb{Q} \subset \mathbb{H}_2$ two nonempty closed convex sets. Let $\mathbb{A} : \mathbb{H}_1 \to \mathbb{H}_2$ be a bounded linear operator with its adjoint $\mathbb{A}^*$. Let $\mathbb{B}$ be a strongly positive bounded linear operator on $\mathbb{H}_1$ with coefficient $\overline{\gamma} > 0$. Let $\mathscr{C} : \mathbb{H}_1 \to \mathbb{H}_1$ be a κ-contraction. Let $\mathbb{V} : \mathbb{Q} \to \mathbb{Q}$ and $\mathbb{U} : \mathbb{C} \to \mathbb{C}$ be two nonexpansive mappings.

In the sequel, our objective is to

$$\text{Find } x^{\dagger} \in \mathbb{C} \cap \text{Fix}(\mathbb{U}) \text{ such that } \mathbb{A}x^{\dagger} \in \mathbb{Q} \cap \text{Fix}(\mathbb{V}). \quad (25)$$

We use Ω to denote the solution set of (25); that is,

$$\Omega = \left\{ x^{\dagger} \mid x^{\dagger} \in \mathbb{C} \cap \text{Fix}(\mathbb{U}), \mathbb{A}x^{\dagger} \in \mathbb{Q} \cap \text{Fix}(\mathbb{V}) \right\}. \quad (26)$$

Now, we introduce the following implicit algorithm.

Algorithm 6. Define an implicit algorithm $\{x_t\}$ as follows:

$$\begin{aligned} x_t &= t\zeta\mathscr{C}(x_t) + (I - t\mathbb{B})\,\mathbb{U}\text{proj}_{\mathbb{C}} \\ &\quad \times \left(x_t - \delta\mathbb{A}^*\left(I - \mathbb{V}\text{proj}_{\mathbb{Q}}\right)\mathbb{A}x_t\right), \quad t \in (0,1), \end{aligned} \quad (27)$$

where $\zeta \in (0, \overline{\gamma}/\kappa)$ and $\delta \in (0, 1/\|A\|^2)$ are two constants.

Remark 7. $\{x_t\}$ is well-defined. Define a mapping $\mathbb{R} : \mathbb{C} \to \mathbb{C}$ as

$$\mathbb{R}x = \text{proj}_{\mathbb{C}}\left(x - \delta\mathbb{A}^*\left(I - \mathbb{V}\text{proj}_{\mathbb{Q}}\right)\mathbb{A}x\right), \quad \forall x \in \mathbb{C}. \quad (28)$$

Then, we have

$$\begin{aligned}
\|\mathbb{R}x - \mathbb{R}y\|^2 &= \big\|\text{proj}_{\mathbb{C}}\left(x - \delta\mathbb{A}^*\left(I - \mathbb{V}\text{proj}_{\mathbb{Q}}\right)\mathbb{A}x\right) \\
&\qquad - \text{proj}_{\mathbb{C}}\left(y - \delta\mathbb{A}^*\left(I - \mathbb{V}\text{proj}_{\mathbb{Q}}\right)\mathbb{A}y\right)\big\|^2 \\
&\le \big\|\left(x - \delta\mathbb{A}^*\left(I - \mathbb{V}\text{proj}_{\mathbb{Q}}\right)\mathbb{A}x\right) \\
&\qquad - \left(y - \delta\mathbb{A}^*\left(I - \mathbb{V}\text{proj}_{\mathbb{Q}}\right)\mathbb{A}y\right)\big\|^2 \\
&= \big\|(x - y) + \delta\mathbb{A}^* \\
&\qquad \times \left[\left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{A}x\right) - \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y - \mathbb{A}y\right)\right]\big\|^2 \\
&= \|x - y\|^2 \\
&\quad + 2\delta\big\langle x - y, \mathbb{A}^*\left[\left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{A}x\right)\right. \\
&\qquad \left.- \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y - \mathbb{A}y\right)\right]\big\rangle \\
&\quad + \delta^2\big\|\mathbb{A}^*\left[\left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{A}x\right)\right. \\
&\qquad \left.- \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y - \mathbb{A}y\right)\right]\big\|^2 \\
&= \|x - y\|^2 + 2\delta\big\langle \mathbb{A}(x - y), \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{A}x\right) \\
&\qquad - \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y - \mathbb{A}y\right)\big\rangle \\
&\quad + \delta^2\big\|\mathbb{A}^*\left[\left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{A}x\right)\right. \\
&\qquad \left.- \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y - \mathbb{A}y\right)\right]\big\|^2 \\
&= \|x - y\|^2 \\
&\quad + 2\delta\big\langle \mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y, \\
&\qquad \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{A}x\right) - \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y - \mathbb{A}y\right)\big\rangle \\
&\quad - 2\delta\big\|\left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{A}x\right) \\
&\qquad - \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y - \mathbb{A}y\right)\big\|^2 \\
&\quad + \delta^2\big\|\mathbb{A}^*\left[\left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{A}x\right)\right. \\
&\qquad \left.- \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y - \mathbb{A}y\right)\right]\big\|^2 \\
&= \|x - y\|^2 + \delta\Big(\big\|\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y\big\|^2 \\
&\qquad + \big\|\left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{A}x\right) \\
&\qquad - \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y - \mathbb{A}y\right)\big\|^2 \\
&\qquad - \|\mathbb{A}x - \mathbb{A}y\|^2\Big) \\
&\quad - 2\delta\big\|\left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{A}x\right) \\
&\qquad - \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y - \mathbb{A}y\right)\big\|^2 \\
&\quad + \delta^2\big\|\mathbb{A}^*\left[\left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}x - \mathbb{A}x\right)\right. \\
&\qquad \left.- \left(\mathbb{V}\text{proj}_{\mathbb{Q}}\mathbb{A}y - \mathbb{A}y\right)\right]\big\|^2
\end{aligned}$$

$$\begin{aligned}&\leq \|x-y\|^2-\delta\|(\mathbb{V}\mathrm{proj}_{\mathbb{Q}}\mathbb{A}x-\mathbb{A}x)\\&\qquad-(\mathbb{V}\mathrm{proj}_{\mathbb{Q}}\mathbb{A}y-\mathbb{A}y)\|^2\\&\quad+\delta^2\|\mathbb{A}\|^2\|(\mathbb{V}\mathrm{proj}_{\mathbb{Q}}\mathbb{A}x-\mathbb{A}x)\\&\qquad-(\mathbb{V}\mathrm{proj}_{\mathbb{Q}}\mathbb{A}y-\mathbb{A}y)\|^2\\&\leq\|x-y\|^2.\end{aligned}\tag{29}$$

This indicates that R is nonexpansive. Consequently, for fixed $t\in(0,1)$, we have that the mapping $t\zeta\mathscr{C}+(I-t\mathbb{B})\mathbb{U}R$ is contractive due to the facts that $\mathscr{C}$ is a κ-contraction and $\mathbb{U}$ is nonexpansive. Therefore, $\{x_t\}$ is well-defined.

Next, we prove the convergence of (27).

Theorem 8. *Suppose that $\Omega\neq\emptyset$. Then the net $\{x_t\}$ generated by algorithm* (25) *converges strongly to $p^{\dagger}=\mathrm{proj}_{\Omega}(\zeta\mathscr{C}+I-\mathbb{B})p^{\dagger}$ which solves the following variational inequality:*

$$\langle(\zeta\mathscr{C}-\mathbb{B})x,y-x\rangle\leq 0,\quad\forall y\in\Omega.\tag{30}$$

Proof. Set $r_t=\mathrm{proj}_{\mathbb{Q}}\mathbb{A}x_t$, $v_t=x_t-\delta\mathbb{A}^*(I-\mathbb{V}\mathrm{proj}_{\mathbb{Q}})\mathbb{A}x_t$, and $u_t=\mathrm{proj}_{\mathbb{C}}(x_t-\delta\mathbb{A}^*(I-\mathbb{V}\mathrm{proj}_{\mathbb{Q}})\mathbb{A}x_t)$ for all $t\in(0,1)$. Then $u_t=\mathrm{proj}_{\mathbb{C}}v_t$. It is clear that the solution of (30) is unique. Let $p^{\dagger}=\mathrm{proj}_{\Omega}(\zeta\mathscr{C}+I-\mathbb{B})p^{\dagger}$. Then, we have $p^{\dagger}\in\mathbb{C}\cap\mathrm{Fix}(\mathbb{U})$ and $\mathbb{A}p^{\dagger}\in\mathbb{Q}\cap\mathrm{Fix}(\mathbb{V})$. First, we easily deduce the following three inequalities:

$$\|r_t-\mathbb{A}p^{\dagger}\|=\|\mathrm{proj}_{\mathbb{Q}}\mathbb{A}x_t-\mathbb{A}p^{\dagger}\|\leq\|\mathbb{A}x_t-\mathbb{A}p^{\dagger}\|,\tag{31}$$

$$\|u_t-p^{\dagger}\|=\|\mathrm{proj}_{\mathbb{C}}v_t-p^{\dagger}\|\leq\|v_t-p^{\dagger}\|,\tag{32}$$

$$\|\mathbb{V}r_t-\mathbb{A}p^{\dagger}\|^2\leq\|r_t-\mathbb{A}p^{\dagger}\|^2\leq\|\mathbb{A}x_t-\mathbb{A}p^{\dagger}\|^2-\|r_t-\mathbb{A}x_t\|^2.\tag{33}$$

From (25), we have

$$\begin{aligned}\|x_t-p^{\dagger}\|&=\|t(\zeta\mathscr{C}(x_t)-\mathbb{B}p^{\dagger})\\&\qquad+(I-t\mathbb{B})(\mathbb{U}u_t-p^{\dagger})\|\\&\leq t\zeta\|\mathscr{C}(x_t)-\mathscr{C}(p^{\dagger})\|\\&\quad+t\|\zeta\mathscr{C}(p^{\dagger})-\mathbb{B}p^{\dagger}\|\\&\quad+(1-t\overline{\gamma})\|u_t-p^{\dagger}\|.\end{aligned}\tag{34}$$

Note that

$$\begin{aligned}\|v_t-p^{\dagger}\|^2&=\|x_t-p^{\dagger}+\delta\mathbb{A}^*(\mathbb{V}r_t-\mathbb{A}x_t)\|^2\\&=\|x_t-p^{\dagger}\|^2+\delta^2\|\mathbb{A}^*(\mathbb{V}r_t-\mathbb{A}x_t)\|^2\\&\quad+2\delta\langle x_t-p^{\dagger},\mathbb{A}^*(\mathbb{V}r_t-\mathbb{A}x_t)\rangle.\end{aligned}\tag{35}$$

Since A is a linear operator and $\mathbb{A}^*$ is the adjoint of $\mathbb{A}$, we get

$$\begin{aligned}&\langle x_t-p^{\dagger},\mathbb{A}^*(\mathbb{V}r_t-\mathbb{A}x_t)\rangle\\&\quad=\langle\mathbb{A}(x_t-p^{\dagger}),\mathbb{V}r_t-\mathbb{A}x_t\rangle\\&\quad=\langle\mathbb{A}x_t-\mathbb{A}p^{\dagger}+\mathbb{V}r_t-\mathbb{A}x_t\\&\qquad-(\mathbb{V}r_t-\mathbb{A}x_t),\mathbb{V}r_t-\mathbb{A}x_t\rangle\\&\quad=\langle\mathbb{V}r_t-\mathbb{A}p^{\dagger},\mathbb{V}r_t-\mathbb{A}x_t\rangle-\|\mathbb{V}r_t-\mathbb{A}x_t\|^2.\end{aligned}\tag{36}$$

At the same time, we know

$$\begin{aligned}&\langle\mathbb{V}r_t-\mathbb{A}p^{\dagger},\mathbb{V}r_t-\mathbb{A}x_t\rangle\\&\quad=\frac{1}{2}\left(\|\mathbb{V}r_t-\mathbb{A}p^{\dagger}\|^2+\|\mathbb{V}r_t-\mathbb{A}x_t\|^2-\|\mathbb{A}x_t-\mathbb{A}p^{\dagger}\|^2\right).\end{aligned}\tag{37}$$

By (33), (36), and (37), we get

$$\begin{aligned}&\langle x_t-p^{\dagger},\mathbb{A}^*(\mathbb{V}r_t-\mathbb{A}x_t)\rangle\\&\quad=\frac{1}{2}\left(\|\mathbb{V}r_t-\mathbb{A}p^{\dagger}\|^2+\|\mathbb{V}r_t-\mathbb{A}x_t\|^2\right.\\&\qquad\left.-\|\mathbb{A}x_t-\mathbb{A}p^{\dagger}\|^2\right)-\|\mathbb{V}r_t-\mathbb{A}x_t\|^2\\&\quad\leq\frac{1}{2}\left(\|\mathbb{A}x_t-\mathbb{A}p^{\dagger}\|^2-\|r_t-\mathbb{A}x_t\|^2\right.\\&\qquad\left.+\|\mathbb{V}r_t-\mathbb{A}x_t\|^2-\|\mathbb{A}x_t-\mathbb{A}p^{\dagger}\|^2\right)\\&\qquad-\|\mathbb{V}r_t-\mathbb{A}x_t\|^2\\&\quad=-\frac{1}{2}\|r_t-\mathbb{A}x_t\|^2-\frac{1}{2}\|\mathbb{V}r_t-\mathbb{A}x_t\|^2.\end{aligned}\tag{38}$$

Substituting (38) into (35) to deduce

$$\begin{aligned}\|v_t-p^{\dagger}\|^2&\leq\|x_t-p^{\dagger}\|^2+\delta^2\|\mathbb{A}\|^2\|\mathbb{V}r_t-\mathbb{A}x_t\|^2\\&\quad+2\delta\left(-\frac{1}{2}\|r_t-\mathbb{A}x_t\|^2-\frac{1}{2}\|\mathbb{V}r_t-\mathbb{A}x_t\|^2\right)\\&=\|x_t-p^{\dagger}\|^2+(\delta^2\|\mathbb{A}\|^2-\delta)\|\mathbb{V}r_{t-\mathbb{A}x_t}\|^2\\&\quad-\delta\|r_t-\mathbb{A}x_t\|^2\\&\leq\|x_t-p^{\dagger}\|^2.\end{aligned}\tag{39}$$

It follows that

$$\|v_t-p^{\dagger}\|\leq\|x_t-p^{\dagger}\|.\tag{40}$$

Thus, from (34), we get

$$\begin{aligned}\left\|x_t-p^{\dagger}\right\| \le\ & t\zeta\kappa\left\|x_t-p^{\dagger}\right\| \\ & + t\left\|\zeta\mathscr{C}\left(p^{\dagger}\right)-\mathbb{B}p^{\dagger}\right\| \\ & + (1-t\overline{\gamma})\left\|x_t-p^{\dagger}\right\| \\ =\ & \left[1-(\overline{\gamma}-\zeta\kappa)t\right]\left\|x_t-p^{\dagger}\right\| \\ & + t\left\|\zeta\mathscr{C}\left(p^{\dagger}\right)-Bp^{\dagger}\right\|.\end{aligned} \tag{41}$$

So,

$$\left\|x_t-p^{\dagger}\right\| \le \frac{\left\|\zeta\mathscr{C}\left(p^{\dagger}\right)-Bp^{\dagger}\right\|}{\overline{\gamma}-\zeta\kappa}. \tag{42}$$

The boundedness of the net $\{x_t\}$ yields.

Since $x_t-\mathbb{U}u_t=t(\zeta\mathscr{C}(x_t)-\mathbb{B}\mathbb{U}u_t)$, we obtain

$$\lim_{t\to 0}\left\|x_t-\mathbb{U}u_t\right\|=0. \tag{43}$$

Using the firmly nonexpansive necessity of $\text{proj}_{\mathbb{C}}$, we have

$$\begin{aligned}\left\|u_t-p^{\dagger}\right\|^2 &= \left\|\text{proj}_{\mathbb{C}}v_t-p^{\dagger}\right\|^2 \\ &\le \left\|v_t-p^{\dagger}\right\|^2-\left\|\text{proj}_{\mathbb{C}}v_t-v_t\right\|^2 \\ &= \left\|v_t-p^{\dagger}\right\|^2-\left\|u_t-v_t\right\|^2.\end{aligned} \tag{44}$$

From (34), we derive that

$$\begin{aligned}\left\|x_t-p^{\dagger}\right\|^2 &= \left\|t\left(\zeta\mathscr{C}(x_t)-\mathbb{B}p^{\dagger}\right)+(I-t\mathbb{B})\left(\mathbb{U}u_t-p^{\dagger}\right)\right\|^2 \\ &\le \left\|(I-t\mathbb{B})\left(\mathbb{U}u_t-p^{\dagger}\right)\right\|^2 \\ &\quad + 2t\left\langle\zeta\mathscr{C}(x_t)-\mathbb{B}p^{\dagger},\ x_t-p^{\dagger}\right\rangle \\ &\le \left[\|I-t\mathbb{B}\|\left\|\mathbb{U}u_t-p^{\dagger}\right\|\right]^2 \\ &\quad + 2t\left\|\zeta\mathscr{C}(x_t)-\mathbb{B}p^{\dagger}\right\|\left\|x_t-p^{\dagger}\right\| \\ &\le (1-t\overline{\gamma})^2\left\|u_t-p^{\dagger}\right\|^2 \\ &\quad + 2t\left\|\zeta\mathscr{C}(x_t)-\mathbb{B}p^{\dagger}\right\|\left\|x_t-p^{\dagger}\right\|.\end{aligned} \tag{45}$$

This together with (44) implies that

$$\begin{aligned}\left\|x_t-p^{\dagger}\right\|^2 &\le \left\|v_t-p^{\dagger}\right\|^2-\left\|u_t-v_t\right\|^2 \\ &\quad + 2t\left\|\zeta\mathscr{C}(x_t)-\mathbb{B}p^{\dagger}\right\|\left\|x_t-p^{\dagger}\right\| \\ &\le \left\|x_t-p^{\dagger}\right\|^2-\left\|u_t-v_t\right\|^2 \\ &\quad + 2t\left\|\zeta\mathscr{C}(x_t)-\mathbb{B}p^{\dagger}\right\|\left\|x_t-p^{\dagger}\right\|.\end{aligned} \tag{46}$$

It follows that

$$\left\|u_t-v_t\right\|^2 \le 2t\left\|\zeta\mathscr{C}(x_t)-Bp^{\dagger}\right\|\left\|x_t-p^{\dagger}\right\|. \tag{47}$$

Hence,

$$\lim_{t\to 0}\left\|u_t-v_t\right\|=0. \tag{48}$$

Returning to (45) and using (39), we have

$$\begin{aligned}\left\|x_t-p^{\dagger}\right\|^2 &\le (1-t\overline{\gamma})^2\left\|u_t-p^{\dagger}\right\|^2 \\ &\quad + 2t\left\|\zeta\mathscr{C}(x_t)-\mathbb{B}p^{\dagger}\right\|\left\|x_t-p^{\dagger}\right\| \\ &\le \left\|v_t-p^{\dagger}\right\|^2+2t\left\|\zeta\mathscr{C}(x_t)-\mathbb{B}p^{\dagger}\right\|\left\|x_t-p^{\dagger}\right\| \\ &\le \left\|x_t-p^{\dagger}\right\|^2+\left(\delta^2\|\mathbb{A}\|^2-\delta\right)\left\|\mathbb{V}r_t-\mathbb{A}x_t\right\|^2 \\ &\quad - \delta\left\|r_t-\mathbb{A}x_t\right\|^2 \\ &\quad + 2t\left\|\zeta\mathscr{C}(x_t)-\mathbb{B}p^{\dagger}\right\|\left\|x_t-p^{\dagger}\right\|.\end{aligned} \tag{49}$$

Thus,

$$\begin{aligned}&\left(\delta-\delta^2\|\mathbb{A}\|^2\right)\left\|\mathbb{V}r_t-\mathbb{A}x_t\right\|^2+\delta\left\|r_t-\mathbb{A}x_t\right\|^2 \\ &\quad \le 2t\left\|\zeta\mathscr{C}(x_t)-\mathbb{B}p^{\dagger}\right\|\left\|x_t-p^{\dagger}\right\|,\end{aligned} \tag{50}$$

which implies that

$$\lim_{t\to 0}\left\|\mathbb{V}r_t-\mathbb{A}x_t\right\|=\lim_{t\to 0}\left\|r_t-\mathbb{A}x_t\right\|=0. \tag{51}$$

So,

$$\lim_{t\to 0}\left\|\mathbb{V}r_t-r_t\right\|=0. \tag{52}$$

Note that

$$\begin{aligned}\left\|v_t-x_t\right\| &= \left\|\delta\mathbb{A}^*\left(\mathbb{V}\text{proj}_{\mathbb{Q}}-I\right)\mathbb{A}x_t\right\| \\ &\le \delta\|\mathbb{A}\|\left\|\mathbb{V}r_t-\mathbb{A}x_t\right\|.\end{aligned} \tag{53}$$

It follows from (51) that

$$\lim_{t\to 0}\left\|x_t-v_t\right\|=0. \tag{54}$$

From (43), (48), and (54), we get

$$\lim_{t\to 0}\left\|x_t-\mathbb{U}x_t\right\|=0. \tag{55}$$

Next we show that the net $\{x_t\}$ is relatively norm-compact as $t\to 0^+$. Assume that $\{t_n\}\subset(0,1)$ is such that $t_n\to 0^+$ as $n\to\infty$. Put $x_n:=x_{t_n}$ and $v_n:=v_{t_n}$.

From (45), we have

$$\begin{aligned}\left\|x_t - p^\dagger\right\|^2 &\le (1-t\overline{\gamma})^2\left\|u_t - p^\dagger\right\|^2 \\ &\quad + 2t\left\langle \zeta\mathscr{C}(x_t) - \mathbb{B}p^\dagger, x_t - p^\dagger\right\rangle \\ &\le (1-t\overline{\gamma})^2\left\|x_t - p^\dagger\right\|^2 \\ &\quad + 2t\zeta\left\langle \mathscr{C}(x_t) - \mathscr{C}(p^\dagger), x_t - p^\dagger\right\rangle \\ &\quad + 2t\left\langle \zeta\mathscr{C}(p^\dagger) - Bp^\dagger, x_t - p^\dagger\right\rangle \\ &\le (1-t\overline{\gamma})^2\left\|x_t - p^\dagger\right\|^2 \\ &\quad + 2t\zeta\kappa\left\|x_t - p^\dagger\right\|^2 \\ &\quad + 2t\left\langle \zeta\mathscr{C}(p^\dagger) - \mathbb{B}p^\dagger, x_t - p^\dagger\right\rangle.\end{aligned} \tag{56}$$

It follows that

$$\begin{aligned}\left\|x_t - p^\dagger\right\|^2 &\le \frac{1}{\overline{\gamma}-\zeta\kappa}\left\langle \zeta\mathscr{C}(p^\dagger) - \mathbb{B}p^\dagger, x_t - p^\dagger\right\rangle \\ &\quad + \frac{\overline{\gamma}^2\left\|x_t - p^\dagger\right\|^2}{2(\overline{\gamma}-\zeta\kappa)}t \\ &\le \frac{1}{\overline{\gamma}-\zeta\kappa}\left\langle \zeta\mathscr{C}(p^\dagger) - \mathbb{B}p^\dagger, x_t - p^\dagger\right\rangle + Mt,\end{aligned} \tag{57}$$

where $M > 0$ is a constant satisfying $\sup_{t\in(0,1)}(\overline{\gamma}^2\|x_t - p^\dagger\|^2/2(\overline{\gamma}-\zeta\kappa)) \le M$. In particular, we have

$$\left\|x_n - p^\dagger\right\|^2 \le \frac{1}{\overline{\gamma}-\zeta\kappa}\left\langle \zeta\mathscr{C}(p^\dagger) - \mathbb{B}p^\dagger, x_n - p^\dagger\right\rangle + Mt_n, \tag{58}$$

since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $q^\dagger \in C$. We deduce from the above results that

$$v_n \rightharpoonup q^\dagger, \qquad u_n \rightharpoonup q^\dagger, \qquad \mathbb{A}x_n \rightharpoonup q^\dagger, \qquad r_n \rightharpoonup \mathbb{A}q^\dagger. \tag{59}$$

By the demiclosed principle of the nonexpansive mappings $\mathbb{V}$ and $\mathbb{U}$ (see Lemma 5), we deduce $q^\dagger \in \mathrm{Fix}(\mathbb{U})$ and $\mathbb{A}q^\dagger \in \mathrm{Fix}(\mathbb{V})$. Note that $u_{n_i} = \mathrm{proj}_{\mathbb{C}} v_{n_i} \in \mathbb{C}$ and $r_{n_i} = \mathrm{proj}_{\mathbb{Q}}\mathbb{A}x_{n_i} \in \mathbb{Q}$. From (58), we deduce $q^\dagger \in \mathbb{C}$ and $\mathbb{A}q^\dagger \in \mathbb{Q}$. To this end, we deduce $q^\dagger \in \mathbb{C}\cap\mathrm{Fix}(\mathbb{U})$ and $\mathbb{A}q^\dagger \in \mathbb{Q}\cap\mathrm{Fix}(\mathbb{V})$. So, $q^\dagger \in \Omega$. We substitute $p^\dagger$ for $q^\dagger$ in (58) to obtain

$$\left\|x_n - q^\dagger\right\|^2 \le \frac{1}{\overline{\gamma}-\zeta\kappa}\left\langle \zeta\mathscr{C}(q^\dagger) - \mathbb{B}q^\dagger, x_n - q^\dagger\right\rangle + Mt_n, \tag{60}$$

since x_n weakly converges to $q^\dagger$, we deduce that $x_n \to q^\dagger$ strongly. Therefore, the net $\{x_t\}$ is relatively norm-compact.

In (58), we take the limit as $n \to \infty$ to deduce

$$\begin{aligned}\left\|q^\dagger - p^\dagger\right\|^2 &\le \frac{1}{\overline{\gamma}-\zeta\kappa} \\ &\quad \times \left\langle \zeta\mathscr{C}(p^\dagger) - \mathbb{B}p^\dagger, q^\dagger - p^\dagger\right\rangle, \quad p^\dagger \in \Omega.\end{aligned} \tag{61}$$

Hence, $q^\dagger$ solves the variational inequality

$$q^\dagger \in \Omega, \quad \left\langle \zeta\mathscr{C}(p^\dagger) - \mathbb{B}p^\dagger, q^\dagger - p^\dagger\right\rangle \ge 0, \quad p^\dagger \in \Omega, \tag{62}$$

which is equivalent to its dual variational inequality

$$\begin{gathered} q^\dagger \in \Omega, \quad \left\langle \zeta\mathscr{C}(q^\dagger) - Bq^\dagger, q^\dagger - p^\dagger\right\rangle \ge 0, \\ p^\dagger \in \Omega. \end{gathered} \tag{63}$$

Therefore, $q^\dagger = \mathrm{proj}_\Omega(\zeta\mathscr{C} + I - \mathbb{B})q^\dagger$. That is, $q^\dagger$ is the unique solution in $\mathrm{VI}(\mathbb{C}, \mathbb{A})$ of the contraction $\mathrm{proj}_\Omega(\zeta\mathscr{C} + I - \mathbb{B})$. Clearly this is sufficient to deduce that $\{x_t\}$ converges strongly to $q^\dagger$ as $t \to 0^+$. The proof is completed. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This study was supported by research funds from Dong-A University.

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